COEFFICIENT ESTIMATES FOR SOME CLASSES OF $p$-VALENT FUNCTIONS

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ABSTRACT. Let $A_p$, where $p$ is a positive integer, denote the class of functions

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$$

which are analytic in $U = \{z : |z| < 1\}$.

For $0 < \lambda \leq 1$, $|a| < \frac{\pi}{2}$, $0 \leq \beta < p$, let $F_\lambda(a, \beta, p)$ denote the class of functions $f(z) \in A_p$ which satisfy the condition

$$|H(f(z)) - 1| < \lambda$$

for $z \in U$,

where

$$H(f(z)) = e^{\frac{iazf'(z)}{f(z) - \beta \cos \alpha - ip \sin \alpha}}$$

Also let $C_\lambda(b, p)$, where $p$ is a positive integer, $0 < \lambda < 1$, and $b \neq 0$ is any complex number, denote the class of functions $g(z) \in A_p$ which satisfy the condition

$$|H(g(z)) - 1| < \lambda$$

for $z \in U$, where

$$H(g(z)) = 1 + \frac{1}{pb}(1 + \frac{zg''(z)}{g'(z) - p})$$

In this paper we obtain sharp coefficient estimates for the above mentioned classes.

KEY WORDS AND PHRASES. $p$-valent, starlike, convex, spirallike functions.

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1. INTRODUCTION.

Let $A_p$, where $p$ is a positive integer, denote the class of functions

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$$

which are analytic in $U = \{z : |z| < 1\}$. We use $U_\lambda$, $0 < \lambda \leq 1$, to denote the class of analytic functions $w(z)$ in $U$ satisfying the conditions $w(0) = 0$ and $|w(z)| < \lambda$, $0 < \lambda \leq 1$.

Padmanabhan introduced the class of starlike functions of bounded order $\lambda$, $0 < \lambda \leq 1$, defined as follows [11]:

...
DEFINITION 1. A function \( f \in A_\lambda \) and satisfying

\[
\left| \frac{zf'(z) - 1}{f(z)} \right| < \lambda
\]

for a given \( \lambda, 0 < \lambda \leq 1 \), \( |z| < 1 \) is said to be starlike of bounded order \( \lambda \) in \( |z| < 1 \) and this class is denoted \( S(\lambda) \), the class of all such functions for a given \( \lambda \).

Let \( F(\alpha, \beta, p) \) denote the class of functions \( f(z) \in A_\lambda \) for which there exists a \( \rho = \rho(f) \) such that

\[
\Re \left( \frac{zf'(z) - \beta \cos \alpha}{f(z)} \right) > 0\quad (1.2)
\]

and

\[
\oint_0^{2\pi} \Re \left( \frac{zf'(z)}{f(z)} \right) d\Theta = 2\pi \rho \quad \text{for} \quad z = re^{i\Theta}, \rho < r < 1. \quad (1.3)
\]

Functions in \( F(\alpha, \beta, p) \) are called \( p \)-valent \( \alpha \)-spirallike functions of order \( \beta \). The class \( F(\alpha, \beta, p) \) was introduced by Patil and Thakare [12].

In this paper we use a method of Clunie [3] to obtain sharp bounds for the coefficients of functions \( F_\lambda (\alpha, \beta, p) \) and \( C_\lambda (b, p) \), where \( p \) is a positive integer, \( 0 < \lambda \leq 1 \), \( |\alpha| < \frac{\pi}{2} \), \( 0 \leq \beta < p \), and \( b \) is any complex number, where \( F_\lambda (\alpha, \beta, p) \) and \( C_\lambda (b, p) \) are defined as follows:

DEFINITION 2. For \( 0 < \lambda \leq 1 \), \( |\alpha| < \frac{\pi}{2} \), and \( 0 \leq \beta < p \), let \( F_\lambda (\alpha, \beta, p) \) denote the class of functions \( f(z) \in A_\lambda \) which satisfy the condition

\[
\left| \frac{H(f(z)) - 1}{H(f(z)) + 1} \right| < \lambda
\]

for \( z \in U \), where

\[
H(f(z)) = e^{i\alpha} \frac{zf'(z) - \beta \cos \alpha}{f(z)} - ip \sin \alpha
\]

DEFINITION 3. For \( p \) is a positive integer, \( 0 < \lambda \leq 1 \), and \( b \neq 0 \) is any complex number, let \( C_\lambda (p, b) \) denote the class of functions \( g(z) \in A_\lambda \) which satisfy the condition

\[
\left| \frac{H(g(z)) - 1}{H(g(z)) + 1} \right| < \lambda
\]

for \( z \in U \), where

\[
H(g(z)) = 1 + \frac{1}{pb} (1 + \frac{zg''(z)}{g'(z)} - p).
\]

We note that by giving specific values to \( \lambda, \alpha, \beta, p \) and \( b \), we obtain the following important subclasses studied by various authors in earlier papers:

(1) \( F_1(0,0,1) = S^* \) and \( C_1(1,1) = C \), are respectively the well-known classes of starlike functions and convex functions, \( F_1(0,\beta,1) = S_\beta \) and \( C_1(1-\beta,1) = C_\beta \), \( 0 \leq \beta < 1 \), are respectively the classes of starlike functions of order \( \beta \) and convex functions of order \( \beta \) introduced by Robertson [14], \( F_\lambda (0,0,1) = S(\lambda) \) and \( C_\lambda (1,1) = C(\lambda) \), is the class of functions \( g \) for which \( zg'(z) \in S(\lambda) \).
COEFFICIENT ESTIMATES FOR SOME CLASSES OF p-VALENT FUNCTIONS

(2) \( F_1(\alpha, 0, 1) = S^\alpha \) and \( C_1(\cos \alpha e^{-i\alpha}, 1) = C^\alpha \), \( |\alpha| < \frac{\pi}{2} \), are respectively the class of \( \alpha \)-spirallike functions introduced by Spacek [18] and the class of functions \( g \) for which \( zg'(z) \) is \( \alpha \)-spirallike introduced by Robertson [15], \( F_1(\alpha, \beta, 1) = S^\beta \) and \( C_1[(1 - \beta) \cos \alpha e^{-i\alpha}, 1] = C^\beta \), \( |\alpha| < \frac{\pi}{2} \), \( 0 < \beta \leq 1 \), are respectively the class of \( \alpha \)-spirallike functions of order \( \beta \) introduced by Libera [8] and the class of functions \( g \) for which \( zg'(z) \) is \( \alpha \)-spirallike of order \( \beta \) by Chichra [2] and Sizik [17].

(3) \( C_1(b, 1) = C(b) \) is the class of functions \( g \in A_1 \) satisfying

\[
\text{Re}\left\{ 1 + \frac{1}{b} \frac{g^{(n)}(z)}{g'(z)} \right\} > 0
\]

introduced by Wiatrowski [19] and studied by [9] and [10].

(4) \( F_1(0, 0, p) = S(p) \), \( C_1(1, p) = C(p) \), \( F_1(0, \beta, p) = S^\beta(p) \) and \( C_1[(1 - \beta) \cos \alpha e^{-i\alpha}, p] \), \( |\alpha| < \frac{\pi}{2} \), \( 0 \leq \beta < p \), are respectively the classes of \( p \)-valent starlike functions, \( p \)-valent convex functions, \( p \)-valent starlike functions of order \( \beta \) and \( p \)-valent convex functions of order \( \beta \) considered by Goodman [6] and the class \( S^\beta(p) \) investigated by Goluzina [5].

(5) \( F_1(\alpha, 0, p) = S^\alpha(p) \) and \( C_1(\cos \alpha e^{-i\alpha}, p) \), \( |\alpha| < \frac{\pi}{2} \), are respectively the class of \( p \)-valent \( \alpha \)-spirallike functions and the class of \( p \)-valent functions \( g \in A_\rho \) satisfying

\[
\text{Re} e^{i\alpha}(1 + \frac{g^{(n)}(z)}{g'(z)}) > 0, \, z \in U
\]

i.e., the class of \( p \)-valent functions \( g \) for which \( \frac{g'(z)}{p} \) is \( p \)-valent \( \alpha \)-spirallike.

(6) \( F_1(\alpha, \beta, p) = F(\alpha, \beta, p) \) and \( C_1[(1 - \beta) \cos \alpha e^{-i\alpha}, p] \), \( |\alpha| < \frac{\pi}{2} \), \( 0 \leq \beta < p \), is the class of \( p \)-valent functions \( g \) for which \( \frac{g'(z)}{p} \) is \( p \)-valent \( \alpha \)-spirallike of order \( \beta \).

(7) \( C_1(b, p) \), is the class of functions \( g \in A_\rho \) satisfying

\[
\text{Re} \{(p + \frac{1}{b}(1 + \frac{g^{(n)}(z)}{g'(z)} - p)) > 0, \, z \in U,
\]

the class \( C(b, p) \) was introduced by the author [1].

(8) \( F_\lambda(\alpha, \beta, 1) = F_\lambda(\alpha, \beta) \), is the class of functions investigated by Gopalakrishna and Umarani [7].

(9) \( C_1[(1 - \beta) \cos \alpha e^{i\alpha}, p] \), \( |\alpha| < \frac{\pi}{2} \), \( 0 \leq \beta < p \), is the class of \( p \)-valent functions \( g(z) \) for which \( \frac{g'(z)}{p} \in F_\lambda(\alpha, \beta, p) \).

We state the following lemma that is needed in our investigation.

**LEMMA 1**[11]. Let \( f(z) \) be analytic for \( |z| < 1 \) and let \( f(0) = 0 \). Then \( f(z) \in S(\lambda) \) if and only if

\[
f(z) = z \exp \left\{ -2 \int_0^z \frac{\phi(t)}{1 + t\phi(t)} \, dt \right\},
\]

where \( \phi(z) \) is analytic and satisfies \( |\phi(z)| \leq \lambda \), \( 0 < \lambda \leq 1 \), for \( |z| < 1 \).
In the rest of the paper we always assume that \( p \) is a positive integer, \( 0 < \lambda \leq 1, \quad |\alpha| < \frac{\pi}{2}, \quad 0 \leq \beta < p, \) and \( b \neq 0 \) is any complex number.

2. REPRESENTATION FORMULAS FOR THE CLASS \( F_\lambda(\alpha, \beta, p) \).

**LEMMA 2.** \( f(z) \in F_\lambda(\alpha, \beta, p) \) if and only if for \( z \in U \)

\[
e ^{i\alpha} \frac{zf'(z)}{f(z)} = \frac{\cos(a) (p-(p-2\beta)\omega(z)) + ip \sin(a)}{1 + \omega(z)},
\]

\( \omega \in \Omega_\lambda \).

**PROOF.** If \( f(z) \) is given by \( 2.1 \), then

\[
H(f(z)) = \frac{1 - \omega(z)}{1 + \omega(z)}
\]

so that \( H(f(z)) - \frac{1}{1 + H(f(z))} = -\omega(z) \)

and so \( 1.4 \) holds. Thus \( f(z) \in F_\lambda(\alpha, \beta, p) \).

Conversely, if \( f(z) \in F_\lambda(\alpha, \beta, p) \), then \( 1.4 \) holds.

Defining \( \omega(z) = \frac{1 - H(f(z))}{1 + H(f(z))} \) we obtain \( 2.1 \) and the proof is complete.

**LEMMA 3.** \( f(z) \in F_\lambda(\alpha, \beta, p) \) if and only if

\[
f(z) = z^p \left[ \frac{f_1(z)}{z} \right]^p
\]

for some \( f_1 \in F_\lambda(\alpha, \beta, p) \).

**PROOF.** Let \( f(z) = z^p \left[ \frac{f_1(z)}{z} \right] \) for \( f_1(z) = z + \frac{\beta}{n} \sum_{n=2}^{\infty} c_n z^n \in F_\lambda(\alpha, \beta, p-1), z \in U \).

By direct computation, we obtain

\[
e ^{i\alpha} \frac{zf'(z)}{f(z)} - \beta \cos(a) - ip \sin(a)
\]

\[
(\beta p) \cos(a)
\]

and the result follows from \( 1.4 \).

In a similar way we can prove the following lemma:

**LEMMA 4.** \( f(z) \in F_\lambda(\alpha, \beta, p) \) if and only if

\[
f(z) = z^p \left[ \frac{f_2(z)}{z} \right]^{(p-\beta) \cos(a)} e^{-i\alpha}
\]

for some \( f_2 \in S(\lambda) \).

An immediate consequence of lemmas 1 and 4 is

**THEOREM 1.** \( f(z) \in F_\lambda(\alpha, \beta, p) \) if and only if

\[
f(z) = z^p \exp[-(p-\beta) \cos(a) e^{-i\alpha} \int \frac{\phi(t)}{1 + t \phi(t)} \, dt]
\]

where \( \phi(z) \) is analytic and satisfies \( |\phi(z)| \leq \lambda, \quad 0 < \lambda \leq 1, \) for \( |z| < 1 \).

3. COEFFICIENT ESTIMATES FOR THE CLASS \( F_\lambda(\alpha, \beta, p) \).

**LEMMA 5.** If integers \( p \) and \( m \) are greater than zero, \( 0 < \beta < p \) and \( |\alpha| < \frac{\pi}{2} \), then
PROOF. We prove the lemma by induction on \( m \). For \( m = 1 \), (3.1) is easily verified directly.

Next suppose that (3.1) is true for \( m = q - 1 \). We have

\[
\begin{align*}
\frac{\cos^2 a}{q^2} (4\lambda^2(p-\beta)^2 + \sum_{k=1}^{q-2} \lambda^2(2p-2\beta+k)^2 + \lambda^2k^2\tan^2 a - k^2\sec^2 a) & + \lambda^2k^2\tan^2 a - k^2\sec^2 a \sum_{j=0}^{k-1} \frac{\lambda^2|2(p-\beta)\cos a e^{-i\alpha} + j|^2}{(j+1)^2} \\
& + \left[ \lambda^2(2p-2\beta+q-1)^2 + \lambda^2(q-1)^2\tan^2 a - (q-1)^2\sec^2 a \right] \sum_{j=0}^{q-2} \frac{\lambda^2|2(p-\beta)\cos a e^{-i\alpha} + j|^2}{(j+1)^2} \\
& = \frac{q-1}{q^2} \lambda^2|2(p-\beta)\cos a e^{-i\alpha} + j|^2 \times \\
& \left( \lambda^2(2p-2\beta+q-1)^2\tan^2 a + \lambda^2(q-1)^2\sec^2 a \right) \sum_{j=0}^{q-2} \\
& = \frac{q-1}{q^2} \lambda^2|2(p-\beta)\cos a e^{-i\alpha} + j|^2 \\
\end{align*}
\]

Thus (3.1) holds for \( m = q \) which proves lemma 5.

THEOREM 2. If \( f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in F_\lambda(\alpha, \beta, p) \), then

\[
|a_n| \leq \frac{\lambda}{k+1} \left| \frac{(p+k) e^{i\alpha} \sec a zf'(z) + (p-2\beta-ip \tan a)f(z)}{w(z)} \right| \\
\]

for \( n \geq p+1 \) and these bounds are sharp for all admissible \( \alpha, \beta \) and \( \lambda \) for each \( n \).

PROOF. As \( f \in F_\lambda(\alpha, \beta, p) \), from Lemma 2, we have

\[
\begin{align*}
(e^{i\alpha} \sec a zf'(z) + (p-2\beta-ip \tan a)f(z))w(z) \\
= (p+ip \tan a)f(z) - e^{-i\alpha} \sec azf'(z) \\
\end{align*}
\]

for \( z \in U, w \in \Omega_\lambda \). Hence we have

\[
\sum_{k=0}^{p} \left[ ((p+k)e^{i\alpha} \sec a + (p-2\beta-ip \tan a))a_{p+k}z^k \right] w(z) \\
= \sum_{k=0}^{p} [p + (p+k)e^{i\alpha} \sec a]a_{p+k}z^k \\
\]

where \( a_p = 1 \) and \( w(z) = \sum_{k=0}^{p} b_{k+1}z^{k+1} \).
Equating coefficients of \( z^m \) on both sides of (3.3), we obtain

\[
\begin{align*}
&\sum_{k=0}^{m-1} \left[ (p+k)e^{ia} \sec a + (p-2\beta - ip \tan a) \right] a_{p+k} \ b_{m-k} \\
&= \left\{ p + ip \tan a - (p+m)e^{ia} \sec a \right\} a_{p+m};
\end{align*}
\]

which shows that \( a_{p+m} \) on right hand side depends only on \( a_{p}, a_{p+1}, \ldots, a_{p+(m-1)} \) of left-hand side. Hence we can write

\[
\begin{align*}
&\sum_{k=0}^{m-1} \left[ (p+k)e^{ia} \sec a + (p-2\beta - ip \tan a) \right] a_{p+k} z^k \\
&= \sum_{k=0}^{m-1} \left[ (p + ip \tan a - (p+k)e^{ia} \sec a \right] a_{p+k} z^k + \sum_{k=m+1}^{\infty} A_k z^k
\end{align*}
\]

(3.4)

for \( m = 1,2,3\ldots \) and a proper choice of \( A_k \) \((k \geq 0)\).

Denoting the right member of (3.4) by \( G(z) \) and the factor multiplying \( w(z) \) in the left member of (3.4) by \( F(z) \), (3.4) assumes the form

\[ G(z) = F(z) \ w(z) \text{ for } z \in U. \]

Since \( |w(z)| < \lambda \) for \( z \in U \) this yields for \( 0 < r < 1, \)

\[
\frac{1}{2\pi} \int_0^{2\pi} |G(re^{i\theta})|^2 \ d\theta \leq \frac{\lambda^2}{2\pi} \cdot \int_0^{2\pi} |F(re^{i\theta})|^2 \ d\theta,
\]

hence, using the definitions of \( G(z) \) and \( F(z) \)

\[
\begin{align*}
&\sum_{k=0}^{m-1} |(p+k)e^{ia} \sec a + (p-2\beta - ip \tan a)|^2 \ |a_{p+k}|^2 r^{2k} \\
&+ \sum_{k=m+1}^{\infty} |A_k|^2 r^{2k} \leq \lambda^2 \sum_{k=0}^{m-1} |(p+k)e^{ia} \sec a + (p-2\beta - ip \tan a)|^2 \ |a_{p+k}|^2 r^{2k}.
\end{align*}
\]

(3.5)

Setting \( r = 1 \) in (3.5), the inequality (3.5) may be written as

\[
\begin{align*}
&\sum_{k=0}^{m-1} \left( \lambda^2 \ |(p+k)e^{ia} \sec a + (p-2\beta - ip \tan a)|^2 - \right. \\
&\left. |p + ip \tan a - (p+k)e^{ia} \sec a|^2 \right] |a_{p+k}|^2 \\
&\geq \sum_{k=m+1}^{\infty} |A_k|^2 r^{2k}.
\end{align*}
\]

(3.6)

Simplification of (3.6) leads to

\[
|a_{p+m}|^2 \leq \frac{\cos^2 a}{m^2} \cdot \sum_{k=0}^{m-1} \left( \lambda^2 (2p-2\beta+k)^2 + \right.
\]

\[
\left. \lambda^2 \ k^2 \ tan^2 a - k^2 \ sec^2 a \right] |a_{p+k}|^2.
\]

(3.7)

Replacing \( p+m \) by \( n \) in (3.7), we are led to

\[
|a_n|^2 \leq \frac{\cos^2 a}{(n-p+1)^2} \cdot \sum_{k=0}^{n-1} \left( \lambda^2 (2p-2\beta+k)^2 + \right.
\]

\[
\left. \lambda^2 \ k^2 \ tan^2 a - k^2 \ sec^2 a \right] |a_{p+k}|^2
\]

(3.8)

where \( n \geq p + 1. \)
For \( n = p + 1 \), (3.8) reduces to
\[
|a_{p+1}|^2 \leq 4(p-8)^2 \lambda^2 \cos^2 \alpha
\]
or
\[
|a_{p+1}| \leq 2(p-8) \lambda \cos \alpha
\] (3.9)
which is equivalent to (3.2).

To establish (3.2) for \( n > p+1 \), we will apply induction argument.

Fix \( n, \ n \geq p + 2 \), and suppose (3.2) holds for \( k = 1, 2, \ldots, n-(p+1) \). Then
\[
\frac{\cos^2 \alpha}{(n-p)^2} \sum_{k=0}^{n-(p+1)} \left[ 2(2p-2k+k)^2 + k^2 \tan^2 \alpha - k^2 \sec^2 \alpha \right] x
\]
Thus from (3.8), (3.10) and Lemma 5 with \( m = n - p \), we obtain
\[
|a_n|^2 \leq \frac{\lambda^2 |2(p-8)\cos \alpha e^{-i\alpha} + j|^2}{(n-p)^2}.
\] (3.10)
This completes the proof of Theorem 2.

Equality holds in (3.2) for \( n \geq p + 1 \) for the function \( f(z) = A_p \) defined by (2.1) with \( w(z) = \lambda z \).

REMARK ON THEOREM 2. For various choices of the parameters, known results can be regained: \([7, 8, 12, 13, 14, 16] \).

In a similar way we can prove the following: Lemma 6, 7, and Theorem 3 for functions in \( C^{b,p} \).

4. REPRESENTATION FORMULAS FOR THE CLASS \( C^{b,p} \)

LEMMA 6. \( \mathfrak{g}(z) \in C^{b,p} \) if and only if for \( z \in U \)
(i) \( \frac{\mathfrak{g}''(z)}{\mathfrak{g}'(z)} = \frac{(p-1)+(p-2pb-1)w(z)}{1+w(z)} \), \( w \in \mathfrak{G}_\lambda \).
(ii) \( \mathfrak{g}'(z) = p(z) \left[ \frac{g_1(z)}{z} \right]^p \) \( \) (4.2)
for some \( g_1 \in S(\lambda) \).
(iii) \( \mathfrak{g}'(z) = p(z) \exp[-2pb \int_0^z \frac{\mathfrak{h}(t)}{1+t} dt] \), \( \) (4.3)
where \( \mathfrak{h}(z) \) is analytic and satisfies \( |\mathfrak{h}(z)| \leq \lambda \), \( 0 < \lambda < 1 \), for \( |z| < 1 \).

5. COEFFICIENT ESTIMATES FOR THE CLASS \( C^{b,p} \)

LEMMA 7. If \( p \) and \( m \) are greater than zero; \( b \neq 0 \) and complex, then
\[
\sum_{j=0}^{m-1} \frac{\lambda^2 |2pb+1|^2}{(j+1)^2} = \frac{1}{m^2} \left( 4p^2 |b|^2 + \lambda^2 \right) + \sum_{k=1}^{m-1} \frac{\lambda^2 |2pb+1|^2}{(j+1)^2}.
\] (5.1)
THEOREM 3. If \( g(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in C_\lambda(b,p) \), then

\[
|d_n| \leq \frac{p}{n} \cdot \frac{n-(p+1)}{k=0} \sum_{k=0}^{\lambda \cdot 2p+b+k} \frac{p}{k+1}
\]

for \( n \geq p+1 \). Equality holds in (5.2) for the function \( g(z) = A_p \) defined by (4.1) with \( \omega(z) = \lambda z \).

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Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

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