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SOME SERIES WHOSE COEFFICIENTS INVOLVE THE VALUES $\zeta(n)$ FOR n ODD

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ABSTRACT. By using two basic formulas for the digamma function, we derive a variety of series that involve as coefficients the values (2n + 1), n = 1,2,..., of the Riemann-zeta function. A number of these have a combinatorial flavor which we also express in a trignometric form for special choices of the underlying variable. We briefly touch upon their use in the representation of solutions of the wave equation.

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1. INTRODUCTION.

For the Riemann-zeta function (z), it is well known that $\zeta(2n) = \frac{1}{2}\left|B_{2n}\right| (2\pi)^{2n}/(2n)!$ in which B_{2n} denotes a Bernoulli number [1]. However, there are no known analogous closed formulas for the numbers $\zeta(2n + 1)$, n = 1,2,... In this brief paper, we call upon two basic formulas for the digamma function $\psi(z)$ to derive series of polynomials and constants that involve these numbers. An example of such a series is the following:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4^n} \zeta(4n-1) = \frac{1}{4}\,. \tag{1.1}$$

Sums of this type provide insights about these numbers and the relationships among them. Aside from number theoretic aspects, series that involve evaluations of various zeta functions play a role in the foundations of combinatorics [2]. The formulas we derive involve polynomials that permit connections with solutions of the wave equation for certain types of singular data. While such representations have little practical value, they illustrate how an intrinsically arithmetic function assumes a meaningful role in a physical problem.

In section 2, we make use of the formulas

$$\text{(a)} \quad \psi(1 + z) = -\gamma + \sum_{n=2}^{\infty} (-1)^n \zeta(n) z^{n-1}, \; |z| < 1 \qquad (1.2)$$

$$\text{(b)} \quad \psi(z + 1) - \psi(z) = 1/z$$

(See [3]) to develop a pair of infinite series identities, in a lens shaped convergence region, that involve the values $\zeta(n)$ as well as certain polynomials. The one of these of interest to us involves only the values $\zeta(2n + 1)$. By using differentiation and integration properties of the polynomials entering these, we derive a number of related series that have a combinatoric flavor. When the basic complex variable has the form $z = \frac{1}{2} + iy$, one can express these series in terms of trignometric functions which are more convenient for obtaining certain evaluations (such as (1.1)). We do this in section 3 and then note the connections with wave solutions in section 4.

While the results obtained appear to be novel, the mathematical tools used involve little beyond elementary complex variables. Further relationships of the type constructed can easily be developed. We leave it to the reader to develop corresponding results for series involving the values $\zeta(2n)$ by using the formula (2.4b) obtained in the following section.

2. BASIC SERIES IDENTITIES.

We first define polynomial sets $\{f_n(z)\}$ and $\{g_n(z)\}$ by means of the relations

$$f_n(z) = z^{2n} - (1-z)^{2n}$$
$$g_n(z) = z^{2n+1} + (1-z)^{2n+1} \qquad (2.1)$$

for $n \geq 0$ and take $f_n(z) = g_n(z) = 0$ for $n < 0$.

Using (1.2a) to express $\psi(1 + (1-z))$ and $\psi(1-z)$ in powers of $(1-z)$ and $(-z)$ respectively, we have, by (1.2b),

$$\psi(2-z) - \psi(1-z) = \psi(1 + (1-z)) - \psi(1 + (-z))$$
$$= \sum_{n=0}^{\infty} g_n(z)\, \zeta(2n + 2) + \sum_{n=1}^{\infty} f_n(z)\, \zeta(2n + 1) = 1/(1-z) \qquad (2.2)$$

for $z \in R$ where $R = \{(|z| < 1) \cap (|1-z| < 1)\}$. Similarly, (1.2) also gives

$$\psi(1+z) - \psi(z) = \psi(1+z) - \psi(1 + (z-1))$$
$$= \sum_{n=0}^{\infty} g_n(z)\, \zeta(2n + 2) - \sum_{n=1}^{\infty} f_n(z)\, \zeta(2n + 1) = 1/z \qquad (2.3)$$

for $z \in R$. Upon adding (subtracting) the last two members of (2.3) to (from) the last two members of (2.2), we obtain

$$\text{(a)} \quad \sum_{n=1}^{\infty} f_n(z)\ \zeta(2n+1) = \frac{1}{2}\left(\frac{1}{1-z} - \frac{1}{z}\right)$$
$$\text{(b)} \quad \sum_{n=0}^{\infty} g_n(z)\ \zeta(2n+2) = \frac{1}{2}\left(\frac{1}{1-z} + \frac{1}{z}\right) \qquad (2.4)$$

for $z \in R$. These serve as the basic starting series. The subsequent discussion makes use of only the first of these.

From (2.1), it readily follows that

$$\text{(a)} \quad D_z^{2p} f_n(z) = [(2n)! \,/\, (2n-2p)!\ f_{n-p}(z)$$
$$\text{(b)} \quad D_z^{2p+1} f_n(z) = [(2n)! \,/\, (2n-2p-1)!]\ g_{n-p-1}(z) \qquad (2.5)$$

Using these, it follows that if we differentiate (2.4a) $2p$ and $2p+1$ times with respect to z, we get, for $z \in R$,

$$\text{(a)} \quad \sum_{n=1}^{\infty} \binom{2n+2p}{2p} f_n(z)\ \zeta(2n+2p+1) = \frac{1}{2}\left[(1-z)^{-(2p+1)} - z^{-(2p+1)}\right]$$
$$\text{(b)} \quad \sum_{n=1}^{\infty} \binom{2n+2p+2}{2p+1} g_n(z)\ \zeta(2n+2p+3) = \frac{1}{2}\left[(1-z)^{-(2p+2)} + z^{-(2p+2)}\right] \qquad (2.6)$$

Similarly, if we multiply (2.4a) by $z^p(1-z)^q$, p,q positive integers, and integrate with respect to z from 0 to 1, we get using the definition of the beta function,

$$\sum_{n=1}^{\infty} \{B(2n+p+1,\ q+1) - B(p+1,\ 2n+q+1)\}\ \zeta(2n+1) = \frac{1}{2}\left[B(p+1,\ q) - B(p,\ q+1)\right] \qquad (2.7)$$

One can obtain formulas analogous to (2.7) by using (2.6).

For $z \in R$, the formulas (2.6) are convenient for obtaining a number of specific series. For example, if we take $z = 1/2$ in (2.6b), we get

$$\sum_{n=0}^{\infty} \frac{1}{2^{2n}} \binom{2n+2p+2}{2p+1} \zeta(2n+2p+3) = 2^{2p+2} \qquad (2.8)$$

Similarly, if we select $q = p + 1$ in (2.7) and simplify the corresponding beta functions, we have for $p \geq 1$,

$$\sum_{n=1}^{\infty} \frac{n \quad \zeta(2n+1)}{(2n+p+1)(2n+p+2)\ldots(2n+2p+2)} = \frac{(p-1)!}{4\,(2p+1)!} \qquad (2.9)$$

3. TRIGNOMETRIC FORMS.

Suppose $z = 1/2 + iy$ with $|y| < \sqrt{3/2}$. Then $z = \rho e^{i\theta}$ with $\rho^2 = \frac{1}{4} + y^2$, $\theta = \tan^{-1}(2y)$, and $(1-z) = \bar{z} = \rho e^{-i\theta}$. Note that $|\theta| < \pi/3$. Then we have

$$\begin{aligned} &(a) \quad f_n(z) = 2\,i\,\rho^{2n} \sin(2n\theta) \\ &(b) \quad g_n(z) = 2\rho^{2n+1} \cos(2n+1)\theta \\ &(c) \quad (1-z)^{-k} \pm z^{-k} = \rho^{-k}[e^{ik\theta} \pm e^{-ik\theta}] \end{aligned} \qquad (3.1)$$

Using these in (2.6), we deduce that

$$\begin{aligned} &(a) \quad \sum_{n=1}^{\infty} \binom{2n+2p}{2p} \rho^{2n} \sin(2n\theta)\,\zeta(2n+2p+1) \\ &\qquad = \frac{1}{2}\rho^{-(2p+1)} \sin(2p+1)\theta \\ &(b) \quad \sum_{n=0}^{\infty} \binom{2n+2p+2}{2p+1} \rho^{2n+1} \cos((2n+1)\theta)\,\zeta(2n+2p+3) \\ &\qquad = \frac{1}{2}\rho^{-(2p+2)} \cos(2p+2)\theta \end{aligned} \qquad (3.2)$$

If we select $\theta = \pi/4$, then $\rho = 1/\sqrt{2}$. Then from (3.2a), we get

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^{2n}} \binom{4n+2p-2}{2p} \zeta(4n+2p-1) = 2^{p-2}\sqrt{2}\,\{\sin(2p+1)\pi/4\}. \qquad (3.3)$$

For $p = 0$, this reduces to (1.1). Similarly, this choice for θ in (3.2b) with $p = 0$ gives

$$\zeta(3) + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2n-1}} [4n\,\zeta(4n+1) + (2n+1)\,\zeta(4n+3)] = \frac{1}{\sqrt{2}} \qquad (3.4)$$

Other choices for θ (such as $\pi/6$) and p in the formulas (3.2) will lead to additional identities.

4. CONNECTIONS WITH WAVE FUNCTIONS.

The wave polynomials $w_{1,n}(x,t)$ and $w_{2,n}(x,t)$, $n = 0,1,2,\ldots$, are solutions of the equation

$$\frac{\partial^2 w(x,t)}{\partial t^2} = \frac{\partial^2 w(x,t)}{\partial x^2} \qquad (4.1)$$

that correspond, respectively, to the initial conditions $w_{1,n}(x,0) = x^n$, $\frac{\partial}{\partial t} w_{1,n}(x,0) = 0$ and $w_{2,n}(x,0) = 0$, $\frac{\partial}{\partial t} w_{2,n}(x,0) = x^n$ ([4], [5]). They are given explicitly by

$$\begin{aligned} &\text{(a)} \quad w_{1,n}(x,t) = \frac{1}{2}[(x+t)^n + (x-t)^n] \\ &\text{(b)} \quad w_{2,n}(x,t) = \frac{1}{2(n+1)}[(x+t)^{n+1} - (x-t)^{n+1}] \,. \end{aligned} \tag{4.2}$$

Suppose we select $0 < x < 1$ and t real so that $|x \pm t| < 1$ and $|1-(x \pm t)| < 1$. If we evaluate (2.4a) first at $z = x + t$ and then at $x - t$ and add, it follows from the definition of $f_n(z)$ and (4.2a) that

$$\sum_{n=1}^{\infty} [w_{1,2n}(x,t) - w_{1,2n}(1-x,t)]\,\zeta(2n+1) = \frac{1}{2}\left[\frac{1-x}{(1-x)^2-t^2} - \frac{x}{x^2-t^2}\right] \tag{4.3}$$

The series in the left member of this converges in the interior S^o of the square having vertices at (0,0), (1,0), $(1/2, \sqrt{2}/2)$, and $(1/2, -\sqrt{2}/2)$. Note that the right member of (4.3) reduces to $\frac{1}{2}(\frac{1}{1-x} - 1/x)$ at $t = 0$ which has singularities at $x = 0$ and $x = 1$. The characteristics of the equation (4.1) through the points (0,0) and (1,0) determine this same region S^o. The formula (2.6a) can be used to construct other such series involving the $\zeta(2n + 1)$. Finally, one can use (2.4b) to construct examples of series of the wave polynomials $w_{2,n}(x,t)$ that involve the values $\zeta(2n)$ as coefficients.

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