Internat. J. Math. & Math. Sci.
VOL. 15 NO. 2 (1992) 279-290

ON SUBCLASSES OF CLOSE-TO-CONVEX FUNCTIONS OF HIGHER ORDER

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(Received October 30, 1990 and in revised form October 21, 1991)

ABSTRACT. The classes $T_k(\rho)$, $0 \le \rho < 1$, $k \ge 2$, of analytic functions, using the class $V_k(\rho)$ of functions of bounded boundary rotation, are defined and it is shown that the functions in these classes are close-to-convex of higher order. Covering theorem, arc-length result and some radii problems are solved. We also discuss some properties of the class $V_k(\rho)$ including distortion and coefficient results.

1980 AMS SUBJECT CLASSIFICATION. 30C45.

KEY WORDS AND PHRASES: Analytic functions, close-to-convex, univalent, bounded boundary rotation, coefficient, positive real part.

1. THE CLASS $P_k(\rho)$

Let $P_k(\rho)$ be the class of functions $p(z)$ analytic in the unit disc $E = \{z:|z|<1\}$ satisfying the properties $p(0) = 1$ and

$$\int_0^{2\pi} \left|\frac{\text{Re } p(z) - \rho}{1 - \rho}\right| d\theta \le k\pi, \tag{1.1}$$

where $z = re^{i\theta}$, $k \ge 2$ and $0 \le \rho < 1$. This class has been introduced in [1]. We note that, for $\rho=0$, we obtain the class P_k defined by Pinchuk [2] and for $\rho = 0$, $k = 2$, we have the class P of functions with positive real part. The case $k = 2$ gives us the class $P(\rho)$ of functions with positive real part greater than ρ.

Also we can write

$$p(z) = \frac{1}{2} \int_0^{2\pi} \frac{1 + (1 - 2\rho) ze^{-it}}{1 - ze^{-it}} d\mu(t), \tag{1.2}$$

where $\mu(t)$ is a function with bounded variation on $[0,2\pi]$ such that

$$\left.\begin{aligned} &\int_0^{2\pi} d\mu(t) = 2 \\ \text{and} \quad &\int_0^{2\pi} |d\mu(t)| \le k \end{aligned}\right\} \tag{1.3}$$

From (1.1), we have the following.

THEOREM 1.1. Let $p \in P_k(\rho)$. Then

$$p(z) = (\frac{k}{4} + \frac{1}{2})\ p_1(z) - (\frac{k}{4} - \frac{1}{2})\ p_2(z),$$

whee $p_i \in P(\rho)$, $i = 1,2$.

We now prove.

THEOREM 1.2. The class $P_k(\rho)$ is a convex set.

PROOF. Let H_1, $H_2 \in P_k(\rho)$. We shall show that, for α, $\beta > 0$

$$H(z) = \frac{1}{\alpha+\beta}\left[\alpha\ H_1(z) + \beta\ H_2(z)\right]$$

belongs to $P_k(\rho)$.

From Theorem 1.1, we can write

$$H(z) = \frac{1}{\alpha+\beta}\left[\alpha\{(\frac{k}{4} + \frac{1}{2})\ p_1(z) - (\frac{k}{4} - \frac{1}{2})\ p_2(z)\}\right.$$

$$\left. + \beta\{(\frac{k}{4} + \frac{1}{2})\ p_3(z) - (\frac{k}{4} - \frac{1}{2})\ p_4(z)\}\right],$$

where $p_i \in P(\rho)$, $i = 1,2,3,4$.

Now, writing $p_i(z) = (1-\rho)\ h_i(z) + \rho$, $i=1,2,3,4$, see [3], we have

$$\frac{H(z)-\rho}{1-\rho} = (\frac{k}{4} + \frac{1}{2})\ \{\frac{1}{\alpha+\beta}\ (\alpha h_1(z) + \beta h_3(z))\} - (\frac{k}{4} - \frac{1}{2})\ \{\frac{1}{\alpha+\beta}(\alpha h_2(z) + \beta h_4(z))\}$$

$$= (\frac{k}{4} + \frac{1}{2})\ f_1(z) - (\frac{k}{4} - \frac{1}{2})\ f_2(z),$$

where f_1 and $f_2 \in P$, since P is a convex set, see [2] and this gives us the required result.

THEOREM 1.3. Let $p \in P_k(\rho)$ and be given by $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$. Then

(i) $$\frac{1}{2\pi}\int_0^{2\pi} |p(re^{i\theta})|^2\, d\theta < \frac{1+[k^2(1-\rho)^2-1]r^2}{1-r^2}$$

and

(ii) $$\frac{1}{2\pi}\int_0^{2\pi} |p'(re^{i\theta})|d\theta < \frac{k(1-\rho)}{1-r^2}$$

PROOF. (i) Using Parseval's identity, we have

$$\frac{1}{2\pi}\int_0^{2\pi} |p(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2\ r^{2n}$$

$$= 1 + k^2(1-\rho)^2 \sum_{n=1}^{\infty} r^{2n} = \frac{1+[k^2(1-\rho)^2-1]r^2}{(1-r^2)},$$

where we have used an easily established sharp result $|c_n| \le k(1-\rho)$, for all $n \geq 1$.

(ii) By using Theorem 1.1, we can write

$$p(z) - \rho = (\frac{k}{4} + \frac{1}{2})(1-\rho)\, h_1(z) - (\frac{k}{4} - \frac{1}{2})\, (1-\rho)\, h_2(z),$$

where h_1, $h_2 \in P$.

Therefore,

$$p'(z) = (\frac{k}{4} + \frac{1}{2})\, (1-\rho)\, h_1'(z) - (\frac{k}{4} - \frac{1}{2})\, (1-\rho)\, h_2'(z) \qquad (1.4)$$

Now, for all $h \in P$, we have

$$h'(z) = \frac{2w'(z)}{(1+w(z))^2},$$

where $w(z)$ is a Schwarz function [3], and

$$\frac{1}{2\pi} \int_0^{2\pi} |h'(re^{i\theta})| d\theta = \frac{1}{2\pi} \int_0^{2\pi} \frac{2|w'(re^{i\theta})|}{|1 + w(re^{i\theta})|^2} d\theta$$
$$\leq \frac{2}{1-r^2}. \qquad (1.5)$$

Hence, from (1.4) and (1.5), we have

$$\frac{1}{2\pi} \int_0^{2\pi} |p'(re^{i\theta})| d\theta \leq \frac{k(1-\rho)}{1-r^2},$$

which is the required result.

From Theorem 1.1 and the properties of the class $P(\rho)$, we immediately have the following.

THEOREM 1.4. Let $p \in P_k(\rho)$. Then

$$\frac{1 - k(1-\rho)r + (1-2\rho)r^2}{1 - r^2} \leq \operatorname{Re} p(z) \leq \frac{1 + k(1-\rho)r + (1-2\rho)r^2}{1 - r^2}$$

THEOREM 1.5. Let $p \in P_k(\rho)$. Then $p \in P$ for $|z| < r_0$, where r_0 is given by

$$r_0 = 2/[k(1-\rho) + \sqrt{k^2(1-\rho)^2 - 4(1-2\rho)}], \quad \rho \neq \frac{1}{2} \qquad (1.6)$$

When $\rho = 0$, we obtain the results proved in [2].

2. THE CLASS $V_k(\rho)$

DEFINITION 2.1. Let $V_k(\rho)$ denote the class of analytic and locally univalent functions f in E with normalization $f(0) = 0$, $f'(0) = 1$ and satisfying the condition

$$\frac{(zf'(z))'}{f'(z)} \in P_k(\rho), \quad 0 \leq \rho < 1, \quad k \geq 2$$

When $\rho = 0$, we obtain the class V_k of functions with bounded boundary rotation. The class $V_k(\rho)$ also generalizes the class $C(\rho)$ of convex functions of order ρ.

It can easily be seen [1] that $f \in V_k(\rho)$ if and only if there exists $F \in V_k$ such that

$$f'(z) = (F'(z))^{1-\rho} \qquad (2.1)$$

In the following, we will study the distortion theorems for the class $V_k(\rho)$. We will use the hypergeometric functions

$$G(a,b;\ c,z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\ \Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 u^{a-1}\ (1-u)^{c-a-1}\ (1-zu)^{-b}\ du,$$

where Re a>0 and Re(c-a)>0. These functions are analytic for $z \in E$ [4]. In addition, we define the functions

$$M_1(a,b;\ c,r) = \frac{2^{b-1}}{a} \left[G(a,b;\ c,-1) - r_1^{-a}\ G(a,b;\ c,-r_1^{-1})\right]$$

and (2.2)

$$M_2(a,b;\ c,r) = \frac{2^{b-1}}{a} \left[G(a,b;\ c,-1) - r_1^{a}\ G(a,b;\ c,-r_1)\right] \quad ,$$

where $\quad r_1 = \frac{1-r}{1+r}$

THEOREM 2.1. Let $f \in V_k(\rho)$. Then, for $|z| = r$ $(0 < r < 1)$, we have

$$M_2(a,b;\ c,r) \le |f(z)| \le M_1(a,b;\ c,r), \qquad (2.3)$$

where

$$\left.\begin{aligned} a &= (\tfrac{k}{2} - 1)\ (1 - \rho) + 1, \\ b &= 2\rho \\ c &= (\tfrac{k}{2} - 1)\ (1 - \rho) + 2 \end{aligned}\right\} \qquad (2.4)$$

and M_1, M_2 are as defined in (2.2).

This result is sharp.

PROOF. Using (2.1) and the well-known bounds for $|F'(z)|$ with $F \in V_k$, see [2], we have

$$\frac{(1-|z|)^{(\frac{k}{2}-1)(1-\rho)}}{(1+|z|)^{(\frac{k}{2}+1)(1-\rho)}} \le |f'(z)| \le \frac{(1+|z|)^{(\frac{k}{2}-1)(1-\rho)}}{(1-|z|)^{(\frac{k}{2}+1)(1-\rho)}} \qquad (2.5)$$

Let d_r denote the radius of the largest schlicht disk centered at the origin contained in the image of $|z| < r$ under $f(z)$. Then there is a point z_o, $|z_o| = r$, such that $|f(z_o)| = d_r$. The ray from 0 to $f(z_o)$ lies entirely in the image of E and the inverse image of this ray is a curve in $|z| < r$.

Thus

$$d_r = |f(z_o)| = \int_C |f'(z)|\ |dz|$$

$$> \int_C \frac{(1-|z|)^{(\frac{k}{2}-1)(1-\rho)}}{(1+|z|)^{(\frac{k}{2}+1)(1-\rho)}} |dz|$$

$$> \int_0^{|z|} \frac{(1-t)^{(\frac{k}{2}-1)(1-\rho)}}{(1+t)^{(\frac{k}{2}+1)(1-\rho)}} dt$$

$$= \int_0^{|z|} \left(\frac{1-t}{1+t}\right)^{(\frac{k}{2}-1)(1-\rho)} \frac{dt}{(1+t)^{2(1-\rho)}}$$

Let $\frac{1-t}{1+t} = \xi$. Then $\frac{-2}{(1+t)^2} dt = d\xi$.

So

$$|f(z_o)| > 2^{2\rho-1} \left\{ \int_0^1 \xi^{(\frac{k}{2}-1)(1-\rho)} (1+\xi)^{-2\rho} d\xi - \int_0^{\frac{1-|z|}{1+|z|}} \xi^{(\frac{k}{2}-1)(1-\rho)} (1+\xi)^{-2\rho} d\xi \right\}$$

Put $\frac{1-|z|}{1+|z|} = \frac{1-r}{1+r} = r_1$ and $\xi = r_1 u$.
This gives

$$|f(z_o)| > \frac{2^{b-1}}{a} \{G(a,b;\ c,-1) - r_1^a\ G(a,b;\ c,-r_1)\}$$
$$= M_2(a,b,\ c,r),$$

where a,b,c and M_2 are respectively defined by (2.4) and (2.2). Similarly we can calculate the lower bound for $|f(z)|$ and this establishes our result.

Equality is attained in (2.3) for the function $f_o \varepsilon V_k(\rho)$ defined by

$$f_o'(z) = \frac{(1+\delta_1 z)^{(\frac{k}{2}-1)(1-\rho)}}{(1-\delta_2 z)^{(\frac{k}{2}+1)(1-\rho)}}, \quad |\delta_1| = |\delta_2| = 1 \qquad (2.6)$$

We now study the behaviour of the integral transform

$$f_\alpha(z) = \int_0^z (f'(\xi))^\alpha d\xi \qquad (2.7)$$

for $f \varepsilon V_k(\rho)$

This problem has been studied for the class of univalent normalized functions in E and for the close-to-convex functions, see [3]. We have

THEOREM 2.2. Let $f \in V_k(\rho)$, $0 \le \rho < 1$, $k \ge 2$ and let α, $0<\alpha<1$ be given. Then $f_\alpha \in V_m$ for $m \le \{\alpha(1-\rho)(k-2)+2\}$.

PROOF. From (2.1), we have

$$f'(z) = (F'(z))^{1-\rho}, \quad F \in V_k$$

Now

$$f_\alpha'(z) = (f'(z))^\alpha = (F'(z))^{\alpha(1-\rho)}$$

$$= \exp \int_{-\pi}^{\pi} -\log(1-\xi e^{-it})\, \alpha(1-\rho)\, dm(t)$$

$$= \exp \int_{-\pi}^{\pi} -\log(1-\xi e^{-it})\, d\mu(t),$$

where $d\mu(t) = \alpha(1-\rho)\, dm(t) + [1-\alpha(1-\rho)]\frac{dt}{\pi}$

Also

$$\int_{-\pi}^{\pi} d\mu(t) = \alpha(1-\rho)\int_{-\pi}^{\pi} dm(t) + \frac{1-\alpha(1-\rho)}{\pi}\int_{-\pi}^{\pi} dt = 2,$$

and

$$\int_{-\pi}^{\pi} |d\mu(t)| \le \alpha(1-\rho)\int_{-\pi}^{\pi} |dm(t)| + \frac{1-\alpha(1-\rho)}{\pi}\int_{-\pi}^{\pi} dt$$

$$\le \alpha(1-\rho)k + 2[1-\alpha(1-\rho)] .$$

Hence the result.

We note that f_α is univalent for $\alpha \le \frac{2}{(1-\rho)(k-2)}$, since V_m consists of univalent functions for $2 \le m \le 4$. Hence f_α is univalent even if f is not univalent provided $\alpha \le \frac{2}{(1-\rho)(k-2)}$.

Using the standard technique, we can easily prove the following.

THEOREM 2.3. Let $g, h \in V_k(\rho)$ and let $\alpha \ge 0$, $\beta \ge 0$ and $\alpha + \beta \le 1-\rho$. Then

$$H(z) = \int_0^z (g'(t))^\alpha (h'(t))^\beta\, dt$$

is convex of order $\rho_1 = (1 - \frac{\alpha+\beta}{1-\rho})$ for $|z| < r_1$,

where

$$r_1 = \frac{1}{2}\left[k - \sqrt{k^2-4}\right] \tag{2.8}$$

The result is sharp when

$$g'(z) = h'(z) = \left[\frac{(1-z)^{(\frac{k}{2}-1)(1-\rho)}}{(1+z)^{(\frac{k}{2}+1)(1-\rho)}}\right].$$

We now prove the following.

THEOREM 2.4. Let $f : f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in V_k(\rho)$. Then, for all $n > 3$, $2 \leq k < \infty$.

$$|a_n| < [k^2(1-\rho)^2 + k(1-\rho)]\, 2^{-2\rho} \left(\frac{2n}{3}\right)^{(1-\rho)(\frac{k}{2}+1)-2}$$

The function f_0 defined by (2.6) shows that the exponent $[(1-\rho)(\frac{k}{2}+1)-2]$ is best possible.

PROOF. By definition, we have

$$(zf'(z))' = f'(z)\, p(z), \qquad p \in P_k(\rho) .$$

Set

$$F(z) = (z(zf'(z))')'$$
$$= f'(z)\, [p^2(z) + zp'(z)] .$$

For $z = re^{i\theta}$, we have

$$n^3|a_n| \leq \frac{1}{2\pi\, r^{n-3}} \int_0^{2\pi} |f'(z)|\, |p^2(z) + zp'(z)|\, d\theta$$

Using (2.5) and theorem 1.3, we obtain

$$n^3|a_n| \leq \frac{1}{r^{n-3}} \frac{(1+r)^{(1-\rho)(\frac{k-2}{2})}}{(1-r)^{(1-\rho)(\frac{k+2}{2})}} \left\{\frac{1+\{k^2(1-\rho)^2-1\}r^2 + k(1-\rho)}{1-r^2}\right\}$$

$$= \frac{1}{r^{n-3}} \frac{(1+r)^{(1-\rho)(\frac{k-2}{2})-1}}{(1-r)^{(1-\rho)(\frac{k+2}{2})+1}} \{1+k(1-\rho) + \{k^2(1-\rho)^2-1\}r^2\}$$

Let $r = 1 - \frac{3}{n}$, $n > 3$. Then

$$n^3|a_n| \leq [k^2(1-\rho)^2 + k(1-\rho)]e^3 . \left(2-\frac{3}{n}\right)^{(1-\rho)(\frac{k-2}{2})} \left(\frac{n}{3}\right)^{(1-\rho)(\frac{k+2}{2})+1}$$

$$= [k^2(1-\rho)^2 + k(1-\rho)]e^3 . \left(\frac{n}{3}\right)^{[(1-\rho)(\frac{k+2}{2})-2]} . \frac{n^3}{27} \left(2-\frac{3}{n}\right)^{(1-\rho)(\frac{k}{2}-1)-1}$$

Thus, for $n>3$,

$$|a_n| < [k^2(1-\rho)^2 + k(1-\rho)](2)^{-2\rho} . \left(\frac{2n}{3}\right)^{(1-\rho)(\frac{k}{2}+1)-2}$$

THEOREM 2.5. Let $f \in V_k(\rho)$, $\rho \neq 1/2$. Then f maps $|z| < r_0$ onto a convex domain where r_0 is given by (1.6). The function f_o, defined by (2.6) shows that this result is sharp.

The proof is straightforward and follows immediately from the definition and Theorem 1.5.

Furthermore it can easily be shown that if $f \in V_k(\rho)$, then f is convex of order ρ for $|z| < r_1$ where r_1 is given by (2.8).

3. THE CLASS $T_k(\rho)$.

A class T_k of analytic functions related with the class V_k has been introduced and studied in [5]. We now define the following.

DEFINITION 3.1. Let f with $f(0) = 0$, $f'(0) = 1$ be analytic in E. Then $f \in T_k(\rho)$, $k \geq 2$, $0 \leq \rho < 1$, if there exists a function $g \in V_k(\rho)$ such that $\frac{f'(z)}{g'(z)} \in P$ for $z \in E$.

Note that $T_k(0) = T_k$ and $T_2(0)$ is the class of close-to-convex functions.

THEOREM 3.1. Let $f \in T_k(\rho)$. Then

$$|f(z)| \geq M_2(a+1, b; c+1, r),$$

where $M_2(a,b; c,r)$ is defined by (2.2) and a,b,c are given by (2.4). This result is sharp.

PROOF. Since $f \in T_k(\rho)$, we can write

$$f'(z) = g'(z)\, h(z), \quad g \in V_k(\rho),\ h \in P.$$

It is well-known that for $h \in P$

$$|h(z)| \geq \frac{1 - |z|}{1 + |z|} \tag{3.1}$$

Thus, using (3.1) and (2.5), we have

$$|f'(z)| \geq \frac{(1 - |z|)^{(\frac{k}{2} - 1)(1-\rho) + 1}}{(1 + |z|)^{(\frac{k}{2} + 1)(1-\rho) + 1}}$$

Proceeding in the same way as in Theorem 2.1, we obtain the required result.

REMARK 3.1. When $\rho = 0$, $f \in T_k$ and since in this case $b = 0 < 1$, $c = 1+a-b$, we have $G(a,b; c, -1) = 1$. Letting $r \to 1$, with $\rho = 0$, in Theorem 3.1, we see that the image of E under functions f in T_k constains the schlicht disk $|z| < \frac{1}{k+2}$.

We now give a necessary condition for a function f to belong to the class $T_k(\rho)$.

THEOREM 3.2. Let $f \in T_k(\rho)$. Then, with $z = re^{i\theta}$ and $\theta_1 < \theta_2$; $0 < \rho < 1$,

$$\int_{\theta_1}^{\theta_2} \text{Re}\, \frac{(zf'(z))'}{f'(z)}\, d\theta > -k(1-\rho)\, \frac{\pi}{2}\, .$$

PROOF. We can write

$$f'(z) = (g_1'(z))^{1-\rho}\, (h_1(z))^{1-\rho}, \quad \text{for some } g_1 \in V_k,\ h_1 \in P.$$

So

$$f'(z) = (g_1'(z)\, h_1(z))^{1-\rho} = (f_1'(z))^{1-\rho}, \tag{3.3}$$

for some $f_1 \in T_k$.

Hence

$$\frac{(zf'(z))'}{f'(z)} = (1-\rho)\, \frac{(zf_1'(z))'}{f_1'(z)} + \rho$$

The required result follows on noting that, for $\theta_1 < \theta_2$, $f_1 \in T_k$

$$\int_{\theta_1}^{\theta_2} \text{Re}\, \frac{(zf_1'(z))'}{f_1'(z)}\, d\theta > -\frac{k}{2}\, \pi\, , \quad \text{see [5].}$$

REMARK 3.2. In [1], Goodman introduced the class $K(\beta)$ of normalized analytic functions which are close-to-convex of order $\beta > 0$ and showed that if f is analytic in E and $f'(z) \neq 0$, then for $\beta > 0$, $f \in K(\beta)$ if for $z = re^{i\theta}$ and $\theta_1 < \theta_2$

$$\int_{\theta_1}^{\theta_2} \text{Re}\, \frac{(zf'(z))'}{f'(z)}\, d\theta > -\beta\pi$$

When $0 < \beta < 1$, $K(\beta)$ consists of univalent functions, whilst if $\beta > 1$, f need not even be finitely-valent.

We note that Theorem 3.2 shows that

$$T_k(\rho) \subset K\left(\frac{k(1-\rho)}{2}\right)\, .$$

Hence $T_k(\rho)$ consists entirely of univalent functions if $2 < k < \frac{2}{1-\rho}$. It also follows easily from the definition that the class $T_k(\rho)$ forms a subset of a linear-invariant family of order $[\frac{k}{2}(1-\rho)+1]$.

Using the method of Clunie and Pommerenke as modified by Thomas [7], we can easily prove the following:

THEOREM 3.3. Denote by $L(r,f)$ the length of the image of the circle $|u|=r$ under f and by $M(r) = \max_{\theta} |f(re^{i\theta})|$. Then, for $0 < r < 1$,

$$L(r) < A(k,\rho)\, M(r)\, \log \frac{1}{1-r}\, ,$$

where $A(k,\rho)$ is a constant depending only on k and ρ.

Let $P_{\alpha,1}$ denote the class of functions $p(z)$ in E given by

$$p(z) = 1 + c_1 z + c_2 z^2 + \ldots\ldots\ldots\ldots\ldots$$

which satisfy the inequality

$$\left|p(z) - \frac{1}{2\alpha}\right| < \frac{1}{2\alpha}, \quad 0 < \alpha < 1$$

The class $P_{\alpha,1}$ has been introduced in [8] and it is shown there that, for $p \in P_{\alpha,1}$, $|z| = r < 1$.

$$\left|\frac{p'(z)}{p(z)}\right| < \frac{(1 + c)}{(1 + cr)(1 - r)}, \tag{3.4}$$

where $c = 1 - 2\alpha$

We now prove the following.

THEOREM 3.4. Let $g \in V_k(\rho)$ and let $\frac{f'(z)}{g'(z)} \in P_{\alpha,1}$. Then f is a convex function of order ρ for $|z| < r$ where $r \in (0,1)$ is the least positive root of the equation

$$(1-\rho)cx^3 - [(\rho+c) + ck(1-\rho)]x^2 + [\rho(k-c) - (1+k)]x + (1-\rho) = 0$$

PROOF. We can write

$$f'(z) = (g_1'(z))^{1-\rho} p(z), \quad g_1 \in V_k, \quad p \in P_{\alpha,1}$$

So

$$\mathrm{Re}\left[\frac{(zf'(z))'}{f'(z)} - \rho\right] > (1-\rho)\ \mathrm{Re}\left[\frac{(zg_1'(z))'}{g_1'(z)}\right] - \left|\frac{zp'(z)}{p(z)}\right|$$

Using Theorem 1.4 with $\rho = 0$ and (3.4), we have the required result.

Furthermore, if

$$T(r) = (1-\rho)cr^3 - [(\rho+c) + ck(1-\rho)]r^2 + [\rho(k-c) - (1+k)]r + (1-\rho),$$

then we note that

$$T(0) = (1-\rho) > 0$$

$$T(1) = -2\rho c - 2\rho - ck(1-\rho) - k(1-\rho) < 0$$

Thus $r \in (0,1)$.

COROLLARY 3.1. When $\alpha = 0$, $c = 1$ and $\rho = 0$, $f \in T_k$. Thus f maps $|z| < r = \frac{1}{2}[(k+2) - \sqrt{k^2 + 4k}]$ onto a convex domain and this result is sharp, see [5].

COROLLARY 3.2. When $\rho=0$, $\alpha = \frac{1}{2}$, and then we have $\left|\frac{f'(z)}{g'(z0} - 1\right| < 1$ for $g \in V_k$. Then f is convex for $|z| < r = \frac{1}{k+1}$. For k=4, V_k consists of univalent functions and in this case $r = \frac{1}{5}$. This result is proved in [8]. For $\alpha = 0$, $k = 4$ and $\rho = 0$, we obtain the known result $r = 3 - 2\sqrt{2}$ of Ratti [9] and when $k = 2$, we have the well-known result giving us the radius of convexity for close-to-convex functions.

Finally we have

THEOREM 3.5. Let $f \in V_k(\rho)$ and let

$$F(z) = \frac{1}{1+m} z^{1-m} [z^m f(z)]', \quad m = 1,2,3,\ldots\ldots .$$

Then $F \in T_k(\rho)$ for all $|z| < r_2$, where, for $(1-2\rho-m)\neq 0$, $0<\rho<1$,

$$r_2 = 2(1+m)/\left[(1-\rho)k + \sqrt{(1-\rho)^2k^2 - 4(1-2\rho-m)(1+m)}\right],$$

The proof is straightforward when we note that

$$\mathrm{Re}\,\frac{F'(z)}{f'(z)} = \frac{1}{1+m}\left[\left\{\mathrm{Re}\,\frac{(zf'(z))'}{f'(z)}\right\} + m\right]$$

and then use theorem 1.4.

ACKNOWLEDGEMENT. The author is grateful to the referee for his valuable comments and suggestions.

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