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ON SEMI-HOMEOMORPHISMS

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ABSTRACT. In the first part of our work we show a condition for a semi-homeomorphism in the sense of Crossley and Hildebrand (s.h.C.H) to be a semi-homeomorphism in the sense of Biswas (s.h.B). Certain relevant examples are provided. Next, we define strong semi-homeomorphisms via "nice" restrictions of semi-homeomorphisms ("global condition") and we show that the new class of functions actually coincides with semi-homeomorphisms. Then, in the third part we introduce local semi-homeomorphisms (l.s.h.C.H.) via a corresponding "local condition" for restrictions. A few results pertaining to the preservation of some topological properties under this new class of functions are examined.

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1. s.h.C.H. VERSUS s.h.B.

We shall start with the following definitions.

A subset $S \subset X$ is said to be *semi-open* if there is an open set $U \subset X$ such that $U \subset S \subset \bar{U}$.

A function $f: X \to Y$ is said to be a *semi-homeomorphism in the sense of Crossley and Hildebrand* (or simply, s.h.C.H.) [1] if:

1. f is bijective
2. f is irresolute (i.e. inverse images of semi-open sets are semi-open)
3. f is pre-semi-open (i.e. images of semi-open sets are semi-open)

Further, a function $f: X \to Y$ is said to be a *semi-homeomorphism in the sense of Biswas* (or simply s.h.B. ([2]), if

1. f is bijective
2. f is continuous
3. f is semi-open

Clearly every homeomorphism is both s.h.B and s.h.C.H. .

T. Neubrunn [3] has shown that there are s.h.C.H. that are not s.h.B. . Answering his question Z. Piotrowski [4] has shown an example of a s.h.B. which is not s.h.C.H. Further he also obtained certain conditions for a s.h.C.H. to be a s.h.B.

In this paragraph we shall prove the following.

Proposition 1. Assume Y has a clopen base. If $f: X \to Y$ is one-to-one, semi-open and somewhat continuous then f is irresolute.

PROOF: Let $A \subset Y$ be semi-open. Let $x \in f^{-1}(A)$ i.e. $f(x) = y \in A$. We shall show that $x \in \overline{\text{Int } f^{-1}(A)}$.

For any open set U containing x, the set f(U) is semi-open and contains y. Further $f(U) \cap A \neq \emptyset$. Since f(U) is open, Y having a clopen base and A is open - all semi-open and open sets coincide, under the assumption upon Y, there is a nonempty open set G such that

$$G = f(U) \cap A. \tag{1.1}$$

Clearly,

$$f^{-1}(G) \subset f^{-1}(f(U) \cap A) \subset f^{-1}(f(U)) = U \tag{1.2}$$

f being one-to-one.

Now, somewhat continuity of f implies that there is an open set $V \subset f^{-1}(G)$, $V \neq \emptyset$. Therefore $V \subset U$, $V \subset f^{-1}(A)$. And since U is an arbitrary neighborhood of x, we have $x \in \overline{\text{Int } f^{-1}(A)}$. Thus $f^{-1}(A)$ is semi-open. □

REMARK: The author is indebted to the referee for pointing out that Proposition 1 generalizes Theorem 2.2 of [5].

The assumption upon Y to have a *clopen base* is essential. In fact:

EXAMPLE 2. There is a semi-open, semi-continuous (hence somewhat continuous!) bijection $f: [0,1] \longrightarrow [0,1]$ which is *not* irresolute. Take $f(x) = x$, if $x \in \left[0,\frac{1}{3}\right)$, $f(x) = x + \frac{1}{3}$, if $x \in \left[\frac{1}{3}, \frac{2}{3}\right]$. $f(x) = -x + \frac{4}{3}$, if $x \in \left(\frac{2}{3},1\right]$. Observe that $f^{-1}\left[\frac{1}{3}, \frac{2}{3}\right]$ is not semi-open.

In fact, there is even a *continuous*, semi-open injective function between two topological spaces which is not irresolute. We shall provide here such an example, originally designed for a different purpose.

EXAMPLE 3. ([4], Example 19, p. 8) Let X = Y = {a, b, c, d}. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ denote the topologies for X and Y, respectively, such that $\mathcal{O}_1$ = {∅, X, {a}, {b}, {a,b}, {b,c,d}} and $\mathcal{O}_2$ = {∅, Y, {a}, {b}, {a,b}}. Let $f:(X,\mathcal{O}_1) \longrightarrow (Y,\mathcal{O}_2)$ be the identity function. It is easy to see that f is continuous and semi-open but not irresolute, since {a,c} is semi-open in Y while it is not semi-open in X.

REMARK 4. Example 2 above is *the best possible* in the class of semi-continuous bijections $f: [0,1] \longrightarrow [0,1]$ (or more generally, $f: X \to Y$, X-compact, Hausdorff and Y being Hausdorff,) in the sense that if f is to be additionally continuous, then, being continuous bijection from a compact, Hausdorff space onto a Hausdorff space, it is a homeomorphism, (see [6], Thm 2.1, p. 226). Now, every homeomorphism (actually openess and continuity suffices) implies irresoluteness of f - we leave the proof of this fact to the reader, also see [1].

Since it is well-known that continuity and somewhat openess imply pre-semi-openess, see also [1] we have the following Corollary from Proposition 1.

COROLLARY 5. Assume Y has a clopen base. If $f: X \to Y$ is s.h.B, then f is s.h.C.H. .

2. STRONG SEMI-HOMEOMORPHISMS ARE PRECISELY SEMI-HOMEOMORPHISMS.

In this paragraph a "semi-homeomorphism" stands for s.h.C.H. .

The following, seemingly stronger conditions (*) and (**) which define - what we call - a *strong semi-homeomorphism* are actually equivalent (!) to the semi-homeomorphicity of f, see the following.

THEOREM 6. A function $f: X \to Y$ is a semi-homeomorphism if and only if:

(*) f is bijective and

(**) $\forall U \subset X$, U-open, $f|U$ is a semi-homeomorphism

PROOF: In fact, it is easy to see that if f satisfies (*) and (**), then f is a semi-homeomorphism - take $U = X$ in (**). Conversely, let $X = \cup \{U_\alpha : \alpha \in A\}$, where each U_α is open and suppose that each restriction $f|U_\alpha$ is both pre-semi-open and irresolute. We shall show that f is also such.

Let $(f|U_\alpha) : U_\alpha \to Y$ denote the restriction of f to U_α. We shall show that f is irresolute. Really, given a semi-open set $K \subset Y$ we have:

$$f^{-1}(K) = \cup \{f^{-1}(K) \cap U_\alpha : \alpha \in A\} = \cup \{(f|U_\alpha)^{-1}(K) : \alpha \in A\}. \qquad (2.1)$$

The latter set is semi-open as the sum of semi-open sets.

Similarly, we shall prove that f is pre-semi-open. Let $L \subset X$ be semi-open, in X. Then $L = \cup \{L \cap U_\alpha : \alpha \in A\}$. Then:

$$\begin{aligned} f(L) &= f(\cup \{L \cap U_\alpha : \alpha \in A\}) = \\ &= \cup f\{L \cap U_\alpha : \alpha \in A\} = \\ &= \cup \{(f|U_\alpha)(L) : \alpha \in A\}. \end{aligned} \qquad (2.2)$$

And again, the latter set is semi-open, in Y. □

The following example shows that the assumption "for *every* open" in (**) is real. As one can see, the restrictions $f|U_\alpha$, $\alpha \in A$ are even *homeomorphisms* (!) for every $U_\alpha \neq X$.

EXAMPLE 7. (See Example 3 of §1.) There is a function $f: X \to Y$ such that

1. f is bijective and
2. $\forall U_\alpha \subsetneq X$, U_α-open, $\alpha \in A$, $f|U_\alpha$ is a homeomorphism

(hence, a semi-homeomorphism) whereas f: $X \to Y$ is
not a semi-homeomorphism.

Really, f|{a}, f|{b} and f|{a,b} are homeomorphisms. Now, consider f|{b,c,d}. We have X = Y = {b,c,d} and $\mathcal{O}_1 \cap X$ = {∅, {b,c,d}, {b}}, whereas $\mathcal{O}_2 \cap Y$ = {∅, {b,c,d}, {b}}. And, here again, f|{b,c,d} is a homeomorphism.

3. LOCAL SEMI-HOMEOMORPHISMS.

Local homeomorphisms, being a very natural generalization of homeomorphisms, occupy an important place in topology, especially in the theory of 1-dimensional continua (curves) as well as some parts of algebraic topology, see also [7] for an extensive treatment of this topic.

Let us define our new class of functions. We say that a function f: $X \to Y$ is a *local semi-homeomorphism* in the sense of Crossley and Hildebrand if:

1. f is bijective and
2. $\forall x \in X\ \exists$ U-open, $x \in U \subset X$ such that f|U is a semi-homeomorphism in the sense of Crossley and Hildebrand.

Well, it is easy to see that every semi-homeomorphism is a local semi-homeomorphism; take U = X. Since every homeomorphism is a semi-homeomorphism, see [1] we have the following diagram:

homeomorphism → strong semi-homeomorphism ⇔ semi-homeomorphism → local semi-homeomorphism

We shall now provide an example of a local semi-homeomorphism which is not a semi-homeomorphism, showing that the arrow to the right is, in general, not reversable.

EXAMPLE 8. Consider Example 3, see §1. Take {a}, {b}, {b,c,d}, {b,c,d}, respectively for open neighborhoods of a, b, c and d, respectively. Using arguments similar to ones applied in Example 7 we prove that f is a local semi-homeomorphism; it has been shown in [4], p. 508 that f is not a semi-homeomorphism.

LEMMA 9. If for every $x \in X$ there is an open set $U \subset X$, $x \in U$ such that f|U is a semi-homeomorphism in the sense of Crossley and Hildebrand, then f is somewhat continuous (inverse images of every nonempty open set if nonempty it has the nonempty interior) and f is somewhat open (image of every open nonempty set has the nonempty interior).

PROOF: Let D be a dense set in X. We shall show that f(D) is dense in f(X). This, in turn, shows that f is somewhat continuous.

In fact, suppose $y \in f(X) \setminus f(D)$ and assume further that there is an open neighborhood V containing y, such that:

$$(*) \qquad V \cap f(D) = \emptyset. \qquad (3.1)$$

Since f is "onto", there is $x \in X$, such that f(x) = y. There is an open set $U \ni x$ such that f|U is a semi-homeomorphism, f being a local semi-homeomorphism. Clearly

$D \cap U$ is dense in U; further $f(D \cap U)$ is dense in $f(U)$, f being semi-homeompriphism on U. Now, $f(U)$ is a semi-open set containing $f(x) = y$. By an elementary property of semi-open sets, $f(D \cap U)$ is dense in $V \cap \text{Int } f(U)$, and hence, also in $V \cap f(U)$. So, $V \cap f(D) \neq \emptyset$, contradicting (*).

Now, for somewhat openess part, consider a dense set D contained in $f(X)$. We shall show that $f^{-1}(D)$ is dense (in X). Suppose $f^{-1}(D)$ is not dense. So, there is a point $x \in X$ and an open neighborhood $U \ni x$ such that

$$(**) \qquad U \cap f^{-1}(D) = \emptyset . \qquad (3.2)$$

Without loss of generality we may assume that U is the open neighborhood of x from the definition of local homeomorphism (or, simply, take the intersection of the two sets, in question). Then $f(U)$ is a semi-open set, free of points of D. For otherwise the set:

$$f^{-}(f(U) \cap D) = f^{-1}(f(U)) \cap f^{-1}(D) = U \cap f^{-1}(D) \neq \emptyset, \qquad (3.3)$$

contradicting (**), which finishes the proof.

COROLLARY 10. Baireness is a local semi-topological property.

PROOF: See [8], Corollary 2, p. 410 and Lemma 9, above.

COROLLARY 11. Separability is a local semi-topological property.

PROOF: Every local semi-homeomorphism is somewhat continuous, and this implies (see [9]) that dense subsets are preserved, which in turn proves our claim.

We will close this work with the following natural

Question 12. What are the topological conditions for X and/or Y so that every local semi-homeomorphism $f: X \to Y$ is a semi-homeomorphism?

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