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TRANSLATIVITY OF ROGOSINSKI SUMMABILITY METHODS OF DIFFERENT ORDERS

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ABSTRACT. The paper deals with the problem of translativity of Rogosinski summability methods $(R_{h,r})$ of orders h, r; $0 < h-r \leq 1$. It has been shown by the author [1] that when $r = 0$, $h \in [1/2, 1]$, $(R_{h,0})$ is translative, and when $r = 0$, $h \in (0, 1/2(\sqrt{3}-1))$, $(R_{h,0})$ is neither translative to the left nor to the right. The problem is left unsettled for the rest of the interval $(0, 1/2)$ with the conjecture that if $h \in (0, 1/2)$, $(R_{h,0})$ is neither translative to the right nor to the left. In this paper we prove that when $h-r \in [1/2, 1]$, $(R_{h,r})$ is translative, and when $h-r \in (0, 1/2)$, $(R_{h,r})$ is neither translative to the right nor to the left. These results establish both, the open problem and its conjecture which have been given by the author [1].

KEY WORDS AND PHRASES. Translative, Rogosinski methods, (C,1) method, left and right translativity, equivalent to convergence.
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1. INTRODUCTION.

The series $a_0 + a_1 + \dots$ of real or complex terms with its partial sums $S_n = a_0 + a_1 + \dots + a_n$ is said to be summable by the Rogosinski method $(R_{h,r})$; $0 < h-r \leq 1$, if $t_n \to t$, as $n \to \infty$, where

$$t_n = \sum_{k=0}^{n} \cos(k+r)\theta_n a_k, \tag{1.1}$$

and

$$\theta_n = \frac{\pi}{2(n+H)}, \quad n = 0,1,2,\dots \tag{1.2}$$

In the special case in which $r = 0$, this method will reduce to the Rogosinski-Bernstein method (R_h) which has been the subject of many papers (see Al-Madi [1], Agnew [2] and Petersen [3]). The series is evaluable to t_M by the (C,1) method, if $M_n \to t_M$ as $n \to \infty$, where

$$M_n = \frac{S_0 + S_1 + \dots + S_n}{n + 1}. \tag{1.3}$$

For a summability method $A \in T_l$ (translative to the left) if, whenever $\sum a_n$ is summable A to s, then so is $\sum a_{n-1}$. $A \in T_r$ if the converse holds. $A \in T$ if, and only if $A \in T_l \cap T_r$.

Much work has already been done on translative summability methods (see Al-Madi [1,4,5], Chowdhury [6], and Kuttner [7-10].

The author [1] proved that if $h \in [\frac{1}{2}, 1]$, $(R_{h,0}) \in T$, and if $0 < h < \frac{1}{2}(\sqrt{3}-1)$, then $(R_{h,0}) \notin T_l \cup T_r$. The author left the problem unsettled for the rest of the interval $(0, \frac{1}{2})$. He conjectured that when $h \in (0, \frac{1}{2})$, then $(R_{h,0}) \notin T_l \cup T_r$.

2. OBJECT OF THE PAPER.

The object of this paper is to prove that when $h-r \in [\frac{1}{2}, 1]$, then $(R_{h,r}) \in T$, and when $h-r \notin (0, \frac{1}{2})$, then $(R_{h,r}) \notin T_l \cup T_r$. The second result establishes both the open problem and its conjecture which have been given in [1].

3. MAIN RESULTS.

In this section we will prove the following three main results:

THEOREM 3.1. If $h-r \in (\frac{1}{2}, 1]$, $(R_{h,r}) \in T$.

THEOREM 3.2. If $h-r = \frac{1}{2}$, then $(R_{h,r}) \in T$.

THEOREM 3.3. If $h-r \in (0, \frac{1}{2})$, then $(R_{h,r}) \notin T_l \cup T_r$.

The following result will be used in the proof of Theorems 3.1 and 3.2.

THEOREM 3.4. (Agnew [11]) If

$$H_n = \sum_{k=0}^{n} d_{n,k} S_k, \tag{3.1}$$

is regular and

$$\lim_{n \to \infty} \inf \left[|d_{n,n}| - \sum_{k=0}^{n-1} |d_{n,k}| \right] > 0, \tag{3.2}$$

then the transformation given by (3.1) is equivalent to convergence.

PROOF OF THEOREM 3.1. We will show that when $h-r \in (\frac{1}{2}, 1]$, then $(R_{h,r})$ and $(C,1)$ are equivalent, and the result follows from the fact that $(C,1) \in T$. Observe that since

$$a_n = (n+1)M_n - 2nM_{n-1} + (n-1)M_{n-2} \qquad n = 0,1,2,\dots \quad M_{-1} = M_{-2} = 0, \tag{3.3}$$

it follows form (1.1) that

$$t_n = \sum_{k=0}^{\infty} A_{n,k} M_k, \tag{3.4}$$

where

$$A_{n,n} = (n+1)\sin(h-r)\Theta_n, \tag{3.5}$$

$$A_{n,n-1} = n[2\sin \tfrac{1}{2}\Theta_n \cos \tfrac{1}{2}(2h-2r+1)\Theta_n - \sin(h-r)\Theta_n], \tag{3.6}$$

$$A_{n,k} = -4(k+1)\sin^2 \tfrac{1}{2}\Theta_n \cos(k+r+1)\Theta_n \quad 0 \leq k \leq n-2, \tag{3.7}$$

and

$$A_{n,k} = 0 \quad k > n. \tag{3.8}$$

Using the same technique as Agnew [2; p. 544-545], we see that if h-r is in $(\tfrac{1}{2}, 1]$, $(R_{h,r})$ is equivalent to (C,1). This completes the proof.

PROOF OF THEOREM 3.2. To prove the result, it is enough to consider some translative summability method, and to show that this method is equivalent to $(R_{h,r})$ in the case in which $h-r = \tfrac{1}{2}$. For this, we consider the sequence-to-sequence method Q given by the transformation

$$Q_n = \frac{n}{2n+1} M_{n-1} + \frac{n+1}{2n+1} M_n, \tag{3.9}$$

where M_n is given by (1.3). Using (1.3) and (3.9) to obtain $\overline{Q}_n$ in terms of Q_n, the result is

$$\overline{Q}_{n+1} = \frac{2n+1}{2n+3} Q_n. \tag{3.10}$$

This implies that $Q \in T$.

Next, put $h-r = \tfrac{1}{2}$, and write $A_{n,k}$ given in (3.7) in the form

$$A_{n,k} = -2(k+1)\sec \tfrac{1}{2}\Theta_n \sin^2 \tfrac{1}{2}\Theta_n [\cos(2k+2r+1)\tfrac{1}{2}\Theta_n + \cos(2k+2r+3)\tfrac{1}{2}\Theta_n], \tag{3.11}$$

and use (3.9) to obtain the transformation $(R_{h,r})(Q)^{-1}$,

$$t_n = \sum_{k=0}^{n} \beta_{n,k} Q_k, \tag{3.12}$$

where

$$\beta_{n,n} = (2n+1)\cos(n+r)\Theta_n, \tag{3.13}$$

$$\beta_{n,k} = -2(2k+1)\sec \tfrac{1}{2}\Theta_n \sin^2 \tfrac{1}{2}\Theta_n \cos(2k+2r+1)\tfrac{1}{2}\Theta_n \quad 0 \leq k \leq n-1. \tag{3.14}$$

It is easily seen that the transformation given by (3.12) is regular. Hence applying Theorem 3.4 to the matrix given in (3.12), one can easily show that the transformation given by (3.12) is equivalent to convergence, and $(R_{h,r})$ is equivalent to Q. This completes the proof.

PROOF OF THEOREM 3.3. We will show first that if $h-r \in (0, 1/2)$, then $(R_{h,r}) \notin T_r$. For this we consider the sequence $\{V_n\}$ given by

$$V_n = S_0 + S_1 + \dots + S_n, \tag{3.15}$$

where

$$a_n = V_n - 2V_{n-1} + V_{n-2}, \ (V_{-k} = 0 \text{ for positive integer } k). \tag{3.16}$$

Using (3.16), it follows from (1.1) that

$$t_n = \sum_{k=0}^{n} C_{n,k} V_k, \tag{3.17}$$

where

$$C_{n,n} = \sin(h+r)\theta_n, \tag{3.18}$$

$$C_{n,n-1} = \sin(1+h+r)\theta_n - 2\sin(h+r)\theta_n, \tag{3.19}$$

and

$$C_{n,k} = -4\sin^2 1/2\,\theta_n \cos(k+r+1)\theta_n, \qquad (0 \leq k \leq n-2). \tag{3.20}$$

Let $\{t_n\}$, $\{\bar{t}_n\}$ be respectively the $(R_{h,r})$ transforms of $\{S_n\}$, $\{S_{n-1}\}$. Obtain t_n in terms of $\bar{t}_n$. The result is

$$t_n = \sum_{k=0}^{n} F_{n,k} \bar{t}_{k+1}, \tag{3.21}$$

where

$$F_{n,k} = \sum_{v=k}^{n} C_{n,v} D_{v+1,k+1}, \qquad (0 \leq k \leq n) \tag{3.22}$$

and $C_{n,v}$ is given by (3.18) - (3.20), and where $D_{n,v}$ is the reciprocal matrix of $C_{n,v}$. Write

$$G_{n,v} = C_{n+1,v+1}\left[\frac{C_{n,v}}{C_{n+1,v+1}} - \frac{C_{n,n}}{C_{n+1,n+1}}\right], \qquad (0 \leq v \leq n) \tag{3.23}$$

and use the fact that

$$D_{n+1,k+1} = -\frac{1}{C_{n+1,n+1}} \sum_{v=k}^{n-1} C_{n+1,v+1} D_{v+1,k+1}, \tag{3.24}$$

to obtain from (3.22) that

$$F_{n,n-j} = \sum_{u=1}^{j} G_{n,n-u} D_{n-u+1,n-j+1} \qquad (0 \leq j \leq n). \tag{3.25}$$

We will show that if $0 < h-r < 1/2$, and $0 < z < 1/4$, then for sufficiently large n and $n^{1/2 z} \leq j \leq n^z$, $F_{n,n-j}$ is not bounded. This implies that the transformation given in (3.21) is not regular, and consequently, when $h-r \in (0, 1/2)$, $(R_{h,r}) \notin T_r$. To prove this, we will show that for sufficiently large n and $2 \leq j \leq n^z$, the terms of the sum (3.25) alternate in sign, and then we will show that the limit of $F_{n,n-j}$ is unbounded.

The inversion formula of (3.17) gives

$$\begin{vmatrix} C_{j+1,j} & C_{j+1,j+1} & 0 & 0 & . & 0 \\ C_{j+2,j} & C_{j+2,j+1} & C_{j+2,j+2} & 0 & . & 0 \\ . & . & . & . & . & . \\ C_{n,j} & . & . & . & . & C_{n,n-1} \end{vmatrix}$$

$$D_{n,j} = (-1)^{n-j}.$$

$$C_{n,n}C_{n-1,n-1}\cdots C_{j,j} \tag{3.26}$$

Using (1.2) and (3.18) - (3.20), we have from (3.26) that

$$\lim_{n\to\infty} (-1)^{u-j} \frac{D_{n-u,n-j}}{n} w^{u-j} = \frac{2}{\pi(h-r)}, \tag{3.27}$$

where $w = \frac{1-h+r}{h-r}$. (3.28)

Next, we will show that when $1 \leq u \leq j \leq n^z$, $G_{n,n-u} < 0$. This together with (3.27) imply that the terms of the sum given in (3.25) alternate in sign for all sufficiently large n uniformly in $2 \leq u \leq j \leq n^z$. Using (3.18) and (3.19), we have from (3.23) that

$$G_{n,n-1} = \sin(h-r)\Theta_n[f(\Theta_n) - f(\Theta_{n+1})], \tag{3.29}$$

where

$$f(\Theta_n) = \frac{\sin(1+h-r)\Theta_n}{\sin(h-r)\Theta_n} . \tag{3.30}$$

Differentiate, and use the mean value theorem to see that there is some x in (Θ_{n+1}, Θ_n) such that

$$G_{n,n-1} = \frac{\sin(h-r)\Theta_n}{\sin^2(h-r)x} [\Theta_n - \Theta_{n+1}] [\sin(1+h-r)x\cos(h-r)x-(1+h-r)\sin x] \tag{3.31}$$

$$< \frac{\sin(h-r)\Theta_n}{\sin^2(h-r)x} [\Theta_n - \Theta_{n+1}][\sin(1+h-r)x - (1+h-r)\sin x] \tag{3.32}$$

$$< -\frac{q}{n^5}, \tag{3.33}$$

where q is some positive constant.

Next, using (3.18) and (3.19), we have from (3.23) and (3.20) that if $2 \leq u \leq j \leq n^z$, then

$$G_{n,n-u} = -4\sin^2 \tfrac{1}{2}\Theta_{n+1}\cos(n-u+r+1)\Theta_{n+1} \cdot$$

$$\left[\frac{\sin^2 \frac{1}{2}\Theta_n \cos(n-u+r+1)\Theta_n}{\sin^2 \frac{1}{2}\Theta_{n+1}\cos(n-u+r+2)\Theta_{n+1}} - \frac{\sin(h-r)\Theta_n}{\sin(h-r)\Theta_{n+1}}\right]. \tag{3.34}$$

Observe that the quantity inside the square brackets of (3.34) is greater than

$$\frac{\sin^2 \frac{1}{2}\Theta_n}{\sin^2 \frac{1}{2}\Theta_{n+1}} - \frac{\sin(h-r)\Theta_n}{\sin(h-r)\Theta_{n+1}} \sim \frac{\Theta_n}{\Theta_{n+1}}\left[\frac{\Theta_n}{\Theta_{n+1}} - 1\right] > 0, \tag{3.35}$$

Using (3.20) and (3.35), we have from (3.34) that

$$G_{n,n-u} < 0 \qquad 2 \leq u \leq j \leq n^z. \tag{3.36}$$

This together with (3.33), imply that

$$G_{n,n-u} < 0 \qquad 1 \leq u \leq j \leq n^z. \tag{3.37}$$

Hence, it follows from (3.27) and (3.37) that the terms of the sum (3.25) alternate in sign for sufficiently large n uniformly in $2 \leq u \leq j \leq n^z$. Finally we will show that when $n^{\frac{1}{2}z} \leq j \leq n^z$, then

$$\lim_{n\to\infty} |F_{n,n-j}| = \infty. \tag{3.38}$$

Using (3.18) - (3.20), and noting that

$$\sin\Theta \sim \Theta - \frac{\Theta^3}{3!} + (\Theta^5);\ (\Theta \to 0), \tag{3.39}$$

we have from (3.23) that

$$G_{n,n-1} \sim -\frac{\pi^3(1+h-r)(1+2h-2r)}{24n^4}, \tag{3.40}$$

and from (3.34) that

$$G_{n,n-u} \sim -\frac{\pi^3(u+h-r-1)}{4n^4}, \tag{3.41}$$

for sufficiently large n uniformly in $2 \leq u \leq j \leq n^z$. Using (3.27), (3.40) and (3.41), we have from (3.25) that

$$F_{n,n-j} \sim -\frac{\pi^2}{2(h-r)n^3}\Big[(-1)^{j-1}\frac{[1+h-r][1+2h-2r]}{6}w^{j-1} + \sum_{u=2}^{j}(-1)^{j-u}(u+h-r-1)w^{j-u}\Big] \tag{3.42}$$

$$= -\frac{\pi^2}{2(h-r)n^3}[A + B], \text{ say.} \tag{3.43}$$

Observe that for any x; $(x \neq -1)$, we have

$$B = (j+h-r-1)k(x) - xk'(x), \text{ (with x instead of w)}, \tag{3.44}$$

where

$$k(x) = 1 - x + x^2 - x^3 + \ldots + (-1)^{j-2}x^{j-2}. \tag{3.45}$$

Observe that $w + 1 = (h-r)^{-1}$. It follows from (3.44) and (3.45) (with w = x) that

$$B = -(-h-r)(-1)^{j-1}w^{j-1} + jh - jr. \tag{3.46}$$

Using this, it follows from (3.43) that for all sufficiently large n and $2 \leq j \leq n^z$,

$$F_{n,n-j} \sim -\frac{\pi^2}{2(h-r)n^3}\Big[(-1)^{j-1}w^{j-1}\frac{[1-h+r][1-2h+2r]}{6} + j(h-r)\Big]. \tag{3.47}$$

Hence, when $h-r \in (0, 1/2)$; that is when $w > 1$, it follows from (3.47) that when $n^{1/2\,z} \leq j \leq n^z$,

$$\lim_{n\to\infty} |F_{n,n-j}| = \infty,$$

and (3.38) is satisfied. This completes the proof of the first part. For the second part, we use (3.17) to obtain $\bar{t}_n$ in terms of t_n. The result is

$$\bar{t}_{n+1} = \sum_{k=0}^{n} H_{n,k}t_k, \tag{3.48}$$

where

$$H_{n,k} = \sum_{v=k}^{n} C_{n+1,v+1}D_{v,k} \qquad (0 \leq k \leq n), \tag{3.49}$$

and where $C_{n,k}$ and $D_{n,k}$ are given in the proof of the first part. Using the identity

$$\frac{1}{C_{n,n}}[C_{n,n}C_{n+1,v+1} - C_{n+1,n+1}C_{n,v}] = -\frac{C_{n+1,n+1}}{C_{n,n}}G_{n,v}, \tag{3.50}$$

where $G_{n,v}$ is given by (3.23), we have from (3.24) and (3.49) that

$$H_{n,n-j} \sim -\frac{C_{n+1,n+1}}{C_{n,n}} \sum_{u=1}^{j} G_{n,n-u} D_{n-u,n-j}. \qquad (3.51)$$

Observe that since $D_{n-u,n-j} \sim D_{n-u+1,n-j+1}$, we have from (3.25) and (3.51) that

$$H_{n,n-j} \sim \frac{C_{n+1,n+1}}{C_{n,n}} F_{n,n-j}.$$

Using (3.18) and (3.47), it follows from (3.52) that when $0 < h-r < 1/2$, $H_{n,n-j}$ is not bounded for all sufficiently large n and and $n^{1/2 z} \leq j \leq n^z$. This implies that the transformation given by (3.48) is not regular; that is when $h-r \in (0, 1/2)$ $(R_{h,r}) \notin T_1$. This completes the proof of Theorem 3.

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