A PAIR OF BIOORTHOGONAL POLYNOMIALS FOR THE
SZEGŐ-HERMITE WEIGHT FUNCTION

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ABSTRACT. A pair of polynomial sequences \{S_n^m(x;k)\} and \{T_m^m(x;k)\} where \(S_n^m(x;k)\) is
of degree \(n\) in \(x^k\) and \(T_m^m(x;k)\) is of degree \(m\) in \(x\), is constructed. It is shown
that this pair is biorthogonal with respect to the Szegő-Hermite weight function
\(|x|^{2\mu}\exp(-x^2), (\mu > -1/2)\) over the interval \((-\infty, \infty)\) in the sense that

\[
\int_{-\infty}^{\infty} |x|^{2\mu} \exp(-x^2) S_n^m(x;k) T_m^m(x;k) dx = 0, \text{ if } m \neq n
\]

\[
\neq 0, \text{ if } m = n
\]

where \(m, n = 0, 1, 2, \ldots\) and \(k\) is an odd positive integer.

Generating functions, mixed recurrence relations for both these sets are
obtained. For \(k = 1\), both the above sets get reduced to the orthogonal polynomials
introduced by professor Szegő.

KEYS WORDS AND PHRASES. Szego-Hermite weight function, Biorthogonal pair, Generating
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1. INTRODUCTION.

The biorthogonality conditions are useful in the computations involving the
penetration of gamma rays through matter as well as in determining the moments of a
hypergeometric distribution function. The notion of biorthogonality dates back to
Didon [1] and Deruyts [2]. The questions of constructing biorthogonal pairs of
polynomials corresponding to the weight functions of classical orthogonal polynomials
were taken up by Konhauser [3] for the Laguerre weight function \(x^\alpha e^{-x}\), by Toscano [4],
Chai [5], Carlitz [6] and Madhekar and Thakare [7] for the Jacobi weight function
\((1-x)^\alpha (1+x)^\beta\) and by Thakare and Madhekar [8] for the Hermite weight function
\(\exp(-x^2)\). The Szegő-Hermite polynomials \(u_n^\mu(x)\) are orthogonal w.r.t. the Szegő-Hermite
weight function \(|x|^{2\mu}\exp(-x^2), (\mu > -1/2)\) over the interval \((-\infty, \infty)\) and these are found
useful in connection with Gauss-Jacobi mechanical quadrature, see Szegö [8]. For
\( \mu = 0 \), Szegö-Hermite polynomials are just the classical Hermite polynomials.

2. A BIORTHOGONAL SYSTEM.

We shall construct a pair of biorthogonal polynomials w.r.t. the Szego-Hermite
weight function \( |x|^2e^{-x^2}, \mu > -1/2 \). Consider the following pair of polynomial
sequences.

\[
S^\mu_n(x; k) = 2^n \sum_{j=0}^{[n/2]} (-1)^{n-2j} \binom{n}{j} x^{kn-2kj}/\Gamma((kn+\mu+1)/2-kj+\mu),
\]

\[
T^\mu_n(x; k) = (-1)^{n/2} 2^n \sum_{r=0}^{[n/2]} x^{n-2r}/((n/2)-r)! \sum_{s=0}^{[n/2]-r} (-1)^s \binom{n/2-r}{s}
\]

\[
\times ((2s+(k+1)c + 2\mu+1)/2k)^{[n/2]},
\]

where the value of \( \epsilon \) is 0 or 1 according to even or odd nature of \( n \). Throughout this
paper \( \epsilon \) always has this meaning; and \([p]\) is the greatest integer less than or equal
to \( p \).

It is fairly easy to verify after reversioning the order of summation for even and odd
integers that

\[
S^\mu_{2n}(x; k) = (-1)^n 2^{2n} \frac{\Gamma((kn+\mu+k)/2)}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} x^{2kj}/\Gamma(kj+\mu+1/2),
\]

\[
= (-1)^n 2^{2n} \frac{\Gamma((kn+\mu+k)/2)}{n!} \frac{\Gamma((kn+\mu+1)/2)}{n!} (x^2; k),
\]

\[
S^\mu_{2n+1}(x; k) = (-1)^n 2^{2n+1} \frac{\Gamma((kn+\mu+k+1)/2)}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} x^{2kj+k}/\Gamma(kj+\mu+1+k/2),
\]

\[
= (-1)^n 2^{2n+1} \frac{\Gamma((kn+\mu+k+1)/2)}{n!} \frac{\Gamma((kn+\mu+1+k)/2)}{n!} (x^2; k),
\]

\[
T^\mu_{2n}(x; k) = (-1)^n 2^{2n} \frac{\Gamma((kn+\mu+1)/2)}{n!} \sum_{r=0}^{n} \frac{2r}{r!} \sum_{s=0}^{r} (-1)^s \binom{r}{s} ((s+\mu+1/2)/k)_n,
\]

\[
= (-1)^n 2^{2n} \frac{\Gamma((kn+\mu+1)/2)}{n!} \frac{\Gamma((kn+\mu+1+k)/2)}{n!} (x^2; k),
\]

\[
T^\mu_{2n+1}(x; k) = (-1)^n 2^{2n+1} \frac{\Gamma((kn+\mu+1+k+1)/2)}{n!} \sum_{r=0}^{n} (x^{2r+1}/r!) \sum_{s=0}^{r} (-1)^s \binom{r}{s} ((s+\mu+1+k)/k)_n,
\]

\[
= (-1)^n 2^{2n+1} \frac{\Gamma((kn+\mu+1+k+1)/2)}{n!} \frac{\Gamma((kn+\mu+1+k+1)/k)}{n!} (x^2; k).
\]

Here \( Z^\alpha_n(x; k) \) and \( Y^\alpha_n(x; k) \) is a pair of Konhauser [3] biorthogonal polynomials w.r.t.
the Laguerre weight function \( x^\alpha e^{-x} \) over \((0, \infty)\) and are given by

\[
Z^\alpha_n(x; k) = \frac{\Gamma(kn + \alpha + 1)}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} x^{kj}/\Gamma(kj + \alpha + 1),
\]

\[
Y^\alpha_n(x; k) = \frac{\Gamma((kn+\mu+1)/2)}{n!} \sum_{r=0}^{n} \frac{2r}{r!} \sum_{s=0}^{r} (-1)^s \binom{r}{s} ((s+\mu+1)/k)_n,
\]
BIORTHOGONAL POLYNOMIALS FOR THE SZEGÖ-HERMITE WEIGHT FUNCTION 765

\( \gamma_n^\alpha(x;k) = \frac{1}{n!} \sum_{r=0}^{n} \frac{x^r}{r!} \sum_{s=0}^{r} (-1)^s \binom{r}{s} \left( \frac{r}{s} \right) \left( \frac{(s+\alpha+1)/k}{r} \right) \); see Carlitz [9] (2.8)

where \( \alpha > -1 \), and \( k \) is a positive integer, and

\[
\int_0^\infty x^\alpha e^{-x} Z_n(x;k) \gamma_m(x;k) \, dx = \frac{\Gamma(kn+\alpha+1)}{n!} \delta(n,m) \quad \text{with} \quad \delta(n,m)
\]

(2.9)

the Kronecker's delta. Using [10] one readily obtains the following biorthogonality condition for the sets \( \{ s_n^\mu(x;k) \} \) and \( \{ t_m^\mu(x;k) \} \):

\[
\int_{-\infty}^{\infty} |x|^{2\mu} \exp(-x^2) s_n^\mu(x;k) t_m^\mu(x;k) \, dx = \pi^{2n} \frac{[n/2]!}{\Gamma(n+\epsilon+(kn+k-ke)/2)} \delta(n,m).
\]

(2.10)

An independent proof of (2.10) is also possible by using the identity of Carlitz [9, p. 249]:

\[
(-j)_m = \sum_{r=0}^{m} \frac{kj+c+m-r}{m-r} \delta(r,m) \left( \frac{m-r}{s} \right) \left( \frac{(s+c+1)/k}{r} \right).
\]

One has to note, however, that \( k \) is involved in \( s_n^\mu(x;k) \) and \( t_m^\mu(x;k) \) must be an odd positive integer in view of the existence theorem for biorthogonality due to Konhauser [10, p. 255].

One readily obtains

\[
\Gamma(kn+k+u+1/2) s_{zn+1}^\mu(x;k) = 2x^k \Gamma(kn+u+1+k/2) s_{zn}^{u+(k+1)/2}(x;k), \quad \text{and}
\]

(2.11)

\[
t_{zn+1}^\mu(x;k) = 2x^{u+(k+1)/2}(x;k),
\]

(2.12)

\[
D s_{zn}^\mu(x;k) = \left( \frac{nk}{2} \right)^{k-1} \frac{\Gamma(kn+k+1/2)}{\Gamma(kn+u+1/2)} s_{zn-1}^{u+(k-1)/2}(x;k).
\]

(2.13)

3. SOME PROPERTIES.

Using the relationship (2.3) to (2.6) it is fairly easy to obtain many results for the Szegö-Hermite biorthogonal pair of polynomials from the known results for the Konhauser biorthogonal sets. The results stated below could also be proved directly. Recall the Calvez and Ge'nin [11] generating function in the form (see also Srivastava [12]):

\[
\sum_{m=0}^{\infty} \binom{m+n}{n} v_m^\alpha(x;k) t^n = R^{1+\alpha+mk} \exp\{-(1-R)x\} v_m^\alpha(xR;k),
\]

(3.1)

where \( m \) is any integer \( \geq 0 \) and \( R = (1-t)^{-1/k} \). By handling even and odd cases separately, from (2.5) and (2.6) respectively, one obtains

\[
\sum_{m=0}^{\infty} t_m^{u+mk+1/2}(x;k) t^n/[n/2]! = VU^{u+mk+1/2} \left[ U^{-k} t_m^u(xU;k) + t \right] t_{m+1}^u(xU;k) \text{ where } U = (1+4t^2)^{-1/2k} \text{ and } V = \exp\{x^2[1-(1+4t^2)^{-1/k}]\}.
\]

The special case with \( m=0 \) is worth noting. Using (3.2) for even case and then applying (2.12) one obtains in a combined form the recurrence relation for the second set
\[ T_n^\mu(x;k) = \sum_{m=0}^{[n/2]} (-1)^m 2^m \binom{n/2}{m} \frac{(\frac{n-\lambda}{2})_m}{(\frac{\mu-\lambda}{2})_m} T_{n-2m}^\lambda(x;k), \lambda \neq \mu \text{ and } \lambda, \mu > -1/2. \tag{3.3} \]

Taking \( \mu = 0 \), and \( n \) even in (3.3) and using the biorthogonality condition (2.10) we have the integral

\[ \int_{-\infty}^{\infty} |x|^{2\lambda} \exp(-x^2) S_{2m}^\lambda(x;k) T_{2n}^\lambda(x;k) \, dx \tag{3.4} \]

\[ = (-1)^n \frac{4^m n!}{(\lambda/k)^n} \Gamma(kn+\lambda+k/2) \] where with \( \mu = 0 \), \( T_{2n}^\lambda(x;k) \) is the second biorthogonal set suggested by the Hermite polynomials; see Thakare and Madhekar [4]. The integral (3.4) says that \( T_{2n}^\lambda(x;k) \) are orthogonal to \( |x|^{2\lambda} S_{2m}^\lambda(x;k) \) w.r.t. the weight function \( \exp(-x^2) \) when \( n > m+\lambda/k \).

Consider the generating function first given by Genin and Calvez [13]; (see also Karande and Thakare [14], Prabhakar [15]):

\[ \Sigma (c)_n^\alpha n^\alpha n (x;k) t^n/(1+\alpha)_{kn} = (1-t)^{-c} \frac{\Gamma(kn+\lambda+\alpha)}{\Gamma(kn)} \left[ \frac{c; \tilde{t}^k/(1-t)k}{\Delta(k,1+\alpha)} \right] \tag{3.5} \]

where \( |t| < 1 \) and \( \Delta(m,\delta) \) stands for the sequence of parameters \( \delta/m, (\delta+1)/m, \ldots, (\delta+m-1)/m, (m=1) \). Using (2.3) one obtains from (3.5), an expression involving even \( S_{2n}^\mu(x;k) \) which after putting to use relation (2.11) gives a corresponding relation for odd \( S_{2n+1}^\mu(x;k) \). This resulting expression further with the help of the relation

\[ \Sigma (c)_n^\alpha n S_{2n+1}^\mu(x;k) t^{2n+1}/n! \frac{(\mu+k/2)_{nk}}{(\mu+1+k/2)_{nk}} \tag{3.6} \]

\[ = t(k+2\mu+k\theta)/(k+2\mu) \Sigma (c)_n^\alpha n S_{2n+1}^\mu(x;k) t^{2n+1}/n! \frac{(\mu+k/2)_{nk}}{(\mu+1+k/2)_{nk}}, \] where \( \theta = t, d/dt \)

yields

\[ \Sigma \frac{(c)_n^\alpha n}{n! \frac{(\mu+k/2)_{nk}}{(\mu+1+k/2)_{nk}}} S_{2n+1}^\mu(x;k) t^{2n+1} = 2tx^k \frac{u^{-2k}(1+c)}{u^{-2k} - \frac{8ck^2}{k+2}}. \tag{3.7} \]

\[ \frac{1}{1F_k} \left[ \frac{c; \tilde{t}^k/(1-t)k}{\Delta(k,1+\alpha)} \right] + \frac{16ck^3}{(k+2\mu)} \frac{2k(c+2)}{(1+\mu+k/2)_{k}} \frac{1}{1F_k} \left[ \frac{c+1; \tilde{t}^k/(1-t)k}{\Delta(k,1+\alpha)} \right] \]

where \( W = \frac{4x^2k^2}{(1+4t^2)k} \).

In fact, one obtains after combining even case with (3.7) the following generating function for the first biorthogonal set \( \{ s_{2n}^\mu(x;k) \} \):

\[ \Sigma \frac{(c)_n^\alpha n}{n! \frac{(\mu+k/2)_{nk}}{(\mu+1+k/2)_{nk}}} s_{2n}^\mu(x;k) t^n = \frac{(\mu+k/2)_{n/2}}{(\mu+1/n/2)} \frac{1}{1F_k} \left[ \frac{c; \tilde{t}^k/(1-t)k}{\Delta(k,1+1/2)} \right] \tag{3.8} \]

\[ + 2tx^k \frac{u^{2k}(1+c)}{u^{-2k} - \frac{8ck^2}{k+2\mu}} \frac{1}{1F_k} \left[ \frac{c+1; \tilde{t}^k/(1-t)k}{\Delta(k,1+1/2)} \right] \]

\[ + \frac{16ck^3}{(k+3\mu)} \frac{2k(c+2)}{(1+\mu+k/2)_{k}} \frac{1}{1F_k} \left[ \frac{c+3; \tilde{t}^k/(1-t)k}{\Delta(k,1+3k/2)} \right]. \]
We finally state the differential equation satisfied by the first set \( \{S_n^\mu(x;k)\} \) in the form

\[
\left[ x^2 \left( xD + 2\mu + 1 + e \right) \right]^k \left( x^{1-2k} \left( D - ek/x \right) S_n^\mu(x;k) \right) = (2x^2)^k \left( x D S_n^\mu(x;k) - nk S_n^\mu(x;k) \right), \quad (3.9)
\]

and a differential recurrence relation for the second set

\[
k T_{n+2}^\mu(x;k) = -2xD T_n^\mu(x;k) - 2(1+m+2\mu-2x^2) T_n^\mu(x;k). \quad (3.10)
\]

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<th>Deadline</th>
<th>Date</th>
</tr>
</thead>
<tbody>
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<td>Manuscript Due</td>
<td>May 1, 2009</td>
</tr>
<tr>
<td>First Round of Reviews</td>
<td>August 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>November 1, 2009</td>
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