A NEW CLASS OF COMPOSITION OPERATORS

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ABSTRACT. A new class of composition operators \( P_\phi : H^2(T) \rightarrow H^2(T) \), with \( \phi : T \rightarrow \overline{T} \) is introduced. Sufficient conditions on \( \phi \) for \( P_\phi \) to be bounded and Hilbert-Schmidt are obtained. Properties of \( P_\phi \) with \( \phi(e^{it}) = ae^{it} + be^{-it} \) for different values of the parameters \( a \) and \( b \) have been investigated. This paper concludes with a discussion on the compactness of \( P_\phi \).

KEY WORDS AND PHRASES. \( H^p \) Space, Composition operator, Hilbert Schmidt operator, Compact operator.

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1. PRELIMINARIES.

For a complex valued function \( f \) analytic in \( D = \{ z : |z| < 1 \} \) and for \( 1 \leq p \leq \infty \) set

\[
M_p(r,f) = \left( \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty
\]

and

\[
M(r,f) = \sup_{0 < \theta < 2\pi} |f(re^{i\theta})|.
\]

The function \( f \) is said to be in \( H^p(D) \), if \( \lim_{r \to 1-} M_p(r,f) < \infty \). Similarly, let \( H^p(T) \), \( T = \{ z : |z| = 1 \} \), be the class of functions in \( L^p(T) \) such that

\[
\int_0^{2\pi} f(e^{i\theta}) \overline{e^{in\theta}} d\theta = 0, \quad n = 1,2,3, \ldots.
\]

It is known [1,2] that for \( f \) in \( H^p(D) \), \( \lim_{r \to 1-} f(re^{i\theta}) = f_\pi(e^{i\theta}) \) exists for almost all \( \theta \) and \( f_\pi \) belongs to \( H^p(T) \). Conversely, the Poisson integral of a function in \( H^p(T) \) is in \( H^p(D) \). Also, if \( f \) in \( H^p(D) \) has the sequence \( \{ a_n \} \) as its Taylor coefficients then, \( f_\pi \) has the same sequence as its Fourier coefficients and vice versa. This correspondence establishes an isometrical isomorphism between \( H^p(D) \) and \( H^p(T) \). Thus, these
two spaces are interchangeably used and are usually referred to as the Hardy Space $H^p[1,2]$.

In the sequel we came across another space familiarly known as the weighted Hardy space $H^p$. Let $p(n)$ be a sequence of positive numbers. An analytic function $f: \mathbb{D} \to \mathbb{C}$, given by $f(z) = \sum a_n z^n$, is said to be in the class $H^p$, if $\|f\|_p = \sum |a_n|^p < \infty$. Also we need the following definition. Let $H$ be a Hilbert space and $T$ be a bounded linear operator on $H$. Then, $T$ is said to be Hilbert Schmidt if there exists an orthonormal basis $\{e_n\}$ in $H$ such that $\sum \|Te_n\|^2 < \infty$.

Throughout in the present paper we denote by $e_n$, $n = 0, 1, 2, \ldots$, the function $e^{int}$. We note that $\{e_n\}$ forms an orthonormal basis for $H^2$.

2. A NEW CLASS OF COMPOSITION OPERATORS.

Let $\phi: \mathbb{D} \to \mathbb{D}$ be analytic and let $C_\phi: H^p(\mathbb{D}) \to H^p(\mathbb{D})$ be defined by $(C_\phi f)(z) = f(\phi(z))$, $z$ in $D$. The operator $C_\phi$ is known as a composition operator on $H^p(\mathbb{D})$ and is extensively studied in the literature [4]. In the present paper we introduce and study a new class of composition operators $P_\phi$ on $H^2(T)$ where $\phi: T \to \overline{\mathbb{D}}$ many be 'non-analytic' also. That is $\phi$ nonvanishing negative Fourier coefficients.

**DEFINITION.** Let $\phi: T \to \overline{\mathbb{D}}$ satisfy the following properties:

(a) for every set $E \subset T$, of linear measure zero, $\phi^{-1}(E) = \{z \in T: \phi(z) = w, w \in E\}$ is also a set of linear measure zero and

(b) for every $f$ in $H^2(T)$, $fo\phi$ is in $L^2(T)$.

Then, define $P_\phi: H^2(T) \to H^2(T)$ by $P_\phi f = P(fo\phi)$ where $P$ is the projection of $L^2(T)$ into $H^2(T)$.

Here some explanations are in order. We observe that a function $f$ in $H^2(T)$ can be extended analytically into $\mathbb{D}$ as described in Section I. So with the condition (a), $fo\phi$ is defined almost everywhere on $T$. Further, let $f$ be represented by the Fourier series $\sum_{n=0}^\infty a_n e^{inx}$. Then by the Weierstrass theorem, $\sum_{n=0}^\infty a_n (\phi(e^{i\theta}))^n$ converges pointwise to $f(\phi(e^{i\theta}))$ for all $\theta$ such that $\phi(e^{i\theta}) \in \mathbb{D}$ and by a result of Carleson [5] $\sum_{n=0}^\infty a_n (\phi(e^{i\theta}))^n$ converges pointwise to $f(\phi(e^{i\theta}))$ for almost all $\theta$ such that $\phi(e^{i\theta}) \in T$. Hence $\sum_{n=0}^\infty a_n (\phi(e^{i\theta}))^n$ converges pointwise almost everywhere on $T$ to $f(\phi(e^{i\theta}))$. Thus throughout in this paper we write $\sum_{n=0}^\infty a_n (\phi(e^{i\theta}))^n$ in place of $f(\phi(e^{i\theta}))$.

We note that if $\phi$ satisfies the conditions of the definition, then by the Closed Graph Theorem, $P_\phi$ is a bounded operator. So a natural question is: under what conditions on $\phi$, $fo\phi$ is in $L^2(T)$ for all $f$ in $H^2(T)$. The present paper primarily deals with this question.

In the following sections we first obtain bounds for the norm of $P_\phi$ under suitable conditions on $\phi$. Then we consider $\phi$ defined by $\phi(e^{it}) = ae^{it} + be^{-it}$ and study
conditions on \(a\) and \(b\) such that \(f \circ \phi \in L^2(T)\) for all \(f\) in \(H^2(T)\). In the last section we have discussed the compactness of \(P_\phi\) with the help of some examples.

3. NORM OF \(P_\phi\).

We have the following results.

THEOREM 1. Let \(\phi: T \to D\) be such that

\[
\int_0^{2\pi} \frac{dt}{1 - |\phi(e^{it})|^2} = M(\phi) < \infty
\]

Then, \(P_\phi\) is Hilbert Schmidt and

\[
\|P_\phi f\| \leq \frac{M(\phi)}{2\pi} \quad .
\]

PROOF. Let \(f\) in \(H^2(T)\) be given by \(f(z) = \sum a_n z^n\), \(z\) in \(T\). Then,

\[
|f(\phi(e^{it}))|^2 = \left| \sum a_n (\phi(e^{it}))^n \right|^2 \leq (\sum |a_n|^2) (\sum |\phi(e^{it})|^{2n})
\]

so,

\[
\|P_\phi f\| \leq \|f\| \leq \frac{M(\phi)}{2\pi}
\]

and we get \(\|P_\phi f\| \leq \frac{M(\phi)}{2\pi}\).

Next, with the orthonormal basis \(e_n, n = 0,1,2,\ldots\), of \(H^2(T)\), we have

\[
\frac{1}{2\pi} \int_0^{2\pi} \left| P_\phi (e_n) \right|^2 \leq \frac{1}{2\pi} \int_0^{2\pi} \left| e_n \right|^2 = \frac{1}{2\pi} \int_0^{2\pi} |\phi(e^{it})|^{2n} dt = \frac{M(\phi)}{2\pi} < \infty.
\]

Thus, \(P_\phi\) is Hilbert Schmidt.

COROLLARY 2. If \(\phi: T \to D\) is continuous then \(P_\phi\) is Hilbert Schmidt.

PROOF. The condition (3.1) is trivially satisfied if \(\phi\) is continuous.

By an example in the next section we will show that (3.1) is only a sufficient condition for \(P_\phi\) to be Hilbert Schmidt. We need the following lemma due to Gabriel [6] for the proof of our next theorem.

LEMMA. Let \(\Gamma\) be a rectifiable convex curve in the closed unit disc. Then, for every \(f\) in \(H^2\)

\[
\int_{\Gamma} |f(w)|^2 \, |dw| \leq 4\pi \|f\|_2^2
\]

THEOREM 2. Let \(\phi: T \to D\) be such that

(i) \(\phi\) describes a closed rectifiable convex curve in \(\overline{D}\) and

(ii) \(m = \inf |\phi'(e^{it})| > 0, 0 \leq t \leq 2\pi\),

Then, \(\|P_\phi f\| \leq 1/(2m)^{1/2}\).

PROOF. By lemma and the condition (ii) we have

\[
4\pi \|f\|_2^2 \geq \int_0^{2\pi} |f(\phi(e^{it}))|^2 \, |\phi'(t)| \, dt \geq m \int_0^{2\pi} |(f \circ \phi)(e^{it})|^2 dt
\]

so that

\[
\|P_\phi f\|^2 \leq \frac{1}{4\pi} \int_0^{2\pi} |(f \circ \phi)(e^{it})|^2 dt \leq \frac{2}{m} \|f\|_2^2
\]
The conditions (i) and (ii) in the above theorem are not necessary for $P_{\phi}$ to be bounded. As an example consider $$\phi(t) = \left\{ \begin{array}{ll} e^{it} & 0 < t < \pi \\ 0 & \pi \leq t \leq 2\pi \end{array} \right.$$ 

so that $\phi$ does not satisfy any of the conditions (i) or (ii) of the theorem. Now,

$$\frac{1}{2\pi} \int_{0}^{2\pi} |f(\phi(e^{it}))|^2 \, dt = \frac{1}{2\pi} \int_{0}^{\pi} |f(e^{it})|^2 \, dt + \frac{1}{2\pi} \int_{\pi}^{2\pi} |f(0)|^2 \, dt \leq ||f||^2_2.$$ 

This shows that $P_{\phi}$ is bounded with $||P_{\phi}|| \leq \sqrt{2}$.

4. A FAMILY OF COMPOSITION OPERATORS.

In this section we study the properties of $P_{\phi}$ for the particular family of functions $\phi : T \to \overline{D}$ given by

$$\phi(z) = az + bz \quad z \in T \quad (4.1)$$

where $|a| + |b| \leq 1$. We note that if $|a| \neq |b|$ then the curve traced by $\phi$ is an ellipse containing the origin in its interior. Also $m = \inf \left| a e^{-it} - b \right| \geq ||a| - |b|| > 0$.

Hence by Theorem 2, $P_{\phi}$ is bounded. It turns out that $P_{\phi}$ has many interesting properties for different values of the parameters $a$ and $b$. We need the following technical lemma.

**Lemma 1.** For all $n, k$ in $\mathbb{Z}^+$

$$\binom{n+2k}{k} < 2^{n+2k} \quad (4.2)$$

**Proof.** We shall prove (4.2) by method of induction on $n$. Let $n = 0$ so that we have to show

$$\binom{2k}{k} < 2^{2k} \quad \text{for } k = 1, 2, 3, \ldots \ldots \quad (4.3)$$

We establish (4.3), also by the process of induction on $k$. For $k = 1$, $\binom{2}{1} = 2 < 2^2$ is trivially true. Next, assume that

$$\binom{2k}{k} < 2^{2k}, \quad \text{i.e.} \quad \frac{(2k)!}{(k!)^2} < 2^{2k}$$

To complete induction on $k$ we consider

$$\frac{2(2k+1)!}{(k+1)!^2} < 2^{2k} \quad \text{or} \quad 2^{2k+1} \frac{(2k+1)!}{(k+1)!} < 2^{2k+2} \quad (4.4)$$

Thus, (4.3) is true for all $k = 1, 2, 3 \ldots \ldots$ Next, let $n = 1$. Then,

$$\binom{2k+1}{k} = \binom{2k}{k} \cdot \frac{(2k+1)}{(k+1)} < 2^{k+1} \cdot 2 = 2^{k+1} \quad (4.5)$$

Now, assume that $\binom{n+2k}{k} < 2^{n+2k}$. To complete the induction we consider

$$\binom{n+1+2k}{k} = \frac{(n+2k)!}{(k!)^2} \frac{(n+1+2k)!}{(n+1+k)!} < 2^{n+2k} \cdot 2 < 2^{n+1+2k} \quad (4.6)$$

Thus (4.3) is true for all $k = 1, 2, \ldots$ and $n = 0, 1, 2 \ldots \ldots$. 
THEOREM 3. Let \( \phi: T \to \overline{D} \) be given by \( \phi(z) = az + b \overline{z} \), \( z \) in \( T \).

(i) If \( |a| + |b| \leq 1 \), \( |a| \neq |b| \), \( b \neq 0 \) and \( |a| < \frac{1}{2} \) then \( P_\phi \) is Hilbert Schmidt.

(ii) If \( |a| + |b| = \frac{1}{2} \) then \( f \circ \phi \) need not be in \( L^2 \) for all \( f \) in \( H^2 \) so that \( P_\phi \) is not defined on the whole of \( H^2 \).

(iii) The inequality \( |a| < \frac{1}{2} \) in (i) is best possible.

PROOF. Consider the orthonormal basis \( e_n \), \( n = 0, 1, 2, \ldots \), for \( H^2 \). With respect to this basis \( P_\phi \) has a matrix representation

\[
T_{mn} = \begin{cases} 
0 & \text{if } n < m \\
\left( \frac{a^n}{k^2} \right) & \text{if } n - m = 2k \\
0 & \text{if } n - m = 2k + 1 
\end{cases}
\]

where \( m, n \), and \( k \in \mathbb{Z}_+ \). Now

\[
|T_{mn}|^2 = \sum_{n=0}^{\infty} \left( \frac{a^n}{k^2} \right)^2 |a|^{2n} |ab|^{2k}
\]

We use Lemma 1 to show that the second sum in the right-hand side is convergent.

\[
\sum_{n=0}^{\infty} \left( \frac{a^n}{k^2} \right)^2 |a|^{2n} |ab|^{2k} \leq \sum_{k=1}^{\infty} \frac{1}{2^n} |a|^2 \left( \frac{1}{2^n} \right)^2 \frac{1}{2^{2n+4k}} |a|^{2n} |ab|^{2k}
\]

since \( |a| < \frac{1}{2} \), which also implies \( |ab| < \frac{1}{4} \). This proves that \( P_\phi \) is Hilbert Schmidt. (i) For the proof of (ii) consider the \( H^2 \) function \( f(z) = (1-z)^{1-\alpha} \), \( 0 < \alpha < \frac{1}{2} \).

For \( a = b = \frac{1}{2} \), \( \phi(e^{i\theta}) = \cos \theta \). Thus,

\[
f(\phi(e^{i\theta})) = f(\cos \theta) = \frac{1}{(1-\cos \theta)^{1-\alpha}} = \frac{1}{2^{-\alpha} \sin^{2\alpha} \frac{\theta}{2}} \quad \text{a.e.}
\]

and

\[
\int_0^{2\pi} |f(\phi(e^{i\theta}))|^2 d\theta = \int_0^{\pi/2} \sin^{-4\alpha} \theta d\theta \geq \int_0^{\pi/2} \frac{2^{2-2\alpha}}{2^{2-2\alpha} \sin^{-4\alpha} \theta} d\theta = \infty
\]

if \( \alpha > \frac{1}{4} \). In the above we have made use of the well-known inequality \( \frac{2\theta}{\pi} < \sin \theta < \theta \) for \( 0 < \theta < \frac{\pi}{2} \).

(iii) In view of (ii), for the proof of (iii), it is sufficient to show that \( P_\phi \) is not Hilbert Schmidt if \( a + b = 1 \) and \( a > b \). In fact, we show that under the above condition

\[
\sum_{m,n} |T_{mn}|^2 = \infty
\]

Observe that
Since \((a + b)^n = 1\), we have \(\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = 1\). Hence considering this sum as the inner product of two vectors

\[
\begin{bmatrix}
\binom{n}{0} a^n b^0, \\
\binom{n}{1} a^{n-1} b^1, \\
\cdots,
\binom{n}{n} a^0 b^n
\end{bmatrix}
\]

and \(1, 1, \ldots, 1\) terms

We see that, since \(||[1, 1, \ldots, 1]||^2 = (n+1)\), by Cauchy Schwarz inequality

\[
\sum_{k=0}^{n} \left| \binom{n}{k} a^{n-k} b^k \right|^2 \geq \frac{1}{(n+1)}.
\]

Further, we observe that if \(a > b\) then \(\binom{n}{r} a^{n-r} b^r > \binom{n}{n-r} a^r b^{n-r}\) so that over half of the above sum is from terms where \(n-k \geq k\) and so \(||P \phi e_n||^2 \geq 1/2(n+1)\), leading us to \(\sum_{n=0}^{\infty} \left| t_{n,n} \right|^2 = \infty\). This completes the proof of the theorem.

Also, with the help of the same function \(\phi(z) = az + \overline{bz}\), we show that the condition (3.1) of Theorem 1 is not a necessary condition for \(P\phi\) to be a Hilbert Schmidt.

For this take \(a, b \in \mathbb{R}\), ab \(> 0\) and \(|a| + |b| = 1\). Then,

\[
\int_{0}^{2\pi} \frac{dt}{1 - \phi(e^{i\theta})} = \int_{0}^{2\pi} \frac{dt}{(1 - (a-b)^2) \sin^2 t} = \infty.
\]

However, in view of Theorem 3, it follows that \(P\phi\) is Hilbert Schmidt.

In the following theorem we present a sufficient condition on \(f\) in \(H^2\) to ensure that \(f\phi\) is in \(L^2(T)\).

THEOREM 4. Let \(\phi: T \rightarrow \mathbb{D}\) be given by \(\phi(z) = az + \overline{bz}\), \(|a| = |b| = \frac{1}{2}\) and \(f\) in \(H^2\) be given by \(f(z) = \sum_{n=0}^{\infty} a_n z^n\). Further if \(a_n = 0\left(\frac{1}{n}\right)\) with \(a > \frac{3}{4}\), then \(f\phi \in L^2\).

PROOF. First let \(a = b = \frac{1}{2}\), so that \(\phi(e^{i\theta}) = \cos \theta\). Now,

\[
|f(\phi(e^{i\theta}))|^2 \leq \left| \sum_{n=0}^{\infty} \cos n \theta \frac{z^n}{n} \right|^2.
\]

We know that

\[
\frac{1}{(1-z)^{\beta+1}} = \sum_{n=0}^{\infty} \binom{\beta}{n} z^n
\]

and [7]

\[
\binom{\beta}{n} \sim \frac{n^\beta}{\Gamma(\beta+1)}.
\]

Taking \(\beta = -\alpha\) in (4.5) and (4.6), we get

\[
\sum_{n=0}^{\infty} \frac{\cos n \theta}{n^\alpha} \sim \frac{\Gamma(-\alpha+1)}{(1-\cos \theta)(1-\alpha)}.
\]

Thus, to complete the proof, it is sufficient to show that

\[
\int_{0}^{2\pi} (1-\cos \theta)(2\alpha-2) d\theta = \int_{0}^{2\pi} \sin (2\alpha-4) \frac{\theta}{2} d\theta < \infty.
\]

However, this is true because of the condition \(\alpha > \frac{3}{4}\). To dispose of the general case
we observe that if $2a = e^1a$ and $2b = e^1y$ then $\phi(e^{i\theta}) = e^{i(\alpha + \gamma)/2} \cos(\frac{\alpha - \gamma + 2\theta}{2})$ and this leads to similar calculations as above.

Taking cue from the above theorem we next show that $f \phi \in L^2$ for $f \in H^2(\rho(n))$ for a suitable choice of the sequence $\rho(n)$.

THEOREM 5. Let $\phi: T \to \mathbb{D}$ be given by $\phi(e^{i\theta}) = ae^{i\theta} + be^{-i\theta}$, $|a| = |b| = \frac{1}{2}$ and $\rho(n) = n^\beta$. Then,

(i) $f \phi$ is in $L^2$ for all $f$ in $H^2(\rho(n))$ if $\beta > \frac{1}{2}$.

(ii) for each $\beta < \frac{1}{2}$ there is a function $f_\beta$ in $H^2(\rho(n))$ such that $f_\beta \phi$ is not in $L^2$.

PROOF. As in the previous theorem we assume $a = b = \frac{1}{2}$ so that $f(\phi(e^{i\theta})) = f(\cos \theta)$.

Let $f$ in $H^2(\rho)$ be given by $f(z) = \sum_{n=0}^{\infty} a_n z^n$. We have

$$|f(\phi(e^{i\theta}))|^2 = |\sum_{n=0}^{\infty} a_n \cos^n \theta|^2 \leq \left( \sum_{n=0}^{\infty} |a_n|^2 \rho(n) \right) \left( \sum_{n=0}^{\infty} \frac{\cos 2n \theta}{\rho(n)} \right).$$

Now using (4.5) and (4.6) as in the previous theorem it can be shown that

$$\sin^{2\beta-2} \theta \sim \frac{n}{n=0} \frac{1}{f(\theta + 1)} \cos 2n \theta.$$

Thus $f \phi$ is in $L^2$ if $\beta > \frac{1}{2}$.

For the proof of (ii) consider the function

$$f(z) = \frac{1}{(1-z)^{\alpha+1}} = \sum_{n=0}^{\infty} A_n z^n.$$

By (4.6),

$$\Sigma |A_n|^2 \propto (\sum_{n=0}^{\infty} 2^n n^{2a+\beta}).$$

The sum on the right hand converges if $2a + \beta < -1$ i.e. $a < -(\beta+1)/2$. Thus, $f$ is in $H^2(\rho)$, $\rho(n) = n^\beta$ for $a < -(\beta+1)/2$. However,

$$\int_0^{2\pi} |f(\phi(\theta))|^2 \, d\theta = \frac{1}{2^{2a+1}} \int_0^{2\pi} \frac{1}{\sin^{4\alpha+4} \theta} \, d\theta = \infty,$$

if $a = -\frac{3}{4}$. Thus, for given $\beta < \frac{1}{2}$, if we chose $a = -(3+2\beta+2)/8$, $f$ is in $H^2(\rho)$ but $f \phi$ is not in $L^2$.

REMARK. The case $\rho(n) = n^{1/2}$, remains open in the above theorem. However, in the next theorem we prove the same result for a sequence $\rho(n)$ having faster rate of growth than $n^{1/2}$ but with slower rate than $n^{1/2+\epsilon}$ for any $\epsilon > 0$.

THEOREM 6. Let $\phi: T \to \mathbb{D}$ be as in the previous theorem and $\rho(n) = n^{1/2}(\log n)^\beta$. Then,

(i) $f \phi$ is in $L^2$ for all $f$ in $H^2(\rho(n))$ if $\beta > 1$,

(ii) for each $\beta < 0$, there is a function $f_\beta$ in $H^2(\rho(n))$ such that $f_\beta \phi$ is not in $L^2$. 
PROOF (i) Let \( f \), given by \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), be in \( H^2(\rho(n)) \) so that
\[
||f||_\rho = \sum_{n=0}^{\infty} n^{1/2} (\log n)^\beta |a_n|^2 < \infty.
\]
By Cauchy Schwarz inequality,
\[
|f(e^{i\theta})|^2 \leq \sum_{n=0}^{\infty} n^{1/2} (\log n)^\beta |a_n|^2 \left( \sum_{n=0}^{\infty} \frac{\cos 2n\theta}{n^{1/2} (\log n)^\beta} \right).
\]
(4.7)

It is known [7, p. 192] that if
\[
\frac{1}{(1-z)^{a+1}} (\log \frac{1}{1-z})^{\beta} = \sum_{n=0}^{\infty} A_n^{(a,\beta)} z^n
\]
then for \( a > 2, \beta \neq -1,-2,-3, \ldots \), \( \alpha, \beta \in \mathbb{R} \)
\[
A_n^{(a,\beta)} \sim \frac{n^{\alpha}}{\Gamma(a+1)} (\log n)^\beta
\]
so that
\[
A_n^{(-1/2,\beta)} \sim \frac{1}{\sqrt{\pi}} n^{1/2} (\log n)^\beta
\]
i.e.
\[
\frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(\cos^2 \theta)^n}{n^{1/2} (\log n)^\beta} \sim \frac{1}{(1-\cos^2 \theta)^{1/2} (\log \frac{1}{1-\cos^2 \theta})^{1/2}} \left( \frac{\log a}{1-\cos^2 \theta} \right)^{-\beta} = \frac{1}{\sin \theta} (\log \frac{\sin \theta}{\sin \theta})^{-\beta},
\]
(4.9)

Further, because of the inequality \( (2\theta/\pi) < \sin \theta < \theta \), it is sufficient to show the integrability, in an interval \((0,\delta)\), of the function
\[
h(\theta) = \frac{1}{\theta} (\log \frac{1}{\theta})^{-\beta}
\]
Making the substitution \( \log \frac{1}{\theta} = u \), we get
\[
\lim_{\epsilon \to 0^+} \int_{\delta/2}^{\delta} (\log \frac{1}{\theta})^{-\beta} d\theta = \lim_{\epsilon \to 0^+} \frac{(\log \frac{1}{\epsilon})^{1-\beta} - (\log \frac{1}{\delta})^{1-\beta}}{1-\beta}
\]
Thus, the above integral converges if \( \beta > 1 \). This completes the proof of (i).

(ii) Let \( \rho(n) = n^{1/2} (\log n)^{-\beta}, \beta > 0 \). Now consider the function
\[
f(z) = \frac{1}{(1-z)^{1/4}} (\log \frac{1}{1-z})^{\gamma} = \sum_{n=0}^{\infty} A_n^{\gamma} z^n
\]
where \(-1/2 > \gamma < (\beta-1)/2\). We first observe that \( f \) is in \( H^2(\rho(n)) \). In fact, a comparison of \( f \) with (4.8) shows that
\[
A_n^{\gamma} \sim \frac{n^{-3/4}}{\Gamma(1/4)} (\log n)^\gamma
\]
Thus,
\[
|A_n^{\gamma}|^2 \rho(n) \sim \sum_{n=0}^{\infty} (\log n)^{2\gamma-\beta}
\]
and the right hand side series converges because \( \gamma < (\beta-1)/2 \). Now,
where \( \beta = \frac{a}{2} \). The above integral diverges with the integral

\[
\int_0^{\pi/2} \frac{1}{b} \log \left( \frac{1}{2} \right) \sin \theta \, d\theta
\]

because of the condition \( \gamma > \frac{1}{2} \). Thus, we prove that although \( f \in H^2(\rho(n)) \), \( fo \phi \) is not in \( L^2 \).

REMARK. The case \( \rho(n) = n^2(\log n)^8 \), \( 0 \leq \beta < 1 \) remains unsettled.

We conclude this section by showing that \( P_\phi \) is an unbounded operator on \( H^2 \) for \( \phi(e^{it}) = (e^{it} + e^{-it})/2 = \cos t \). This we do by exhibiting a sequence of functions \( f_n \) in \( H^2 \) for which \( \lim_{n \to \infty} \|P_\phi f_n\| = \infty \).

Let \( f_n(z) = \sum_{k=1}^{n} \frac{z^k}{k} \) and \( f(z) = \sum_{k=1}^{\infty} \frac{z^k}{k} = \log \left( \frac{1}{1-z} \right) \) so that

\[
g_n(t) = f_n(\phi(e^{it})) = \sum_{k=1}^{n} \frac{\cos t}{k} \quad \text{and} \quad g(t) = f(\phi(e^{it})) = \sum_{k=1}^{\infty} \frac{\cos t}{k}.
\]

Observe that \( g \) is in \( L^1 \) but is not in \( L^2 \). Let \( a_k \) and \( a_{n_k} \) respectively be the \( k \)th Fourier coefficients of \( g \) and \( g_n \). Since

\[
\lim_{n \to \infty} \int_0^{2\pi} |g_n(t) - g(t)| \, dt = 0
\]

we get

\[
\lim_{n \to \infty} a_{n_k} = a_k. \quad \text{Now,} \quad \lim_{n \to \infty} \|P_\phi f_n\|^2 = \lim_{n \to \infty} \sum_{k=n}^{\infty} |a_k|^2 = \sum_{k=1}^{\infty} |a_k|^2 \quad \frac{1}{2\pi} \int |g(t)|^2 \, dt = \infty.
\]

5. COMPACTNESS OF \( P_\phi \)

In this section we discuss some examples illustrating cases when \( P_\phi \) is compact and when it is not. Let \( \phi_1, \phi_2, \phi_3 : \mathbb{T} \to \mathbb{D} \), be defined by

(i) \( \phi_1(e^{it}) = ae^{-it} \), \( |a| = 1 \), \( 0 \leq t \leq 2\pi \)

(ii) \( \phi_2(e^{it}) = \begin{cases} 
 0, & 0 \leq t < \pi \\
 1, & \pi \leq t \leq 2\pi 
\end{cases} \)

(iii) \( \phi_3(e^{it}) = \begin{cases} 
 e^{it}, & 0 \leq t \leq \pi \\
 ae^{it} + be^{-it}, & \pi \leq t \leq 2\pi \quad a+b = 1 \quad a \neq b, \quad a,b \geq 0 
\end{cases} \)

(1) \( P_{\phi_1} \) is a finite rank, hence a compact, operator. For, if \( f \) in \( H^2 \) is given by

\[
f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{then} \quad (P_{\phi_1} f)(e^{it}) = \sum_{n=0}^{\infty} a_n e^{-int} = a_0.
\]
For composition operators with analytic \( \phi \), Schwartz [8] has shown that if \( C_\phi : H^p(D) \to H^p(D) \) is compact then \( |\phi(e^{it})| < 1 \) a.e. where \( \phi(e^{it}) \) is the radial limit of \( \phi(z) \). We observe in this example that \( P_{\phi_1} \) deviates in behaviour from \( C_\phi \).

(ii) We have shown at the end of Section 3 that \( |\|P_{\phi_2}\| | < \sqrt{2} \). We show here that \( P_{\phi_2} \) is not compact.

By Riemann-Lebesgue Lemma the sequence \( e_n', n = 0,1,2,\ldots \) converges to zero weakly in \( H^2 \). However, \( P_{\phi_2}(e_n') = P(e_n \circ \phi_2) \), does not converge strongly to zero. For, if the Fourier series of \( e_n \circ \phi_2 \) is given by \( (e_n \circ \phi_2)(e^{it}) = \sum_{m=-\infty}^{\infty} a_m e^{imt} \), then by direct computation it can be seen that

\[
a_m = \begin{cases} 
\frac{1}{\pi(n-m)} & \text{if } n-m \text{ is odd} \\
0 & \text{if } n-m \text{ is even} \\
\frac{1}{2} & \text{if } n=m
\end{cases}
\]

and \( |\|P_{\phi_2}(e_n')\| |^2 = \sum_{m=0}^{\infty} |a_m|^2 > \frac{1}{4} \). Thus, \( P_{\phi_2} \) is not compact.

By a similar argument as in (ii) it can be shown that \( P_{\phi_3} \) is bounded but not a compact operator.

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