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GEOMETRIC PRESENTATIONS OF CLASSICAL KNOT GROUPS

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ABSTRACT. The question addressed by this paper is, how close is the tunnel number of a knot to the minimum number of relators in a presentation of the knot group? A dubious, but useful conjecture, is that these two invariants are equal. (The analogous assertion applied to 3-manifolds is known to be false. [1]) It has been shown recently [2] that not all presentations of a knot group are "geometric". The main result in this paper asserts that the tunnel number is equal to the minimum number of relators among presentations satisfying a somewhat restrictive condition, that is, that such presentations are always geometric.

1. DEFINITIONS AND CONVENTIONS.

Let K be any non-trivial knot in S^3; V = Cl(N(K) = solid torus containing K; X = $Cl(S^3 - V)$, the knot exterior.

Let G = the knot group $\pi_1(S^3 - K)$. If G has the deficiency-1 presentation $< g_1,\dots,g_{n+1} | r_1,\dots,r_n >$, we will abuse notation and use g_i and r_j to denote loops representing the corresponding group elements. We restrict consideration to presentations having representative loops g_i and r_j such that there are disks D_j X with $\partial D_j = r_j$ and $int(D_j)$ $g_i = \emptyset$ for all i,j. Let s = s(K) = minimum number of relators in such a presentation. $X_0 = Cl(N(g_i))$, a neighborhood of a "bouquet" of generators, is a handlebody of genus s+1.

Let t = t(K) be the tunnel number of K, which is the minimum number of arcs that need to be attached to K so that the complement is an open handlebody. Equivalently, t(K) is the minimum number of 1-handles (tunnels) that need to be removed from the knot exterior to obtain a Heegard decomposition of S^3. t(K) is also called the "Heegard" genus of K [3,2].

2. THEOREM: $s(K) = t(K)$ for any tame knot K.

The direction $s(K) < t(K)$ is easy: The fundamental group of the complement of K together with the $t = t(K)$ arcs is free on $t+1$ generators. For each arc, a loop around that arc provides a relation. Such a presentation is called geometric.

Outline of proof, of $t(K) < s(K)$: For each r_j, the corresponding disk D_j can be moved so that $\partial D_j \subseteq \partial X_0$, int $(D_j) \subseteq S^3 - X_0$. We build up a space X_s from X_0 by stages using 2-handle surgeries along these disks; X_j is the result of the jth surgery. X_s is the result of the last surgery, with $S^3 - X_s = N'(K)$, a solid torus neighborhood of K. Thus a Heegard decomposition of S^3 can be obtained by drilling tunnels through the disks used in the 2-handle surgeries.

The rest of this paper is concerned with filling in the details.

3. THE CONSTRUCTION.

Let $X_1 = X_0 \cup Cl(N(D_1))$, by 2-handle surgery along D_1; $D_1' = D_1$. For each $j > 1$, for each $i < j$, if $N(D_i')$ $D_j \neq \emptyset$ then modify D_j to remove intersections. (Fig. 1a,b). Of course, there may be more intersections, but they must all be simple arcs or circles. In the latter case, the intersections can simply be "capped off". In the case of multiple arcs, start with the outermost arc of the new disk and continue until all of the intersections have been removed. The modification may result in an increase in the number of disks. If any disk is "trivial", simply "throw it out". We consider a disk to be trivial if the union of the disk with X_{j-1} and previous disks contains an incompressible 2-sphere. The determination of which disks are trivial is not unique: after D_j has been modified, if extra disks result, then select one at a time. If you get an incompressible sphere, discard that disk from consideration and move on to the next.

Let D_j' be the union of disks obtained from D_j after modification then removal of trivial disks. For $j > 1$, let X_j be the space obtained from X_{j-1} by 2-handle surgeries along D'_j. By minimality of s, at least s 2-handle surgeries are performed. The boundary of X_j differs from that of X_{j-1} either by a decrease in the genus of one component or by an increase in the number of components. At the same time none of the boundary components can be a 2-sphere. Thus the end result of the construction, X_s, is bounded by tori only: $\partial X_s = T_0 \ \ldots \ T_k$. We need to show that $k = 0$, where T_0 bounds a solid torus $N'(K)$.

4. Before finishing the proof, we make some essential observations.

(i) $\pi_1(X_s) \simeq \pi_1(X) = G$, the isomorphism being induced by inclusion, follows from a standard Seifert-Van Kampen argument.

(ii) There is no 2-sphere $S \subsetneq X_s$ that separates two of these tori. This follows from the construction itself.

(iii)For each i, $\pi_1(T_i) \to \pi_1(X_s)$ is injective. Otherwise there exists an essential loop α in T_i and a disk D in X_s bounded by α. $D \times \{-1,1\}$ together with part of the torus is a sphere that can be moved away from the torus and separates it from the other tori, contradicting remark (ii).

(iv) Each torus bounds a solid torus in S^3 on one side and hence a (possibly trivial) knot exterior on the other [4]. If $i > 0$, then the solid side of T_i must contain the knot; otherwise, there exists a contractible loop in the exterior of K which is not contractible in X_s, thus contradicting (i).

(v) For $i > 1$, let B_i= component of $S^3 - X_s$ bounded by T_i. Let B_0 be the component of

(a) (b)

Figure 1

$X - X_s$ bounded by T_0 $\partial N(K)$. By (iv) B_i is a knot exterior for $i > 1$, hence $\pi_1(T_i) \to \pi_1(B_i)$ is injective. For related reasons, $\pi_1(T_0)$ $\pi_1(B_0)$ is injective.

5. We will now proceed to show that $B_i = \emptyset$ for $i > 0$.

Let $W_0 = X_s$ B_0, $W_j = W_{j-1}$ B_j for $j > 0$. In particular, $W_k = X$.

For $i > 0$, we apply the Seifert-Van Kampen Theorem to the injectivity of $\pi_1(T_i) \to \pi_1(X_s)$ and $\pi_1(T_i) \to \pi_1(B_i)$ to get injectivity of $\pi_1(B_i) \to \pi_1(X_s$ $B_i)$ and $\pi_1(X_s) \to \pi_1(X_s$ $B_i)$. In particular, $\pi_1(X_s) \to \pi_1(W_0)$ is injective. We apply the inductive hypothesis that $\pi_1(X_s) \to \pi_1(W_{j-1})$ is injective, together with another application of Seifert-Van Kampen, to get injectivity of $\pi_1(W_{j-1}) \to \pi_1(W_j)$ and of $\pi_1(X_s \cup B_j) \to \pi_1(W_j)$. Since $W_k = X$, we have

$$\pi_1(X_s) \to \pi_1(W_0) \to \pi_1(W_1) \to \dots \to \pi_1(X) = G.$$

Since this composition is an isomorphism, all of the component injections must also be isomorphisms. From this and the Seifert-Van Kampen construction, it follows that $\pi_1(X_s) \simeq \pi_1(X_s$ $B_i)$ and hence $\pi_1(B_i) \simeq \pi_1(T_i)$ for $i > 0$, which is impossible for a knot exterior. Thus only B_0 is non-empty.

6. It remains to verify that $\mathrm{int}(B_0$ $V) = N'(K)$, just a fatter neighborhood of K, thus completing the proof.

The boundary of B_0 is the union of two tori, one of which is $\partial N(K)$. The inclusions in both cases must induce isomorphisms of fundamental groups. Since $\pi_1(B_0) \simeq \pi_1(T_0) \simeq \pi_1(\partial N(K)) \simeq Z + Z$ is abelian, the Hurewicz homomorphism is an isomorphism onto $H_1(B_0)$. It follows that the meridional loops m_1 and m_2 on the bounding tori are homologous. Let α be a path from the base point of m_1 to the base point of m_2. Then $m_1 \alpha m_2^{-1} \alpha^{-1}$ is homogous to 0 and hence homotopically trivial. Thus the meridians bound an annulus and the torus T_0 is parallel to $\partial N(K)$, which suffices to complete the proof.

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