Internat. J. Math. & Math. Sci.
Vol. 10 No. 2 (1987) 217-226

DIFFERENTIABLE STRUCTURES ON A GENERALIZED PRODUCT OF SPHERES

SAMUEL OMOLOYE AJALA

School of Mathematics
The Institute for Advanced Study
Princeton, New Jersey 08540

and

Mathematics Department
University of Lagos
Akoka-Yaba
Lagos-Nigeria, West Africa

(Received April 21, 1986)

ABSTRACT. In this paper, we give a complete classification of smooth structures on a generalized product of spheres. The result generalizes our result in [1] and R. de Sapio's result in [2].

KEY WORDS AND PHRASES. Differential structures, product of spheres.
1980 AMS SUBJECT CLASSIFICATION CODE. 57R55

1. INTRODUCTION

In [2] a classification of smooth structures on product of spheres of the form $S^k \times S^p$ where $2 \leq k \leq p$, $k+p \geq 6$ was given by R. de Sapio and in [1] this author extended R. de Sapio's result to smooth structures on $S^p \times S^q \times S^r$ where $2 \leq p \leq q \leq r$. The next question is, how many differentiable structures are there in any arbitrary product of ordinary spheres. In this paper, we give a classification under the relation of orientation preserving diffeomorphism of all differentiable structures of spheres $S^{k_1} \times S^{k_2} \times \ldots \times S^{k_r}$ where $2 \leq k_1 < k_2 \leq \ldots \leq k_r$. S^n denotes the unit n-sphere with the usual differential structure in the Euclidean (n+1)-space R^{n+1}. θ^n denotes the group of h-cobordism classes of homotopy n-sphere under the connected sum operation. Σ^n will denote an homotopy n-sphere. H(p,k) denotes the subset of θ^p which consists of those homotopy p-sphere Σ^p such that $\Sigma^p \times S^k$ is diffeomorphic to $S^p \times S^k$. By [2], H(p,k) is a subgroup of θ^p and it is not always zero and in fact in [1], we showed that if $k \geq p-3$, then $H(p,k) = \theta^p$.

By Hauptremutung [3], piecewise linear homoemorphism will be replaced by homeomorphism. Consider two manifolds $S^{k_1} \times S^{k_2} \times S^{k_3} \times S^{k_4}$ and $\Sigma^{k_2+k_4} \times S^{k_1} \times S^{k_3}$, we shall denote the connected sum of the two manifolds along a k_2+k_4-1 cycle by

$(S^{k_1}\times S^{k_2}\times S^{k_3}\times S^{k_4}) \#_{k_2+k_4} (\Sigma^{k_2+k_4}\times S^{k_1}\times S^{k_3})$. This is geometrically achieved by removing $\mathrm{Int}(D^{k_2+k_4})\times S^{k_1}\times S^{k_3}$ from both manifolds and then identify their common boundary. Thus nothing else other than taking the usual connected sum of $S^{k_2}\times S^{k_4}$ and $\Sigma^{k_2+k_4}$ by removing the interior of an embedded disc $D^{k_2+k_4}$ from each manifold and identify the manifolds along their common boundary $S^{k_2+k_4-1}$ to obtain $S^{k_2}\times S^{k_4}\#\Sigma^{k_2+k_4}$. This is a well-defined operation. We then take the cartesian product with $S^{k_1}\times S^{k_3}$ to have $S^{k_1}\times S^{k_3}\times(S^{k_2}\times S^{k_4}\#\Sigma^{k_2+k_4}) =$ $S^{k_1}\times S^{k_3}\times S^{k_2}\times S^{k_4} \#_{k_2+k_4} \Sigma^{k_2+k_4}\times S^{k_1}\times S^{k_3}$. But $S^{k_1}\times S^{k_3}\times S^{k_2}\times S^{k_4}$ is diffeomorphic to $S^{k_1}\times S^{k_2}\times S^{k_3}\times S^{k_4}$ then $(S^{k_1}\times S^{k_2}\times S^{k_3}\times S^{k_4}) \#_{k_2+k_4} \Sigma^{k_2+k_4}\times S^{k_1}\times S^{k_3} = S^{1}\times S^{k_3}\times(S^{k_2}\times S^{k_4}\#\Sigma^{k_2+k_4})$.

We will then prove the following.

CLASSIFICATION THEOREM If M^n is a smooth manifold homeomorphic to $S^{k_1}\times S^{k_2}\times\ldots\times S^{k_r}$ where $2\le k_1<\ldots<K_{r-1}$ and $k_4-3\le k_{r-1}\le k_r$ and $n=k_1+k_2+\ldots+k_r$ then there exists homotopy spheres $\Sigma^{k_1+k_2+k_3},\ldots\Sigma^{n-k_r},\ldots,\Sigma^{n-k_1},\Sigma^n$ such that M^n is diffeomorphic to

$$\Big[(S^{k_1}\times\ldots\times S^{k_r})\#_{k_1+k_r}(\Sigma^{k_1+k_r}\times S^{k_2}\times\ldots\times S^{k_{r-1}})\#_{k_2+k_r}(\Sigma^{k_2+k_r}\times S^{k_1}\times S^{k_3}\times\ldots\times S^{k_{r-1}})$$
$$\#\ldots\#_{k_1+k_2+k_3}(\Sigma^{k_1+k_2+k_3}\times S^{k_4}\times\ldots\times S^{k_r})\#\ldots\#_{n-k_r}(\Sigma^{n-k_r}\times S^{k_r})$$
$$\#\ldots\#_{n-k_1}(\Sigma^{n-k_1}\times S^{k_1})\Big]\#\Sigma^n .$$

We shall use the above classification theorem to give the number of differentiable structures on $S^{k_1}\times S^{k_2}\times\ldots\times S^{k_r}$. We shall lastly compute the number of structures in some simple cases.

2. PRELIMINARY RESULTS

We shall apply obstruction theory of Munkres [4]. Let M and N be smooth n-manifolds and L a closed subset of M when triangulated. A homeomorphism $f:M\to N$ is a diffeomorphism modulo L if $f|(M-L)$ is a diffeomorphism and each simplex α of L has a neighborhood V, such that f is smooth on V-L near α . By [4], if two n-manifolds M and N are combinatorially equivalent then M is diffeomorphic modulo an (n-1)-skeleton L onto N .

If $f:M^n\to N^n$ is a diffeomorphism modulo m-skeleton $m<n$ then Munkres showed that the obstruction to deforming f to a diffeomorphism $g:M^n\to N^n$ modulo (m-1)-skeleton is an element $\lambda_m(f)\in H_m(M,\Gamma^{n-m})=\Gamma^{n-m}$. Where Γ^{n-m} is a group of diffeomorphisms of S^{n-m-1} modulo the diffeomorphisms that are extendable to diffeomorphisms of D^{n-m} . We call g the smoothing of f . If $\lambda_m(f)=0$ then g exists. Recall that in ([1], Lemma 2.1.1) we proved that if $q\ge p$ then $\Sigma^p\times S^q$ is diffeomorphic to $S^p\times S^q$ for any homotopy sphere Σ^p . In Remark (1) following that lemma, we showed further that even when $p-3\le q$ the result is still true.

LEMMA 2.1 Suppose $f:M^n\to S^{k_1}\times S^{k_2}\times\ldots\times S^{k_r}$ is a piecewise linear homeomorphism which is a diffeomorphism modulo $(n-k_i)$-skeleton $1\le i\le r$, then there exists an homotopy sphere Σ^{k_i} and a piecewise linear homeomorphism

$$h : M^n \to S^{k_1} \times S^{k_2} \times \ldots \times S^{k_{i-1}} \times \Sigma^{k_i} \times S^{k_{i+1}} \times \ldots \times S^{k_r}$$

which is a diffeomorphism modulo $(n-k_i-1)$ skeleton.

PROOF. Since $f : M^n \to S^{k_1} \times \ldots \times S^{k_r}$ is a diffeomorphism modulo $(n-k_i)$-skeleton then by Munkres [4], the obstruction to deforming f to a diffeomorphism modulo $(n-k_i-1)$-skeleton is an element $\lambda_{k_i}(f) \in H_{n-k_i}(M^n, \Gamma^{k_i}) = \Gamma^{k_i}$. Let $[\psi] = \lambda_{k_i}(f) \in \Gamma^{k_i}$ where $\psi : S^{k_i-1} \to S^{k_i-1}$ is a diffeomorphism. We define $\Sigma^{k_i} = D_1^{k_i} \cup_{\psi} D_2^{k_i}$ and a homeomorphism $j : S^{k_i} \to \Sigma^{k_i}$ where we have $S^{k_i} = D_1^{k_i} \cup_{id} D_2^{k_i}$ and so j is identity map on $\mathrm{Int}(D_1^{k_i})$ and radial extension of ψ^{-1} on $D_2^{k_i}$. So j is a piecewise linear homeomorphism by the definition and the obstruction to deforming j to a diffeomorphism is $[\psi^{-1}] = -\lambda_{k_i}(f)$. So consider the map

$$id \times j : (S^{k_1} \times \ldots \times S^{k_i-1} \times S^{k_i+1} \times \ldots \times S^{k_r}) \times S^{k_i} \to (S^{k_1} \times \ldots \times S^{k_i-1} \times S^{k_i+1} \times \ldots \times S^{k_r}) \times \Sigma^{k_i}.$$

The map is a piecewise linear homeomorphism and the obstruction to deforming it to a diffeomorphism is $[\psi^{-1}] = -\lambda_{k_i}(f)$. Notice that the manifold $(S^{k_1} \times \ldots \times S^{k_i-1} \times \ldots \times S^{k_r}) \times S^{k_i} = S^{k_1} \times \ldots \times S^{k_{i-1}} \times S^{k_i} \times S^{k_{i+1}} \times \ldots \times S^{k_r}$ and $(S^{k_1} \times \ldots \times S^{k_{i-1}} \times S^{k_{i+1}} \times \ldots \times S^{k_r}) \times \Sigma^{k_i} = S^{k_1} \times \ldots \times S^{k_{i-1}} \times \Sigma^{k_i} \times S^{k_{i+1}} \times \ldots \times S^{k_r}$.

Consider the composite $(id \times j) \cdot f = h$, the obstruction to deforming h to a diffeomorphism modulo $(n-k_i-1)$ skeleton is $\lambda_{k_i}(h) = \lambda_{k_i}((id \times j) \cdot f) = \lambda_{k_i}(id \times j) + \lambda_{k_i}(f) = -\lambda_{k_i}(f) + \lambda_{k_i}(f) = 0$ hence $h : M^n \to S^{k_1} \times \ldots \times S^{k_{i-1}} \times \Sigma^{k_i} \times S^{k_{i+1}} \times \ldots \times S^{k_r}$ is a diffeomorphism modulo $(n-k_i-1)$ skeleton. Hence the lemma.

LEMMA 2.2 Let $f : M^n \to S^{k_1} \times \ldots \times S^{k_r}$ be a diffeomorphism modulo $n-(k_i+k_j)$ skeleton $1 \leq i,j \leq r$ then there exists homotopy sphere $\Sigma^{k_i+k_j}$ and a piecewise linear homeomorphism

$$f : M^n \to (S^{k_1} \times \ldots \times S^{k_r}) \underset{k_i+k_j}{\#} (\Sigma^{k_i+k_j} \times S^{k_1} \times \ldots \times S^{k_{i-1}} \times S^{k_{i+1}} \times \ldots \times S^{k_{j-1}} \times S^{k_{j+1}} \times \ldots \times S^{k_r})$$

which is a diffeomorphism modulo $n-(k_i+k_j)-1$ skeleton.

PROOF. Since f is a diffeomorphism modulo $n-(k_i+k_j)$ skeleton, it follows that the obstruction to deforming f to a diffeomorphism modulo $n-(k_i+k_j)-1$ skeleton is

$$\lambda(f) \in H_{n-(k_i+k_j)}(M^n, \Gamma^{k_i+k_j}) = \Gamma^{k_i+k_j}. \quad \text{Let } [\phi] = \lambda(f) \in \Gamma^{k_i+k_j}$$

where $\phi : S^{k_i+k_j-1} \to S^{k_i+k_j-1}$ is a diffeomorphism and $\Sigma^{k_i+k_j} = D^{k_i+k_j} \cup_{\phi} D^{k_i+k_j}$. We define $j : S^{k_i} \times S^{k_j} \to S^{k_i} \times S^{k_j} \# \Sigma^{k_i+k_j}$ to be identity map on $S^{k_i} \times S^{k_j} - \mathrm{Int}(D^{k_i+k_j})$ and radial extension of ϕ^{-1} on $\mathrm{Int}(D^{k_i+k_j})$ hence j is a piecewise linear homeomorphism and the obstruction to deforming j to a diffeomorphism is $[\phi^{-1}] = -\lambda(f)$. Then consider

$$j \times id : S^{k_i} \times S^{k_j} \times (S^{k_1} \times \ldots \times S^{k_{i-1}} \times S^{k_{i+1}} \times \ldots \times S^{k_{j-1}} \times S^{k_{j+1}} \times \ldots \times S^{k_r}) \longrightarrow$$
$$(S^{k_i} \times S^{k_j} \# \Sigma^{k_i+k_j}) \times (S^{k_1} \times \ldots \times S^{k_{i-1}} \times S^{k_{i+1}} \times \ldots \times S^{k_{j-1}} \times S^{k_{j+1}} \times \ldots \times S^{k_r}) .$$

Note that

$$S^{k_i} \times S^{k_j} \times (S^{k_1} \times \ldots \times S^{k_{i-1}} \times S^{k_{i+1}} \times \ldots \times S^{k_{j-1}} \times S^{k_{j+1}} \times \ldots \times S^{k_r}) =$$
$$(S^{k_1} \times S^{k_2} \times \ldots \times S^{k_i} \times \ldots \times S^{k_j} \times \ldots \times S^{k_r})$$

and

$$(S^{k_i} \times S^{k_j} \# \Sigma^{k_i+k_j}) \times (S^{k_1} \times \ldots \times S^{k_{i-1}} \times S^{k_{i+1}} \times \ldots \times S^{k_r}) =$$
$$(S^{k_1} \times \ldots \times S^{k_r}) \underset{k_i+k_j}{\#} (\Sigma^{k_i+k_j} \times S^{k_1} \times \ldots \times S^{k_{i-1}} \times S^{k_{i+1}} \times \ldots \times S^{k_{j-1}} \times S^{k_{j+1}} \times \ldots \times S^{k_r})$$

hence the above map is

$$id \times j : (S^{k_1} \times \ldots \times S^{k_r}) \longrightarrow$$
$$(S^{k_1} \times \ldots \times S^{k_r}) \underset{k_i+k_j}{\#} (\Sigma^{k_i+k_j} \times S^{k_1} \times \ldots \times S^{k_{i-1}} \times S^{k_{i+1}} \times \ldots \times S^{k_{j-1}} \times S^{k_{j+1}} \times \ldots \times S^{k_r})$$

which is piecewise linear and its obstruction to a diffeomorphism is $-\lambda(f)$ hence the obstruction to deforming the composite $(j \times id)\cdot f$ to a diffeomorphism modulo $n-(k_i+k_j)-1$ skeleton is zero. Hence if $f' = (j \times id)\cdot f$ then $f' : M^n \longrightarrow (S^{k_1} \times \ldots \times S^{k_r}) \underset{k_i+k_j}{\#} (\Sigma^{k_i+k_j} \times S^{k_1} \times \ldots \times S^{k_{i-1}} \times S^{k_{i+1}} \times \ldots \times S^{k_{j-1}} \times S^{k_{j+1}} \times \ldots \times S^{k_r})$ is a diffeomodphism modulo $n-(k_i+k_j)-1$ skeleton.

3. CLASSIFICATION

THEOREM 3.1 If M^n is a smooth manifold homeomorphic to $S^{k_1} \times S^{k_2} \times \ldots \times S^{k_r}$ then there exists homotopy spheres, $\Sigma^{k_1+k_r}, \Sigma^{k_2+k_r}, \ldots, \Sigma^{k_1+k_2+k_3}, \ldots, \ldots \Sigma^{n-k_r}, \ldots \Sigma^{n-k_1}$, and Σ^n such that M^n is diffeomorphic to

$$(S^{k_1} \times \ldots \times S^{k_r}) \underset{k_1+k_r}{\#} (\Sigma^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}}) \underset{k_2+k_r}{\#} (\Sigma^{k_2+k_r} \times S^{k_1} \times S^{k_3} \times \ldots \times S^{k_{r-1}})$$
$$\# \ldots \underset{k_1+k_2+k_3}{\#} (\Sigma^{k_1+k_2+k_3} \times S^{k_4} \times \ldots \times S^{k_r})$$
$$\ldots \underset{n-k_r}{\#} (\Sigma^{n-k_r} \times S^{k_r}) \# \ldots \underset{n-k_1}{\#} (\Sigma^{n-k_1} \times S^{k_1}) \# \Sigma^n$$

where $2 \le k_1 < k_2 < \ldots < k_r$, $k_r-3 \le k_{r-1} \le k_r$ and $n = k_1 + k_2 + \ldots + k_r$.

PROOF. Suppose $M^n \xrightarrow{h} S^{k_1} \times \ldots \times S^{k_r}$ is the homeomorphism. By Munkres theory [4], h is a diffeomorphism modulo (n-1) skeleton. Since the first non-zero homology appears in dimension $n-k_1$, (apart from the zero dimension) it then means that h is a diffeomorphism modulo $(n-k_1)$ skeleton. The obstruction to deforming h to a diffeomorphism modulo $(n-k_1-1)$ skeleton is $\lambda(h) \in H_{n-k_1}(M^n, \Gamma^{k_1}) = \Gamma^{k_1}$. By Lemma 2.1, there exists a piecewise linear homeomorphism h' and a homotopy sphere Σ^{k_1} such that $h' : M^n \to \Sigma^{k_1} \times S^{k_2} \times \ldots \times S^{k_r}$ which is a diffeomorphism modulo $(n-k_1-1)$

skeleton. In [1] Lemma 2.1.1 it was proved that $\Sigma^{k_1} \times S^{k_2}$ is diffeomorphic to $S^{k_1} \times S^{k_2}$ since $k_1 < k_2$. It then follows that $\Sigma^{k_1} \times S^{k_2} \times \ldots \times S^{k_r}$ is diffeomorphic to $S^{k_1} \times \ldots \times S^{k_r}$ hence $h' : M^n \to S^{k_1} \times \ldots \times S^{k_r}$ is a diffeomorphism modulo $(n-k_1-1)$ skeleton. There is no other obstruction to deforming h' to a diffeomorphism until the $(n-k_2-1)$-skeleton. This is because

$$H_i(M^n, Z) = 0$$

for $n-k_2+1 < i < n-k_1$. So we can assume that h' is a diffeomorphism modulo $(n-k_2)$ skeleton. The obstruction to deforming h' to a diffeomorphism modulo $(n-k_2-1)$ skeleton is $\lambda(h') \in H_{n-k_2}(M^n, \Gamma^{k_2}) = \Gamma^{k_2}$. Again by Lemma 2.1, there exists a homotopy sphere Σ^{k_2} and a piecewise linear homeomorphism $h'' : M^n \to S^{k_1} \times \Sigma^{k_2} \times S^{k_3} \times \ldots \times S^{k_r}$ which is a diffeomorphism modulo $(n-k_2-1)$ skeleton. By the same argument as above since $k_2 < k_3$ we see that $\Sigma^{k_2} \times S^{k_3}$ is diffeomorphic to $S^{k_2} \times S^{k_3}$ hence $S^{k_1} \times \Sigma^{k_2} \times S^{k_3} \times \ldots \times S^{k_r}$ is diffeomorphic to $S^{k_1} \times S^{k_2} \times S^{k_3} \times \ldots \times S^{k_r}$. This shows that $h'' : M^n \to S^{k_1} \times \ldots \times S^{k_r}$ is a diffeomorphism modulo $(n-k_2-1)$-skeleton. By the same argument since M^n has no homology between $n-k_3-1$ and $n-k_2-1$ we can assume that h'' is a diffeomorphism modulo $(n-k_3)$-skeleton. Proceeding this way using the same argument we can construct a homeomorphism say $h'' : M^n \to S^{k_1} \times \ldots \times S^{k_r}$ which is a diffeomorphism modulo $(n-k_r)$-skeleton. However, to deform h'' to a diffeomorphism modulo $(n-k_r-1)$-skeleton, there is an obstruction $\lambda(h'') \in H_{n-k_r}(M^n, \Gamma^{k_r}) = \Gamma^{k_r}$. It also follows by Lemma 2.1 that there exists a homotopy sphere Σ^{k_r} and a piecewise linear homeomorphism $f : M^n \to S^{k_1} \times \ldots \times S^{k_{r-1}} \times \Sigma^{k_r}$ which is a diffeomorphism modulo $(n-k_r-1)$-skeleton. Now in Remark (1) of [1] it was shown that even when $p-3 \leq r$, $S^r \times \Sigma^p$ is diffeomorphic to $S^r \times S^p$ and so by our assumption that $k_r - 3 \leq k_{r-1} \leq k_r$ it follows that $S^{k_{r-1}} \times \Sigma^{k_r}$ is diffeomorphic to $S^{k_{r-1}} \times S^{k_r}$. Hence $S^{k_1} \times \ldots \times S^{k_{r-1}} \times \Sigma^{k_r}$ is diffeomorphic to $S^{k_1} \times \ldots \times S^{k_{r-1}} \times S^{k_r}$ and so $f : M^n \to S^{k_1} \times \ldots \times S^{k_r}$ is a diffeomorphism modulo $(n-k_r-1)$-skeleton. The next obstruction to deforming f to a diffeomorphism is on $n-(k_r+k_1)-1$ skeleton and it is $\lambda(f) \in H_{n-(k_1+k_r)}(M^n, \Gamma^{k_1+k_r}) = \Gamma^{k_1+k_r}$. By Lemma 2.2, there exists a piecewise linear homeomorphism

$$f' : M^n \to (S^{k_1} \times S^{k_2} \times \ldots \times S^{k_r}) \underset{k_1+k_r}{\#} (\Sigma^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}})$$

which is a diffeomorphism modulo $n-(k_1+k_r)-1$ skeleton for some homotopy sphere $\Sigma^{k_1+k_r}$ defined using $\lambda(f) \in \Gamma^{k_1+k_r}$. At this point, we want to remark that if $k_1+k_r-3 \leq \max(k_2, \ldots, k_{r-1})$ and suppose $k_j = \max(k_2, \ldots, k_{r-1})$ then it follows from Remark (1) of [1] since $k_1+k_r-3 \leq k_j$, that $\Sigma^{k_1+k_r} \times S^{k_j}$ is diffeomorphic to $S^{k_1+k_r} \times S^{k_j}$ and so $\Sigma^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}}$ is diffeomorphic to $S^{k_1+k_r} \times S^{k_1} \times \ldots \times S^{k_{r-1}}$. This then implies that $(S^{k_1} \times \ldots \times S^{k_r}) \underset{k_1+k_r}{\#} (\Sigma^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}})$ is diffeomorphic to $(S^{k_1} \times S^{k_2} \times \ldots \times S^{k_r}) \underset{k_1+k_r}{\#} (S^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}})$ and this is diffeomorphic to $S^{k_1} \times S^{k_2} \times \ldots \times S^{k_r}$ because $S^{k_1} \times S^{k_r} \# S^{k_1+k_r} = S^{k_1} \times S^{k_r}$. So this means that the factor

$\Sigma^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}}$ will disappear in the above sum if we have the condition $k_1+k_r-3 \le \max(k_2,\ldots,k_{r-1})$.

Anyway, we have $f' : M^n \to (S^{k_1}\times\ldots\times S^{k_r}) \#_{k_1+k_r} (\Sigma^{k_1+k_r}\times S^{k_2}\times\ldots\times S^{k_{r-1}})$ which is a diffeomorphism modulo $n-(k_1+k_r)-1$ skeleton. Since $H_i(M^n,Z) = 0$ for $n-(k_2+k_r) < i \le n-(k_1+k_r)-1$ then there is no obstruction to deforming f' to a diffeomorphism modulo $n-(k_2+k_r)$ skeleton and the obstruction to deforming f' to a diffeomorphism modulo $n-(k_2+k_r)-1$ skeleton is $\lambda(f') \in H_{n-(k_2+k_r)}(M^n,\Gamma^{k_2+k_r}) = \Gamma^{k_2+k_r}$. Using the same technique as in the proof of Lemma 2.2 it can be easily shown that there exists an homotopy sphere $\Sigma^{k_2+k_r} = D^{k_2+k_r} \cup_{\psi} D^{k_2+k_r}$ where $\psi = \lambda(f') \in \Gamma^{k_2+k_r}$ and $\psi : S^{k_2+k_r-1} \to S^{k_2+k_r-1}$ is a diffeomorphism and a piecewise linear homeomorphism

$$j : S^{k_1}\times\ldots\times S^{k_r} \to (S^{k_1}\times\ldots\times S^{k_r}) \#_{k_2+k_r} (\Sigma^{k_2+k_r}\times S^{k_1}\times\ldots\times S^{k_{r-1}})$$

where obstruction to a diffeomorphism is $-\lambda(f')$. We now define a map

$$j' : (S^{k_1}\times\ldots\times S^{k_r}) \#_{k_1+k_r} (\Sigma^{k_1+k_r}\times S^{k_2}\times\ldots\times S^{k_{r-1}})$$

$$\to (S^{k_1}\times\ldots\times S^{k_r}) \#_{k_2+k_r} (\Sigma^{k_2+k_r}\times S^{k_1}\times S^{k_3}\times\ldots\times S^{k_{r-1}}) \#_{k_1+k_r} (\Sigma^{k_1+k_r}\times S^{k_2}\times\ldots\times S^{k_{r-1}})$$

where $j' = j$ on $(S^{k_1}\times\ldots\times S^{k_r}) - \mathrm{Ind}(D^{k_1+k_r}) \times S^{k_2}\times\ldots\times S^{k_{r-1}}$ and identity on $\Sigma^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}} - \mathrm{Int}(D^{k_1+k_r}) \times S^{k_2} \times \ldots \times S^{k_{r-1}}$.

Clearly j' is piecewise linear and its obstruction to a diffeomorphism is $-\lambda(f')$ hence the obstruction to deforming the composite $g = j' \cdot f'$ where $g:M^n \to (S^{k_1}\times\ldots\times S^{k_r}) \#_{k_1+k_r} (\Sigma^{k_1+k_r}\times S^{k_2}\times\ldots\times S^{k_{r-1}}) \#_{k_2+k_r} (\Sigma^{k_2+k_r}\times S^{k_1}\times\ldots\times S^{k_{r-1}})$ is $\lambda(j'\cdot f') = \lambda(j') + \lambda(f') = 0$. Hence $g = j' \cdot f'$ is a diffeomorphism modulo $n-(k_2+k_r)-1$ skeleton. Proceeding in this way, we see that the next obstruction to a diffeomorphism will be on $(n-(k_3+k_r))$-skeleton. Using the above technique continuously, we can construct a piecewise linear homeomorphism

$$g':M^n \to (S^{k_1}\times\ldots\times S^{k_r}) \#_{k_1+k_r} (\Sigma^{k_1+k_r}\times S^{k_2}\times\ldots\times S^{k_{r-1}})$$

$$\#_{k_2+k_r} (\Sigma^{k_2+k_r}\times S^{k_1}\times S^{k_3}\times\ldots\times S^{k_{r-1}}) \# \ldots \#_{\substack{k_{i_1}+\ldots+k_{i_\ell} \\ j_p's \ne i_\ell}} (\Sigma^{k_{i_1}+\ldots+k_{i_2}} \times S^{k_{j_1}}\times\ldots\times S^{k_{j_p}}) ,$$

which is a diffeomorphism modulo $n-(k_{r-1}+\ldots+k_1) = k_r$ skeleton. The obstruction to extending g' to a diffeomorphism modulo (k_r-1) skeleton is $\lambda(g') \in H_{k_r}(M^n,\Gamma^{n-k_r}) = \Gamma^{n-k_r}$. By using the same technique as in the proof of Lemma 2.1, there exists a piecewise linear homeomorphism j and homotopy sphere Σ^{n-k_r} such that

$$j : S^{k_1}\times\ldots\times S^{k_r} \to (S^{k_1}\times\ldots\times S^{k_r}) \#_{n-k_r} (\Sigma^{n-k_r}\times S^{k_r})$$

has an obstruction to a diffeomorphism to be $-\lambda(g')$. From this we define the map,

$$j' : (S^{k_1} \times \ldots \times S^{k_r}) \underset{k_1+k_r}{\#} (\Sigma^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}}) \underset{k_2+k_r}{\#} (\Sigma^{k_2+k_r} \times S^{k_1} \times S^{k_3} \times \ldots \times S^{k_{r-1}}) \# \ldots \underset{\substack{k_{i_1}+\ldots+k_{i_\ell} \\ j_p \neq i_1, i_2 \ldots i_\ell}}{\#} (\Sigma^{k_{i_1}+\ldots+k_{i_\ell}} \times S^{k_{j_1}} \times \ldots \times S^{k_{j_p}})$$

$$\longrightarrow (S^{k_1} \times \ldots \times S^{k_r}) \underset{n-k_r}{\#} (\Sigma^{n-k_r} \times S^{k_r}) \underset{k_1+k_r}{\#} (\Sigma^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}}) \# \ldots \underset{k_{i_1}+\ldots+k_{i_\ell}}{\#} (\Sigma^{k_{i_1}+\ldots+k_{i_\ell}} \times S^{k_{j_1}} \times \ldots \times S^{k_{j_p}})$$

where $j' = j$ on $(S^{k_1} \times \ldots \times S^{k_r}) - (\mathrm{Int}(D^{k_1+k_r}) \times S^{k_2} \times \ldots \times S^{k_{r-1}})$ and identity elsewhere. It is easily seen that j' is piecewise linear homeomorphism and the obstruction to deforming the composite $j' \cdot g'$ to a diffeomorphism is zero. Hence the map $h' = j' \cdot g'$ where

$$h' : M^n \to (S^{k_1} \times \ldots \times S^{k_r}) \underset{k_1+k_r}{\#} (\Sigma^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}}) \underset{k_2+k_r}{\#} (\Sigma^{k_2+k_r} \times S^{k_1} \times S^{k_3} \times \ldots \times S^{k_{r-1}}) \# \ldots \underset{n-k_r}{\#} (\Sigma^{n-k_r} \times S^{k_r})$$

is a diffeomorphism modulo (k_r-1) skeleton. However, since $H_i(M^n, Z) = 0$ for $k_{r-1} < i < k_r-1$, there is no more obstruction to deforming h' to a diffeomorphism modulo k_{r-1}-skeleton. To deform h' to a diffeomorphism modulo $(k_{r-1}-1)$ skeleton, there is an obstruction and this equals $\lambda(h') \in H_{k_{r-1}}(M^n, \Gamma^{n-k_{r-1}}) = \Gamma^{n-k_{r-1}}$. Applying the above technique again, we can get an homotopy sphere $\Sigma^{n-k_{r-1}}$ and a piecewise linear homeomorphism

$$h'' : M^n \to (S^{k_1} \times \ldots \times S^{k_r}) \underset{k_1+k_r}{\#} (\Sigma^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}}) \underset{k_2+k_r}{\#} (\Sigma^{k_2+k_r} \times S^{k_1} \times S^{k_3} \times \ldots \times S^{k_{r-1}}) \# \ldots \underset{k_1+k_2+k_3}{\#} (\Sigma^{k_1+k_2+k_3} \times S^{k_4} \times \ldots \times S^{k_r}) \ldots \underset{n-k_r}{\#} (\Sigma^{n-k_r} \times S^{k_r}) \underset{n-k_{r-1}}{\#} (\Sigma^{n-k_{r-1}} \times S^{k_{r-1}})$$

which is a diffeomorphism modulo $k_{r-1}-1$ skeleton. The next obstruction will be on $k_{r-2}-1$ skeleton. Proceeding this way gradually down the remaining skeleton, we can construct a map

$$g : M^n \to (S^{k_1} \times \ldots \times S^{k_r}) \underset{k_1+k_r}{\#} (\Sigma^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}}) \underset{k_2+k_r}{\#} (\Sigma^{k_2+k_r} \times S^{k_1} \times S^{k_3} \times \ldots \times S^{k_{r-1}}) \# \ldots \underset{k_1+k_2+k_3}{\#} (\Sigma^{k_1+k_2+k_3} \times S^{k_4} \times \ldots \times S^{k_r}) \ldots \underset{n-k_r}{\#} (\Sigma^{n-k_r} \times S^{k_r}) \underset{n-k_{r-1}}{\#} (\Sigma^{n-k_{r-1}} \times S^{k_{r-1}}) \# \ldots \underset{n-k_1}{\#} (\Sigma^{n-k_1} \times S^{k_1})$$

which is a diffeomorphism modulo k_1-skeleton. Since $H_i(M^n, Z) = 0$ for $0 < i < k$, then g is a diffeomorphism modulo one point. It therefore follows that there exist an homotopy sphere Σ^n such that M^n is diffeomorphis to

$$\Big[(S^{k_1}\times\ldots\times S^{k_r} \mathop{\#}_{k_1+k_r} (\Sigma^{k_1+k_r}\times S^{k_2}\times\ldots\times S^{k_{r-1}}) \mathop{\#}_{k_2+k_r} (\Sigma^{k_2+k_r}\times S^{k_1}\times S^{k_3}\times\ldots\times S^{k_{r-1}})$$
$$\# \ldots \mathop{\#}_{k_1+k_2+k_3} (\Sigma^{k_1+k_2+k_3}\times S^{k_4}\times\ldots\times S^{k_r}) \# \ldots \mathop{\#}_{n-k_r} (\Sigma^{n-k_r}\times S^{k_r})$$
$$\# \ldots \mathop{\#}_{n-k_1} (\Sigma^{n-k_1}\times S^{k_1})\Big] \# \Sigma^n \quad .$$

Hence the theorem.

Recall that $H(p,k)$ denotes the subgroup of θ^p consisting of homotopy p-spheres Σ^p such that $\Sigma^p \times S^p$ is diffeomorphic to $S^p \times S^k$.

THEOREM 3.2 The number of differentiable structures on $S^{k_1} \times \ldots \times S^{k_r}$ where $2 \leq k_1 < k_2 < \ldots < k_{r-1}$ and $k_r - 3 \leq k_{r-1} \leq k_r$ equals the order of the group

$$\frac{\theta^{k_1+k_r}}{H((k_1+k_r),(k_3,\ldots,k_{r-1}))} \times \frac{\theta^{k_2+k_r}}{H(k_2+k_r,(k_1,k_3,\ldots,k_{r-1}))} \times \ldots \times$$
$$\frac{\theta^{k_1+k_2+k_3}}{H((k_1+k_2+k_3,(k_4,\ldots,k_r))} \times \ldots \times \frac{\theta^{n-k_r}}{H(n-k_r,k_r)} \times \ldots \times \frac{\theta^{n-k_1}}{H(n-k_1,k_1)} \times \theta^n \quad .$$

PROOF Let $(0(k_1+k_r),0(k_2+k_r),\ldots,0(k_1+k_2+k_3),\ldots,0(n-k_r),\ldots,0(n-k_1),0(n))$ represent the trivial elements of $\theta^{k_1+k_r}, \theta^{k_2+k_r},\ldots,\theta^{k_1+k_2+k_3},\ldots,\theta^{n-k_1},\theta^n$, then we define a map

$$\beta:(\theta^{k_1+k_r}\times\theta^{k_2+k_r}\times\ldots\times\theta^{k_1+k_2+k_3}\times\ldots\times\theta^{n-k_r}\times\ldots\times\theta^{n-k_1}\times\theta^n,0(k_1+k_r),\ldots,0(n-k_1),0(n))$$
$$\longrightarrow (\text{Structures on } S^{k_1}\times\ldots\times S^{k_r},0)$$

where 0 represents the usual structures on $S^{k_1}\times\ldots\times S^{k_r}$. If $\Sigma^{k_1+k_r} \in \theta^{k_1+k_r}$, $\ldots, \Sigma^{k_1+k_2+k_3} \in \theta^{k_1+k_2+k_3},\ldots,\Sigma^{n-k_r} \in \theta^{n-k_r},\ldots,\Sigma^{n-k_1} \in \theta^{n-k_1}$ and $\Sigma^n \in \theta^n$ then we define

$$\beta(\Sigma^{k_1+k_r},\Sigma^{k_2+k_r},\ldots,\Sigma^{k_1+k_2+k_3},\ldots,\Sigma^{n-k_r},\ldots,\Sigma^{n-k_1},\Sigma^n) =$$
$$\Big[(S^{k_1}\times\ldots\times S^{k_r}) \mathop{\#}_{k_1+k_r} (\Sigma^{k_1+k_r}\times S^{k_2}\times\ldots\times S^{k_{r-1}}) \mathop{\#}_{k_2+k_r} (\Sigma^{k_2+k_r}\times S^{k_1}\times S^{k_3}\times\ldots\times S^{k_{r-1}})$$
$$\# \ldots \mathop{\#}_{k_1+k_2+k_3} (\Sigma^{k_1+k_2+k_3}\times S^{k_4}\times\ldots\times S^{k_r}) \# \ldots \mathop{\#}_{n-k_r} (\Sigma^{n-k_r}\times S^{k_r})$$
$$\# \ldots \mathop{\#}_{n-k_1} (\Sigma^{n-k_1}\times S^{k_1})\Big] \# \Sigma^n \quad .$$

β is well-defined because if

$$\Sigma_1^{k_1+k_r}, \Sigma_2^{k_1+k_r} \in \theta^{k_1+k_r}; \Sigma_1^{k_2+k_r}, \Sigma_2^{k_2+k_r} \in \theta^{k_2+k_r}; \ldots \Sigma_1^{k_1+k_2+k_3}, \Sigma_2^{k_1+k_2+k_3} \in$$
$$\theta^{k_1+k_2+k_3}\ldots; \Sigma_1^{n-k_r}, \Sigma_2^{n-k_r} \in \theta^{n-k_r};\ldots; \Sigma_1^{n-k_1}, \Sigma_2^{n-k_1} \in \theta^{n-k_1}; \Sigma_1^n, \Sigma^n \in \theta^n$$

are h-cobordant respectively then they are diffeomorphic. It then follows that $\Sigma_1^{k_1+k_r}\times S^{k_2}\times\ldots\times S^{k_{r-1}}$ is diffeomorphic to $\Sigma_2^{k_1+k_r}\times S^{k_2}\times\ldots\times S^{k_{r-1}}$ and $\Sigma_1^{k_2+k_r}\times S^{k_1}\times S^{k_3}\times\ldots\times S^{k_{r-1}}$ is diffeomorphic to $\Sigma_2^{k_1+k_r}\times S^{k_1}\times S^{k_3}\times\ldots\times S^{k_{r-1}}$ and

$\Sigma_1^{k_1+k_2+k_3} \times S^{k_4} \times S^{k_5} \times \ldots \times S^{k_r}$ is diffeomorphic to $\Sigma_2^{k_1+k_2+k_3} \times S^{k_4} \times \ldots \times S^{k_r}$. Also $\Sigma_1^{n-k_r} \times S^{k_r}$ is diffeomorphic to $\Sigma_2^{n-k_r} \times S^{k_r}$ and $\Sigma_1^{n-k_1} \times S^{k_1}$ is diffeomorphic to $\Sigma_2^{n-k_1} \times S^{k_1}$ and so this means that

$$\Big[(S^{k_1} \times \ldots \times S^{k_r}) \underset{k_1+k_r}{\#} (\Sigma_1^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}}) \underset{k_2+k_r}{\#} (\Sigma_1^{k_2+k_r} \times S^{k_3} \times \ldots \times S^{k_{r-1}}$$
$$\# \ldots \underset{k_1+k_2+k_3}{\#} (\Sigma_1^{k_1+k_2+k_3} \times S^{k_4} \times \ldots \times S^{k_r}) \# \ldots$$
$$\# \ldots \underset{n-k_r}{\#} (\Sigma_1^{n-k_r} \times S^{k_r}) \# \ldots \underset{n-k_1}{\#} (\Sigma_1^{n-k_1} \times S^{k_1}) \# \Sigma_1^n\Big] \text{ is diffeomorphic to}$$
$$\Big[(S^{k_1} \times \ldots \times S^{k_r}) \underset{k_1+k_r}{\#} (\Sigma_2^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}}) \underset{k_2+k_r}{\#} (\Sigma_2^{k_2+k_r} \times S^{k_1} \times S^{k_3} \times \ldots \times S^{k_{r-1}})$$
$$\# \ldots \underset{k_1+k_2+k_3}{\#} (\Sigma_2^{k_1+k_2+k_3} \times S^{k_4} \times \ldots \times S^{k_r}) \# \ldots \underset{n-k_2}{\#} (\Sigma_2^{n-k_2} \times S^{k_r})$$
$$\# \ldots \underset{n-k_1}{\#} (\Sigma_2^{n-k_1} \times S^{k_1})\Big] \# \Sigma_2^n .$$

Hence β is well-defined map.

Clearly β takes the base points $0(k_1+k_r)$, $0(k_2+k_{r_1}), \ldots, 0(k_1+k_2+k_3), \ldots, 0(n-k_r), \ldots, 0(n-k_1)$, $0(n)$ to the base point 0. This is because if all the homotopy spheres Σ^ks are standard spheres, then all the summands involving Σ^is in the image of β will vanish leaving only $S^{k_1} \times \ldots \times S^{k_r}$. By Theorem 3.1, β is onto.

Suppose $\Sigma^{k_1+k_r} \in H((k_1+k_r),(k_2,\ldots,k_{r-1}))$, $\Sigma^{k_2+k_r} \in H((k_2+k_r),(k_1,k_3,\ldots,k_{r-1}))$, $\ldots$, $\Sigma^{k_1+k_2+k_3} \in H((k_1+k_2+k_3),(k_4,k_5,\ldots,k_r)), \ldots$, $\Sigma^{n-k_r} \in H(n-k_r,k_r), \ldots, \Sigma^{n-k_1} \in H(n-k_1,k_1)$ then for $\Sigma^{k_1+k_r} \in H((k_1+k_r),(k_2,\ldots,k_{r-1}))$ this means $\Sigma^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}}$ is diffeomorphic to $S^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}}$ hence $S^{k_1} \times \ldots \times S^{k_r} \# \Sigma^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}}$ is diffeomorphic to $(S^{k_1} \times \ldots \times S^{k_r}) \# S^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}} = S^{k_2} \times S^{k_3} \times \ldots \times S^{k_{r-1}} \times (S^{k_1} \times S^{k_r} \# S^{k_1+k_r})$ which is diffeomorphic to $S^{k_2} \times S^{k_3} \times \ldots \times S^{k_{r-1}} \times S^{k_1} \times S^{k_r}$ since $S^{k_1} \times S^{k_r} \# S^{k_1+k_r} = S^{k_1} \times S^{k_r}$. This means that for $\Sigma^{k_1+k_r} \in H((k_1+k_r),(k_2,\ldots,k_{r-1}))$ then $(S^{k_1} \times \ldots \times S^{k_r}) \underset{k_1+k_r}{\#} (\Sigma^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}})$ is diffeomorphic to $S^{k_1} \times \ldots \times S^{k_r}$ and so the summand $\Sigma^{k_1+k_r} \times S^{k_2} \times \ldots \times S^{k_{r-1}}$ in the image β vanishes. Similar arguments show that all other summands involving the Σ^is in $H(i,n-i)$ vanish hence in this case

$$\beta(\Sigma^{k_1+k_r}, \Sigma^{k_2+k_r}, \ldots, \Sigma^{k_1+k_2+k_3}, \ldots, \Sigma^{n-k_r}, \ldots, \Sigma^{n-k_1}, \Sigma^n) = S^{k_1} \times S^{k_2} \times \ldots \times S^{k_r} .$$

Then β induces a map.

$$\Phi : \Big(\frac{\theta^{k_1+k_r}}{H(k_1+k_r,(k_2,\ldots,k_{r-1}))} \times \frac{\theta^{k_1+k_2}}{H(k_2+k_r,(k_1,k_2,\ldots,k_{r-1}))} \times \ldots \times$$
$$\frac{\theta^{k_1+k_2+k_3}}{H(k_1+k_2+k_3,(k_4,\ldots,k_r))} \times \ldots \times \frac{\theta^{n-k_r}}{H(n-k_r,k_r)} \times \ldots \times \frac{\theta^{n-k_1}}{H(n-k_1,k_1)}$$
$$\times\, \theta^n\Big) \longrightarrow (\text{structures on } S^{k_1} \times \ldots \times S^{k_r})$$

which is onto since β is onto.

If $\Phi(\Sigma^{k_1+k_r}, 0(k_2+k_r), \ldots, 0(k_1+k_2+k_3), \ldots, 0(n-k_r), \ldots, 0(n-k_1), 0(n)) = 0$ then it follows by an easy generalization of Theorem 2.2.1 of [1] that $\Sigma^{k_1+k_r} \in H((k_1+k_r),(k_2,\ldots,k_{r-1}))$ and by the same method if $\Phi(0(k_1+k_r), 0(k_2+k_r), \ldots, \Sigma^{n-k_r}), \ldots, \Sigma^{n-k_r}, \ldots, 0(n-k_1), 0(n)) = 0$ then $\Sigma^{n-k_r} \in H(n-k_r,k_r)$. Also in

[[5], Theorem A], Reinhard Schultz showed that the inertial group of product of any number of ordinary spheres is trivial. This result implies that if $\Phi(O(k_1+k_r), O(k_2+k_r),\ldots,O(n-k_r),\ldots,O(n-k_1), \Sigma^n) = 0$ then Σ^n is diffeomorphic to S^n. It then follows that Φ is one to one and onto hence the number of differentiable structures on $S^{k_1}\times S^{k_2}\times\ldots\times S^{k_r}$ is equal to the order of

$$\frac{\theta^{k_1+k_r}}{H(k_1+k_r,(k_2,\ldots,k_{r-1}))}\times\frac{\theta^{k_2+k_r}}{H(k_2+k_r,k_1,k_3,\ldots,k_{r-1})}\times\cdots\times\frac{\theta^{k_1+k_2+k_3}}{H((k_1+k_2+k_3),(k_4,\ldots,k_r))}$$

$$\cdots\times\frac{\theta^{k_1+k_2+k_3}}{H((k_1+k_2+k_3),(k_4,\ldots,k_r))}\times\cdots\times\frac{\theta^{n-k_1}}{H(n-k_r,k_r)}\times\cdots\times\frac{\theta^{n-k_1}}{H(n-k_1,k_1)}\times\theta^n$$

EXAMPLES

We recall that in Table 7.4 of [5], θ^n_k denotes the number of homotopy spheres which do not embed in R^{n+k}. We shall use the values computed in that table in some of the examples given here. Since $\Gamma^i = 0$ for $1 \le i \le 6$, then the number of smooth structures on $S^2\times S^2\times S^2\times S^2$ is the order of $\theta^8 = 2$. Also since $\theta^8_3 = 2 = |\theta^8|$ then $H(8,2) = 0$ and so the number of smooth structures on $S^2\times S^2\times S^2\times S^2\times S^2 = 12$. By similar reasoning, the number of smooth structures on $S^2\times S^2\times S^2\times S^4 = 12$.

Since $\theta^{12} = 0$ and $H(9,3) = 4$ then the number of smooth structures on $S^3\times S^3\times S^3\times S^3 = 2$ whereas since $\theta^{15} = 16256$ and $\theta^9 = 8$ combined with the fact that $\theta^{12} = 0$ and $H(9,3) = 4$ it follows that the number of smooth structures on $S^3\times S^3\times S^3\times S^3\times S^3$ is 32512. From [3] we see that $\theta^8_5 = 1$ and $H(8,4) = \theta^8$ and $\Gamma^{12}=0$, then the number of smooth structures on $S^4\times S^4\times S^4\times S^4 = 2$. By a similar argument, it is easily seen that the number of smooth structures on $S^4\times S^4\times S^4\times S^4\times S^4$ is the order $\frac{\theta^{16}}{H(16,4)}\times\theta^{20}$. Also since $H(10,5) = \theta^{10}$ then the number of smooth structures on $S^5\times S^5\times S^5\times S^5$ is the order of $\frac{\theta^{15}}{H(15,5)}\times\theta^{20}$.

I am grateful to Professor P. Emery Thomas, University of California, Berkeley, for very useful communications.

REFERENCES

1. Samuel Omoloye Ajala. Differentiable structures on product of spheres, Houston Journal of Mathematics, Vol. 10, No. 1 (1984), 1-14.

2. R. de Sapio. Differentiable structures on product of spheres, Comment Math. Hal. (Vol. 44), 1969.

3. D. Sullivan. On the Hauptvanmuting for manifolds, Bulletin American Math. Soc. 73 (1967), 598-600.

4. J. Munkres. Obstruction to smoothings of piecewise linear homomorphism, Annals of Math. 72 (1960), 521-554.

5. J. Levine. Classification of Differentiable knots, Annals of Math. 86 (1965), 15-50.

6. Reinhard Schultz. On inertial group of a product of spheres, Transactions of American Math. Soc. 156 (1971), 137-157.