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GROWTH OF ENTIRE FUNCTIONS WITH SOME UNIVALENT GELFOND-LEONTEV DERIVATIVES

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1. INTRODUCTION. Let $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$ be analytic in $|z| < R$. For a non-decreasing sequence of positive numbers $\{d_n\}_{n=1}^{\infty}$, the Gelfond-Leontev (G-L) derivative of f is defined as [1]

$$Df(z) = \Sigma_{n=1}^{\infty} d_n a_n z^{n-1} . \tag{1.1}$$

The kth iterate $D^k f$, k=1,2,..., of D is given by

$$D^k f(z) = \Sigma_{n=k}^{\infty} d_n \cdots d_{n-k+1} a_n z^{n-k} \tag{1.2}$$

$$= \Sigma_{n=k}^{\infty} \frac{e_{n-k}}{e_n} a_n z^{n-k}$$

where, $e_o=1$ and $e_n=(d_1 d_2 \cdots d_n)^{-1}$, n=1,2,... . If $d_n \equiv n$, Df is the ordinary derivative of f; whereas, if $d_n \equiv 1$, D is the shift operator L which transforms

$f(z) = \Sigma_{n=0}^{\infty} a_n z^n$ into $Lf(z) = \Sigma_{n=1}^{\infty} a_n z^{n-1}$.

Let,

$$\psi(z) = \Sigma_{n=0}^{\infty} e_n z^n \tag{1.3}$$

and have radius of convergence R_o. From the monotonicity of $\{d_n\}_{n=1}^{\infty}$, we have

$$R_o = \lim_{n \to \infty} d_n = \sup_{n \geq 1} \{d_n\}.$$

Clearly, $\psi(0) = 1$ and $D\psi(z) = \psi(z)$. Thus, $\psi(z)$ bears the same relationship to the operator D that the function exp(z) bears to the ordinary differentiation.

For an entire function f, Nachbin used the function $\psi(z)$ as a comparison function for measuring the growth of maximum modulus of f on $|z| = r$. Thus, the growth parameter ψ-type of f is defined as the infimum of the positive numbers τ such that, for sufficiently large r,

$$|f(z)| \leq M\psi(\tau r) \tag{1.4}$$

where, $\psi(z)$ is entire and M is a positive constant. We denote ψ-type of f as $\tau_\psi(f)$. It is known [2,p.6] that

$$\tau_\psi(f) = \lim_{n\to\infty} \sup \left|\frac{a_n}{e_n}\right|^{1/n} . \tag{1.5}$$

For $d_n \equiv n$, the ψ-type of an entire function f reduces to its classical exponential type and the formula (1.5) gives its well known coefficient characterisation [3, p. 11].

The comparison function $\psi(z)$ can also be used to define a measure of growth analogous to classical order [3, p.8] of an entire function. Thus, for an entire function f, let the ψ-order $\rho_\psi(f)$ of f be defined as the infimum of positive numbers ρ such that, for sufficiently large r,

$$|f(z)| \leq K\psi(r^\rho) \tag{1.6}$$

where $\psi(z)$ is entire and K is a positive constant.

Shah and Trimble [4,5] showed that if f is entire then, the assumption that the classical derivatives $f^{(n_p)}$ are univalent in $\Delta = \{z: |z| < 1\}$ for a suitable increasing sequence $\{n_p\}_{p=1}^{\infty}$ of positive integers affects the growth of the maximum modulus of f. If instead, we assume that the G-L derivatives $D^{n_p}f$ of an entire function f are univalent in Δ, then it is natural to enquire in what way the ψ-type and ψ-order of f are influenced. The present paper is an attempt in this direction. In Theorem 1, we find that if f is entire, $D^{n_p}f$ are univalent in Δ and $\lim_{p\to\infty} \sup (n_p - n_{p-1}) = \mu$, $1 \leq \mu < \infty$, then the ψ-type $\tau_\psi(f)$ of f must satisfy

$$\tau_\psi(f) \leq 2(d(\mu+1)\ldots d(2))^{1/\mu}.$$

Further, if $\mu = \infty$, then f need not be of finite ψ-type. Our Theorem 2 shows that if f is entire, $D^{n_p}f$ are univalent in Δ and $n_p \sim n_{p+1}$ as $p \to \infty$, then

$$\rho_\psi(f) \leq \frac{1}{1 - \lim_{p\to\infty} \sup \dfrac{\log d(n_p - n_{p-1})}{\log d(n_p)}} .$$

It is clear that if $0 \leq \rho_\psi(f) \leq 1$, then the above inequality gives no relationship between $D^{n_p}f$ and the ψ-order of an entire function f. In fact, no such relation of this nature exists. This is illustrated in Theorem 3, wherein for any given

ρ, $0 \le \rho \le 1$, and any given increasing sequence $\{n_p\}_{p=1}^{\infty}$ of positive integers, we construct an entire function h, of ψ-order ρ, such that $D^{n_p}h$ is univalent in Δ if and only if $n=n_p$.

In the sequel, we shall assume throughout that $d_n \to \infty$ as $n \to \infty$.

2. ψ-TYPE AND EXPONENTS OF UNIVALENT G-L DERIVATIVES.

THEOREM 1. *Let* $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$ *be an entire function and* $\{n_p\}_{p=1}^{\infty}$ *be an increasing sequence of positive integers. Let* $D^{n_p}f$ *be analytic and univalent in* Δ. *Suppose* $\limsup_{p \to \infty} (n_p - n_{p-1}) = \mu$, $1 \le \mu < \infty$. *Then, the* ψ*-type* $\tau_\psi(f)$ *of* f *satisfies*

$$\tau_\psi(f) \le 2(d(\mu+1)\ldots d(2))^{1/\mu}. \tag{2.1}$$

PROOF. By the hypothesis,

$$D^{n_p}f(z) = \Sigma_{k=0}^{\infty} d(n_p+k)\ldots d(k+1)a(n_p+k)z^k$$

are univalent in Δ. Since, for any function $G(z) = b_0+b_1z+b_2z+\ldots$, univalent in Δ, it is known [6] that $|b_n| \le n|b_1|$ for $n=2,3,\ldots$, we get

$$|a(n_p+k)| \le k\,\frac{d_k\ldots d_1}{d_{k+n_p}\ldots d_1}\, d(n_p+1)\ldots d(2)\,|a(n_p+1)| \tag{2.2}$$

for $k=1,2,\ldots$ and $p=2,3,\ldots$. In particular, putting $k=n_{p+1}-n_p+1$ and inducting upon p, we get, for $p \ge 2$ and $2 \le k \le n_{p+1}-n_p+1$,

$$|a(n_p+k)| \le Ak\,\frac{d_k\ldots d_1}{d_{k+n_p}\ldots d_1} \prod_{i=2}^{p} (n_i-n_{i-1}+1)d(n_i-n_{i-1}+1)\ldots d(2) \tag{2.3}$$

where $A=d(n_1+1)\ldots d(2)|a(n_1+1)|$. Hence, for sufficiently large p,

$$\left|\frac{a(n_p+k)}{e(n_p+k)}\right|^{1/(n_p+k)}$$

$$\le (1+o(1))(d_k\ldots d_1)^{1/(n_p+k)} \prod_{i=2}^{p} \{(n_i-n_{i-1}+1)\, d(n_i-n_{i-1}+1)\ldots\ldots d(2)\}^{1/(n_p+k)} \tag{2.4}$$

Since, $(d_k\ldots d_1)^{1/(n_p+k)}$ is an increasing function of k, and $(n_{p+1}-n_p) \le \mu'$, $\mu' > \mu$, for sufficiently large p,

$$(d_k\ldots d_1)^{1/(n_p+k)} \le (d(n_{p+1}-n_p+1)\ldots d(1))^{1/n_{p+1}} = (1+o(1))$$

Further [7], for $p > 2$

$$\prod_{i=2}^{p} (n_i-n_{i-1}+1)^{1/(n_p+2)} < (1+\frac{n_p}{p})^{p/n_p} < 2 \ . \tag{2.5}$$

Using (2.5) and the preceding inequality in (2.4), we get for sufficiently large p,

$$\left|\frac{a(n_p+k)}{e(n_p+k)}\right|^{1/(n_p+k)} < 2(1+o(1)) \prod_{i=2}^{p} (d(n_i-n_{i-1}+1)\ldots d(2))^{1/(n_p+k)} . \tag{2.6}$$

Now, if $a_j > 0$, $t_j > 0$, $\Sigma\, t_j > 0$ and $\max_{1<j<N-1} (\frac{a_j}{j}) < \frac{a_N}{N}$ then clearly,

$$\frac{\sum_{j=1}^{N} a_j t_j}{\sum_{j=1}^{N} j t_j} < \frac{a_N}{N} \ . \tag{2.7}$$

Further, $\log(d(j+1)\ldots d(2))/j$ is an increasing function of j for $1 < j < \mu$, $\mu = 1,2,\ldots$. Thus, if $1 < j < \mu$,

$$\frac{\log(d(j+1)\ldots d(2))}{j} < \frac{\log(d(\mu+1)\ldots d(2))}{\mu} \tag{2.8}$$

Let $p > p_o$, $1 < \gamma < \mu$. Suppose t_γ is the number of j_i's in $[p_o,p]$ such that

$n_{j+1}-n_j = \gamma$ for $j = j_i$. Then, by (2.7) and (2.8),

$$\frac{\sum_{p_o+1}^{p} \log(d(n_j-n_{j-1}+1)\ldots d(2))}{\sum_{p_o+1}^{p} (n_j-n_{j-1})} = \frac{\sum_{\gamma=1}^{\mu} t_\gamma\,(\log(d(\gamma+1)\ldots d(2))}{\sum_{\gamma=1}^{\mu} \gamma\, t_\gamma} < \frac{\log(d(\mu+1)\ldots d(2))}{\mu} \ .$$

The above inequality implies that

$$\prod_{i=2}^{p} (d(n_i-n_{i-1}+1)\ldots d(2))^{1/(n_p+k)} < \exp\left\{\frac{\sum_{i=2}^{p} \log(d(n_i-n_{i-1}+1)\ldots d(2))}{n_p}\right\} \tag{2.9}$$

$$< \exp\left\{o(1)+\frac{\sum_{p_o+1}^{p} \log(d(n_i-n_{i-1}+1)\ldots d(2))}{\sum_{p_o+1}^{p} (n_i-n_{i-1})}\right\}$$

$$< \exp\{o(1)+\frac{\log(d(\mu+1)\ldots d(2))}{\mu}\} .$$

Using the estimate (2.9) in (2.6) and proceeding to limits

$$\limsup_{k\to\infty} \left|\frac{a_k}{e_k}\right|^{1/k} = \lim\sup \left\{\left|\frac{a(n_p+k)}{e(n_p+k)}\right|^{1/(n_p+k)} : 2 < k < n_{p+1}-n_p+1,\ p > 2\right\}$$

$$< 2(d(\mu+1)\ldots d(2))^{1/\mu}.$$

This completes the proof of the theorem.

REMARK 1. In Theorem 1, it is sufficient to take the function f to be analytic in $|z| < R$, for some R, $0 < R < \infty$, if the sequence $\{d_n\}_{n=1}^{\infty}$ in the definition of G-L derivative of f satisfies the condition $\lim_{m \to \infty} ((\Sigma_{i=2}^{m} \log d(i))/m)=\infty$. In fact, for an analytic function f in $|z| < R$, if $D^{n_p} f$ are univalent in Δ,

$$\lim_{p \to \infty} \sup (n_p - n_{p-1}) = \mu,\ 1 \leq \mu < \infty, \text{ and}$$

$$\lim_{m \to \infty} \frac{\Sigma_{i=2}^{m} \log d(i)}{m} = \infty$$

holds, then f is necessarily entire. To see this, we use (2.5) and

$$(d_k \ldots d_1)^{1/(n_p+k)} \leq 1+o(1)$$

for sufficiently large p in (2.3) to get

$$|a(n_p+k)|^{1/(n_p+k)} \tag{2.10}$$

$$\leq 2(1+o(1))\exp[\frac{1}{n_p} \sum_{i=2}^{n_p} \log(d(n_i - n_{i-1}+1)\ldots d(2))$$

$$- \frac{1}{n_p+k} \sum_{i=2}^{n_p+k} \log d(i)]$$

for sufficiently large p. But since, for sufficiently large $p, (n_p - n_{p-1}) \leq \mu', \mu' > \mu$,

$$\frac{1}{n_p} \sum_{i=2}^{n_p} \log(d(n_i - n_{i-1}+1)\ldots d(2)) \to 0 \text{ as } p \to \infty .$$

Thus, by (2.10) and the condition $\lim_{m \to \infty} ((\Sigma_{i=2}^{m} \log d(i))/m) = \infty$

$$\lim_{k \to \infty} \sup |a_k|^{1/k} = \lim \sup \{|a(n_p+k)|^{1/(n_p+k)} : 2 \leq k \leq n_{p+1} - n_p + 1,\ p \geq 2\}$$

$$= 0.$$

REMARK 2. The inequality (2.1) can be improved by imposing suitable additional restrictions on the sequence $\{d_n\}_{n=1}^{\infty}$. For example, let the sequence $\{d_n\}_{n=1}^{\infty}$ be such that

$$\frac{\{d(n+2)\}^n}{d(n+1)\ldots d(2)} \geq \frac{2}{3}(n+1),\ n=1,2,3,\ldots . \tag{2.11}$$

Note that (2.11) is satisfied for $d_n = n^{\alpha}, \alpha \geq 1$.

Because of (2.11), the function s(j) defined by

$$s(j) = \frac{\log(d(j+1)\ldots d(2)) + \log(j+1)}{j}$$

is an increasing function of j and so for $j=1,2,\ldots\mu$; $\mu=1,2\ldots$

$$\frac{\log(d(j+1)\dots d(2))+\log(j+1)}{j} \leq \frac{\log(d(\mu+1)\dots d(2))+\log(\mu+1)}{\mu}. \tag{2.12}$$

Let t_γ be the same as in the proof of Theorem 1. Using (2.7) and (2.12), we get

$$\frac{\sum_{p_o+1}^{p} \{\log(d(j+1)\dots d(2))+\log(j+1)\}}{\sum_{p_o+1}^{p} (n_j-n_{j-1})} = \frac{\sum_{\gamma=1}^{\mu} t_\gamma \{\log(d(\gamma+1)\dots d(2))+\log(\gamma+1)\}}{\sum_{\gamma=1}^{\mu} \gamma\, t_\gamma}$$

$$\leq \frac{\log(d(\mu+1)\dots d(2))+\log(\mu+1)}{\mu}.$$

Again, we have

$$\prod_{i=2}^{p} \{(n_i-n_{i-1}+1)d(n_i-n_{i-1}+1)\dots d(2)\}^{1/(n_p+k)}$$

$$\leq \exp\left\{o(1) + \frac{\sum_{p_o+1}^{p} \{\log(d(n_i-n_{i-1}+1)\dots d(2)) + \log(n_i-n_{i-1}+1)\}}{\sum_{p_o+1}^{p} (n_i-n_{i-1})}\right\}.$$

The above inequality, when employed in (2.4), gives

$$\left|\frac{a(n_p+k)}{e(n_p+k)}\right|^{1/(n_p+k)} \leq (1+o(1)) \prod_{i=2}^{p} \{(n_i-n_{i-1}+1)d(n_i-n_{i-1}+1)\dots d(2)\}^{1/(n_p+k)}$$

$$\leq (\mu+1)^{1/\mu}(d(\mu+1)\dots d(2))^{1/\mu}.$$

Now, on proceeding to limits, we get

$$\tau_\psi(f) \leq (\mu+1)^{1/\mu}(d(\mu+1)\dots d(2))^{1/\mu}. \tag{2.13}$$

It is clear that the bound on $\tau_\psi(f)$ in (2.13) is better than that in (2.1).

REMARK 3. By taking $\mu=1$, Theorem 1 gives $\tau_\psi(f) \leq 2d(2)$, a result recently proved in [8].

Theorem 1 shows that if $(n_p-n_{p-1}) = O(1)$, then f is of finite ψ-type. We now give an example to show that if $\lim_{p\to\infty} \sup(n_p-n_{p-1}) = \infty$, then f need not be of finite ψ-type.

EXAMPLE. Let $\{n_p\}_{p=1}^{\infty}$ be an increasing sequence of positive integers such that $(n_{p+1}-n_p) > 2$ for all p. Further, assume that the sequence $\{d_n\}_{n=1}^{\infty}$ is such that

(i) $d_1 \equiv 1$ and $\log d(n) \sim \log n$ as $n \to \infty$

(ii) $n_p = o(\eta_p)$

(iii) $\eta_p = o(n_p \log d(n_p))$

where, $\eta_p = \Sigma_{i=2}^{p} \log(d(n_i - n_{i-1}+1)\dots d(2))$,

Let ψ be a non-decreasing step function such that $\psi(n_1)=\psi(n_2)$,

$$\psi(n_p) = \frac{\exp(\eta_p)}{2^{p-1}}, \quad p > 2$$

and

$$\psi(x) = \psi(n_p) \qquad n_p < x \leq n_{p+1}.$$

Let

$$g_{j+1} = \begin{cases} \dfrac{\psi(j)}{d(j+1)\dots d(2)\ (j-n_p+1)} & \text{if } j=n_p \text{ for some } p \\ 0 & \text{otherwise.} \end{cases}$$

Define

$$g(z) = \Sigma_{j=0}^{\infty} g_j z^j$$

We first show that g is an entire function. We have

$$\limsup_{k \to \infty} |g_k|^{1/k} = \limsup_{p \to \infty} \left[\frac{\psi(n_p)}{d(n_p+1)\dots d(2)}\right]^{1/n_p+1}$$

$$\leq \limsup_{p \to \infty} \left[\frac{\exp(\eta_p/n_p)}{(d(n_p+1)\dots d(2))^{1/n_p+1}}\right]$$

$$= \limsup_{p \to \infty} \left[\exp\left(\frac{\eta_p}{n_p} - \frac{1}{n_p+1} \sum_{i=2}^{n_p+1} \log d(i)\right)\right].$$

Since $\log d(n) \sim \log n$ as $n \to \infty$, using the condition (iii), we get from the above inequality that

$$\limsup_{k \to \infty} |g_k|^{1/k} = 0.$$

Hence g is entire. It is easily seen that g is of order 1. But, by the condition (ii),

$$\limsup_{k \to \infty} \left|\frac{g_k}{e_k}\right|^{1/k} = \limsup_{p \to \infty} \left(\frac{\psi(n_p)}{e(n_p+1)d(n_p+1)\dots d(2)}\right)^{1/n_p+1}$$

$$\geq \limsup_{p \to \infty} \frac{\exp(\eta_p/n_p)}{2} = \infty.$$

Thus, f is not of finite ψ-type. It remains to see that

$$D^{n_p} g(z) = \Sigma_{k=1}^{\infty} d(n_{p+k}+1)\dots d(n_{p+k}-n_p+2)a(n_{p+k}+1)z^{n_{p+k}-n_p+1}$$

are univalent in Δ. To this end, it is enough to prove that

$$\Sigma_{k=1}^{\infty} (n_{p+k}-n_p+1) \frac{d(n_{p+k}+1)\dots d(2)}{d(n_{p+k}-n_p+1)\dots d(2)} |a(n_{p+k}+1)|$$

$$< d(n_p+1)\ldots d(2)\,|a(n_p+1)|\;;$$

or, equivalently to show that

$$\Sigma_{k=1}^{\infty} \frac{\psi(n_{p+k})}{d(n_{p+k}-n_p+1)\ldots d(2)} < \psi(n_p).$$

Using the definition of ψ , the last inequality reads as

$$\Sigma_{k=1}^{\infty} \frac{1}{2^k} \frac{\exp(\eta_{p+k}-\eta_p)}{d(n_{p+k}-n_p+1)\ldots d(2)} < 1 . \tag{2.14}$$

Now, an induction on k, gives, for k=1,2,3,...

$$\exp\,(\eta_{p+k}-\eta_p) = \prod_{p+1}^{p+k} d(n_i-n_{i-1}+1)\ldots d(2) < d(n_{p+k}-n_p+1)\ldots d(2)$$

Hence, (2.14) is clearly satisfied.

3. Ψ-ORDER AND EXPONENTS OF UNIVALENT G-L DERIVATIVES.

A function S(x), continuous on $[1,\infty)$, is said to be Slowly Oscillating (S.O.) if for every positive number $c > 0$,

$$\lim_{x \to \infty} \frac{S(cx)}{S(x)} = 1 .$$

A function H(n) is said to be the restriction of a Slowly Oscillating function S(x) if S(n) = H(n) for every positive integer n. It is known [9] that, as $k \to \infty$

$$\Sigma_{i=1}^{k} H(i) \sim kH(k) . \tag{3.1}$$

THEOREM 2. <u>Let</u> $f(z) = \Sigma_{n=o}^{\infty} a_n z^n$ <u>be an entire function of ψ- order ρ_ψ and $\{n_p\}_{p=1}^{\infty}$ be a strictly increasing sequence of positive integers. Let $D^{n_p}f$ be analytic and univalent in Δ, such that $n_p \sim n_{p+1}$ as $p \to \infty$. If log d(n) is the restriction of a slowly oscillating function on integers, then</u>

$$\rho_\psi(f) < \frac{1}{1-\limsup_{p \to \infty} \left[\frac{\log d(n_p-n_{p-1})}{\log d(n_p)}\right]} . \tag{3.2}$$

We need the following lemmas.

LEMMA 1. <u>Let Ψ be defined by</u> (1.3). <u>Let</u> $\gamma_n = \min_{x>0} \psi(x^a)x^{-n}$, $a > 0$. <u>Then,</u>

$$\gamma_n < e_n d_n^{\,n(1-\frac{1}{a})} \left(\frac{e(n+a)}{a}\right). \tag{3.3}$$

PROOF. Since $\{d_n\}_{n=1}^{\infty}$ is increasing, we note that for any pair of integers k and n, $e_k < e_n d_n^{n-k}$. Thus,

$$\psi(x^a) = \sum_{k=0}^{\infty} e_k x^{ak} \leq e_n d_n^n \sum_{k=0}^{\infty} d_n^{-k} x^{ak}.$$

Let $0 < w < 1$. Setting $x_w = wd_n^{1/a}$, we get

$$\psi(x_w^a)x_w^{-n} \leq e_n d_n^{n(1-\frac{1}{a})} \frac{w^{-n}}{(1-w^a)}.$$

Choosing $w = (n/n+a)^{1/a}$ to minimize the right-hand side of the above inequality, we have

$$\gamma_n \leq \min_{0<w<1} \psi(x_w^a)x_w^{-n} \leq e_n d_n^{n(1-\frac{1}{a})} \left(\frac{e(n+a)}{a}\right).$$

LEMMA 2. Let $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$ be an entire function of ψ-order ρ_ψ, where the sequnce $\{d(n)\}$ in Df is such that $\log d(n)$ is the restriction of a slowly oscillating function on positive integers.
Then,

$$\rho_\psi(f) = \limsup_{n \to \infty} \frac{n \log d(n)}{-\log |a_n|}. \tag{3.4}$$

PROOF. By Cauchy's inequality, we get

$$|a_n| \leq M(r)r^{-n}, \ M(r) = \max_{|z|\leq r} |f(z)|.$$

Since f is of ψ-order $\rho_\psi(f) \equiv \rho$, for any $\varepsilon > 0$, $|f(z)| \leq M\psi(r^{\rho+\varepsilon})$.
So that

$$|a_n| \leq M\psi(r^{\rho+\varepsilon})r^{-n}.$$

Using Lemma 1, we have

$$|a_n| \leq M e_n d_n^{n(1-\frac{1}{\rho+\varepsilon})} \left(\frac{e(n+\rho+\varepsilon)}{(\rho+\varepsilon)}\right). \tag{3.5}$$

But, since $\log d(n)$ is the restriction of a S.O. function, by (3.1),

$\Sigma_{i=2}^{n} \log d(i) \sim n \log d(n)$ as $n \to \infty$. Thus, it follows from (3.5)

$$\limsup_{n \to \infty} \frac{n \log d(n)}{-\log |a_n|} \leq \rho.$$

To prove that equality holds in (3.4), suppose that

$$\limsup_{n \to \infty} \frac{n \log d(n)}{-\log |a_n|} < \rho.$$

Then, there exist $\rho_1 < p$ such that $|a_n| < e_n^{1/\rho_1}$ for $n > n_o$. It now follows that, for $|z| = r$,

$$|f(z)| \leq \Sigma_{n=0}^{n_o} |a_n| r^n + \Sigma_{n_o+1}^{\infty} |a_n| r^n \tag{3.6}$$

$$\leq O(1) + \Sigma_{n_o+1}^{\infty} e_n^{1/\rho_1} r^n.$$

Choose

$$N(r) = \frac{\log \psi(r^{\rho_1})}{\log r}.$$

It is easily seen that $N(r) \to \infty$ as $r \to \infty$. Since for all values of k and n, $e_n < e_k d_k^{k-n}$, we have

$$\Sigma_{n=0}^{\infty} e_n^{1/\rho_1} r^n < \Sigma_{n=0}^{\infty} e_k^{1/\rho_1} d_k^{k-n/\rho_1} r^n$$

$$= d_k^{1/\rho_1} e_k^{1/\rho_1} \Sigma_{n=0}^{\infty} \left(\frac{r}{d_k^{1/\rho_1}}\right)^n.$$

Let k be chosen such that $(r/d_k^{1/\rho_1}) < 1$. Then,

$$\Sigma_{n=0}^{\infty} e_n^{1/\rho_1} r^n < \frac{d_k^{k+1/\rho_1} e_k^{1/\rho_1}}{(d_k^{1/\rho_1} - r)}. \tag{3.7}$$

Since the left hand side of (3.7) is independent of k, letting $k \to \infty$, we get

$\Sigma_{n=0}^{\infty} e_n^{1/\rho_1} r^n < 1.$

Thus

$$\Sigma_{n=N(r)}^{\infty} e_n^{1/\rho_1} r^n = o(1), \text{ as } r \to \infty .$$

Since, $r^{N(r)} = \exp(N(r)\log r) = \psi(r^{\rho_1})$, it now follows from (3.6)

$$|f(z)| \le O(1) + \Sigma_{n_o+1}^{N(r)} e_n^{1/\rho_1} r^n + o(1)$$

$$\le O(1)\, \psi(r^{\rho_1}).$$

Since $\rho_1 < \rho$ and ρ is the ψ-order of f, the above inequality contradicts the definition of ψ-order. Thus, equality must hold in (3.4). This proves the lemma.

PROOF OF THEOREM 2. Since $D^{n_p} f$ are univalent in Δ, from (2.2), we get for sufficiently large p and $2 \le k \le n_{p+1} - n_p + 1$.

$$|a(n_p + k)|^{1/(n_p + k)} \tag{3.8}$$

$$\le (1 + o(1))\left(\frac{d_k \cdots\cdots d_1}{d_{k+n_p} \cdots d_1}\right)^{1/(n_p+k)} \prod_{i=2}^{p} \{(n_i - n_{i-1}+1) d(n_i - n_{i-1}+1) \cdots d(2)\}^{1/(n_p+k)}$$

Further, we have

$$(d_k\cdots d_1)^{1/(n_p+k)} \leq (d(n_{p+1}-n_p+1)\cdots d(1))^{1/n_{p+1}}$$

and

$$(d_{n_p+k}\cdots d_1)^{-1/(n_p+k)} \leq (d(n_p+2)\cdots d(1))^{-1/(n_p+2)}.$$

Using these inequalities, (2.5) and (3.8), it follows that, for sufficiently large p,

$$|a(n_p+k)|^{1/(n_p+k)} \tag{3.9}$$

$$\leq \frac{2(1+o(1))}{(d(n_p)\cdots d(1))^{1/n_p}} \prod_{i=2}^{p+1} (d(n_i-n_{i-1}+1))^{(n_i-n_{i-1})/n_p}$$

Let,

$$M_p = \max\{\log d(n_i-n_{i-1}+1) : 2\leq i\leq p\}.$$

Since $\log d(n)$ is the restriction of a slowly oscillating function on integers, by (3.1)

$$\log \frac{\prod_{i=2}^{p+1} d(n_i-n_{i-1}+1)^{(n_i-n_{i-1})/n_p}}{(d(n_p)\cdots d(1))^{1/n_p}}$$

$$\leq \frac{1}{n_p}\left[\sum_{i=2}^{p+1} (n_i-n_{i-1}) \log d(n_i-n_{i-1}+1) - \sum_{i=1}^{n_p} \log d(i)\right]$$

$$\leq \frac{n_{p+1}}{n_p} M_{p+1} - \log d(n_p).$$

Consequently, for sufficiently large p,

$$\frac{(n_p+k)\log d(n_p+k)}{-\log|a(n_p+k)|} \leq \frac{\log d(n_{p+1}+1)}{\log d(n_p) - \frac{n_{p+1}}{n_p} M_{p+1} - \log 2}$$

Again, from the definition of S.O. function $\log d(n_p) \sim \log d(n_{p+1})$ as $p \to \infty$. Hence,

$$\rho_\psi \leq \frac{1}{1-\limsup_{p\to\infty} \frac{M_p}{\log d(n_p)}}. \tag{3.10}$$

If M_p is bounded, there is nothing to prove. So, let $M_p \to \infty$ as $p \to \infty$. For $p > 2$, let,

$$A_p = \frac{\log d(n_p-n_{p-1}+1)}{\log d(n_p)}$$

and

$$B_p = \frac{M_p}{\log d(n_p)}.$$

But as $M_p = \max\{\log d(n_i-n_{i-1}+1) : 2\leq i\leq p\}$, for each $p > 2$, there is some

q_p, $q_p \leq p$ such that $M_p = \log d(n_{q_p} - n_{q_p - 1} + 1)$. Hence

$B_p \leq A_{q_p}$. Taking $q_p \to \infty$,

$$\limsup_{p \to \infty} B_p \leq \limsup_{p \to \infty} A_p .$$

Now (3.2) follows from (3.10).

COROLLARY. Suppose the conditions of Theorem 2 are satisfied. If as $p \to \infty$,

$$\log d(n_p - n_{p-1}) = o(\log d(n_p))$$

then,

$$\rho_\psi(f) \leq 1 .$$

THEOREM 3. _Let_ $0 \leq \rho \leq 1$. _Let_ $\{n_p\}_{p=1}^{\infty}$ _be a strictly increasing sequence of non-negative integers. Then, there is an entire function_ h _of_ ψ_-order_ ρ _such that_ $D^n h$ _is univalent in_ Δ _if and only if_ $n = n_p$ _for some_ p.

PROOF. Suppose $\rho > 0$ and $\{d_n\}_{n=1}^{\infty}$ is an increasing sequence of positive numbers such that $\log d(n)$ is the restriction of a slowly oscillating function on integers and $d_1 = 1$. Let,

$$h_{j+1} = \begin{cases} \dfrac{1}{2^p d(n_p+1)\ldots d(2)\ (j-n_p+1)} & \text{if } j = n_p \text{ for some } p \\ 0 & \text{otherwise.} \end{cases}$$

Define, $h(z) = \Sigma_{j=0}^{\infty} h_j z^j$. Then, $h(z)$ is an entire function and

$$\rho_\psi(h) = \limsup_{k \to \infty} \frac{k \log d(k)}{-\log |h_k|}$$

$$= \limsup_{p \to \infty} \frac{(n_p+1) \log d(n_p+1)}{p \log 2 + \frac{1}{\rho} \log(d(n_p+1)\ldots d(2))} = \rho ,$$

To show that $D^{n_p} h$ given by

$$D^{n_p} h(z) = \Sigma_{k=0}^{\infty} (n_{p+k} - n_p + 1) \frac{d(n_{p+k}+1)\ldots d(2)}{d(n_{p+k}-n_p+1)\ldots d(2)} h(n_{p+k}+1) z^{n_{p+k}-n_p+1}$$

is univalent in Δ, it is enough to prove that

$$\Sigma_{k=1}^{\infty} (n_{p+k} - n_p + 1) \frac{d(n_{p+k}+1)\ldots d(2)}{d(n_{p+k}-n_p+1)\ldots d(2)} |h(n_{p+k}+1)|$$

$$\leq d(n_p+1)\ldots d(2) |h(n_p+1)| .$$

Since $\rho < 1$,

$$\Sigma_{k=1}^{\infty} (n_{p+k}-n_{p+1}) \frac{d(n_{p+k}+1)\ldots d(2)}{d(n_{p+k}-n_p+1)\ldots d(2)} |h(n_{p+k}+1)|$$

$$< \frac{1}{2^p} \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{(d(n_{p+k}+1)\ldots d(2))^{1-\frac{1}{\rho}}}{d(n_{p+k}-n_p+1)\ldots d(2)}$$

$$< \frac{1}{2^p} \left(d(n_p+1)\ldots d(2)\right)^{1-\frac{1}{\rho}} \sum_{k=1}^{\infty} \frac{1}{2^k}$$

$$= d(n_p+1)\ldots d(2)|h(n_p+1)|.$$

As $D^{n+1}h(0) = 0$ unless $n=n_p$ for some p, only $D^{n_p}h$ are univalent in Δ.

If $\rho=0$, then take h^*_{j+1} defined by

$$h^*_{j+1} = \begin{cases} \dfrac{1}{2^{p+d(n_p+1)\ldots d(2)}\,(j-n_p+1)} & \text{if } j=n_p \text{ for some p.} \\ 0 & \text{otherwise.} \end{cases}$$

in place of h_{j+1} in the Taylor series of the function h(z).

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