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A *-MIXING CONVERGENCE THEOREM FOR CONVEX SET VALUED PROCESSES

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ABSTRACT. In this paper the concept of a *-mixing process is extended to multivalued maps from a probability space into closed, bounded convex sets of a Banach space. The main result, which requires that the Banach space be separable and reflexive, is a convergence theorem for *-mixing sequences which is analogous to the strong law of large numbers. The impetus for studying this problem is provided by a model from information science involving the utilization of feedback data by a decision maker who is uncertain of his goals. The main result is somewhat similar to a theorem for real valued processes and is of interest in its own right.

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1. INTRODUCTION.

Our motivation for the present work arises from our efforts to extend a model of a decision maker using feedback information to decide on an appropriate course of action. This model has been described by R. Alo, R. Kleyle and A. de Korvin in [2] and extended to include goal uncertainty on the part of the decision maker by A. de Korvin and R. Kleyle in [12]. The role of the decision maker in this model is to select an appropriate course of action from a finite set of possible courses of action and to use the information obtained from implementing this course of action to reassess the situation prior to selecting his next course of action. The selection process is determined by the decision maker's current estimate of the expected utility associated with each course of action. The case in which the decision maker has a well defined utility function for each course of action (goal certainty) is developed in detail in [2].

Some preliminary results for the case in which the decision maker is unable to assign an explicit utility function to each course of action (goal uncertainty) are obtained in [12]. In this paper goal uncertainty is expressed by interval-valued functions. In the present paper we extend the goal uncertainty case to the situation in which the decision maker considers a convex set of possible utility functions. This leads us to consider multivalued maps from a probability space to subsets of a Banach function space. The first steps in that direction were taken by A. de Korvin and R. Kleyle in [13].

The key to the results obtained in [13] is that for each course of action the convex valued expected utilities form a supermartingale. In the context of multivalued maps a supermartingale is a sequence $\{F_n, H_n\}$ where F_n is a multivalued map and H_n an expanding sequence of σ-fields such that F_n is H_n-measurable and

$$E[F_{n+1} \mid H_n] \subset F_n.$$

The formal definition for conditional expectations in the present context will be given in the next section. The above property is a consequence of goal shaping which is defined in [12] and in [13].

The purpose of the present work is to obtain a convergence result that would replace the convergence result obtained in [12] for situations in which the goal shaping condition is removed. Consequently we wish to remove the condition that F_n is a supermartingale. Another reason for removing the supermartingale condition is that we do not want to tie the expected utility to the immediate past. A far more reasonable condition is to assume that the dependence of F_n on past history becomes weaker as past history becomes more distant. To accomplish this we will assume that the process satisfies the *-mixing condition which will be defined in the next section. This condition is called *-mixing because of its analogy to the *-mixing condition for real valued processes.

At each cycle of an ongoing decision process the decision maker hopes to improve his estimate of the expected utilities associated with each course of action. It is reasonable to assume that he will in some sense want to average his estimates of the sets of utility functions obtained so far. In this paper we define such an average and show that it satisfies a strong law of large numbers with respect to a metric to be defined later.

2. BACKGROUND AND PRELIMINARIES.

The main purpose of this section is to define the important concepts necessary for understanding the results. The most fundamental concept needed is that of Banach-valued martingales.

Let (Ω,Σ,P) be a probability space, and let Y be a Banach space. Let X_i be a sequence of integrable Y-valued functions and $\mathcal{F}_i$ an expanding sequence of sub σ-fields of Σ. The sequence $(X_i,\mathcal{F}_i)$ is called a martingale if X_i is $\mathcal{F}_i$-measurable and

$$E[X_{i+1} \mid \mathcal{F}_i] = X_i \quad \text{a.s.} \tag{2.1}$$

For properties of real-valued martingales the reader is referred to [8] and for the Banach-valued case to [6].

We now focus attention on multivalued functions defined on Ω whose values are closed, bounded, convex subsets of Y. Let F and G denote any such maps. We now define a new map

$$(F \dot{+} G)(\omega) = \mathrm{cl}\{a + b \;/\; a \in F(\omega),\ b \in G(\omega)\}$$

where cl denotes the closure with respect to the norm.

We set

$$\delta(F,G)(\omega) = \mathrm{Max}\{\sup_{x \in F(\omega)} d(x,G(\omega)),\ \sup_{y \in F(\omega)} d(y,F(\omega))\}$$

where
$$d(x,G(\omega)) = \inf_{t \in G(\omega)} \| x-t \| , \quad d(x,F(\omega)) = \inf_{t \in F(\omega)} \| x-t \| .$$

Of course δ is just the Hausdorff distance of $F(\omega)$ to $G(\omega)$, and when there is no confusion, we will write $\delta(F,G)$. For a finite sequence of maps F_i

$$\dot{\Sigma}_{i=1}^{n} F_i = F_1 \dot{+} F_2 \dot{+} \dots \dot{+} F_n ,$$

and $\dot{\Sigma}_{i=1}^{\infty} F_i$ denotes the limit, if it exists, of $\dot{\Sigma}_{i=1}^{n} F_i$ in the δ-metric.

We define the analog of an L^1 distance by

$$\Delta(F,G) = \int \delta(F,G)dP .$$

In this context the notations δ, Δ and $\dot{+}$ were first introduced by Debreu in [7].

Finally we define

$$|F| = \delta(F,\{0\}).$$

Note that $|F|$ is a non negative function defined on Ω.

A multivalue map F is Σ-measurable if for any open set $B \subset Y$, $F^{-1}(B) \in \Sigma$ where

$$F^{-1}(B) = \{\omega \in \Omega \ / \ F(\omega) \cap B \neq \emptyset \} .$$

It is shown in Himmelberg [11] that $D(F) = \{\omega \in \Omega \ / \ F(\omega) \neq \emptyset\}$ and there exists a sequence $\{f_n\}$ of Σ-measurable functions such that

$$F(\omega) = \mathrm{cl}\{f_n(\omega) \quad \text{for all} \quad \omega \in D(F)\}.$$

In fact the above property can be used to define measurability for Banach spaces. For an equivalent definition of measurability the reader is referred to [11], [5], and [7]. We define F to be integrable if

$$\int |F| \, dP < \infty$$

The space $L^2_{wk}(Y)$ will refer to all multivalued maps $F\colon \Omega \to 2^Y$ such that $F(\omega)$ is a weakly compact non empty convex subset of Y and $\int |F|^2 \, dP < \infty$.

By a selector of F we mean a function $f\colon \Omega \to Y$ such that $f(\omega) \in F(\omega)$ for all $\omega \in \Omega$. If F is integrable, $S^1_F(\mathcal{F})$ will denote all $\mathcal{F}$-measurable and integrable selectors of F. $S^1_F(\Sigma)$ will simply be written as S^1_F. For related kinds of selectors see Alo, de Korvin and Roberts [1]. Following Aumann [3] we define, for F Σ-measurable

$$\int F dP = \{ \int f \, dP \ / \ f \in S^1_F \} .$$

From now on $E(F)$ will denote the Aumann integral of F.

We now define the concept of conditional expectation for an integrable, multivalued function F. We assume that Y is separable, and the values of F are weakly compact subsets of Y. Following Hiai and Umegaki [10] (Th. 5.1, p. 169), the conditional expectation of F relative to a sub σ-field $\mathcal{F}$ of Σ is defined by

$$S^1_{E[F|\mathcal{F}]}(\mathcal{F}) = cl\{ E[f|\mathcal{F}] \;/\; f \in S^1_F\}.$$

The lhs of the above line indicates that the conditional expectation of F with respect to $\mathcal{F}$ is the set of integrable selectors of this set which are $\mathcal{F}$-measurable. For notational convenience we denote this set as $E[F|\mathcal{F}]$.

Now that the conditional expectation of multivalued functions has been defined, the concept of a martingale can be extended to these functions by replacing Y-valued functions X_i in (2.1) with convex set valued functions F_i. For convergence theorems pertaining to multivalued martingales the reader is referred to the work of Hia and Umegaki [10] and for multivalued supermartingales to A. de Korvin and R. Kleyle [13].

Let F_i be a sequence of integrable multivalued functions whose values are weakly compact subsets of Y where Y is separable. We say that the sequence is *-mixing if there exists some positive integer N and some function ϕ defined on $[N,\infty)$ such that ϕ is strictly decreasing to 0, and for all $n \geq N$ and $m \geq 1$ we have

$$\Delta[E[F_{n+m} \mid \mathcal{F}_m]],\ E[F_{n+m}]] \leq \phi(n)E|F_{n+m}|. \tag{2.2}$$

A law of large numbers was shown for a somewhat similar real-valued process by Blum et al [4].

The concept of a *-mixing sequence is central to our result. What (2.2) really says is that on the average the dependence of F_{n+m} on $\mathcal{F}_m$ grows weaker as $n \to \infty$ provided $E|F_{n+m}|$ is reasonably bounded. If the interpretation of F_{n+m} is the estimate of the average utility at trial n+m when goal uncertainty is present, then it is reasonable to expect that the dependence of this average on the early history of the process (i.e. the history up to trial m) grows weaker as $n \to \infty$.

For technical reasons we will need the following σ-fields. Let w_n be a family of Σ-measurable selectors of F; then

$$G_F(w_n) = \mathcal{F}[w_{n(\cdot)} \;/\; n(\cdot) \in N^{\Omega}\ ].$$

That is, we consider the σ-field generated by all functions obtained from w_n by allowing n to be a variable index.

If F_t is a family of measurable multivalued maps, $G_t(w_n,t)$ will be used to denote $G_{F_t}(w_n,t)$.

By the σ-field generated by $F_{t_1}, F_{t_2}, \ldots F_{t_k}$ we mean $\sigma[\bigcup_{i=1}^{k} G_{t_i}(w_n,t_i)]$. Given a sequence $F_1, F_2, \ldots$ of Σ-measurable multivalued maps, we will replace Σ by

$$\Sigma' = \sigma[\Sigma \cup \bigcup_{i=1}^{\infty} G_i(w_{n,i})].$$

3. RESULTS.

From now on Y is a separable, reflexive Banach space and F_n denotes a *-mixing sequence with $F_n \in L^2_{wk}(Y)$. We replace Σ by the larger σ-field Σ' as defined in the previous section. We start by listing a result on martingales known to be true for the real valued case [9].

LEMMA 1. Let f_i be a sequence of Y-valued random variables such that $S_n = \sum_{i=1}^{n} f_i$ is a martingale relative to the expanding sequence of σ-fields F_n. Let b_n be an increasing sequence of positive reals such that $\lim b_n = \infty$. Then if

$$\sum_{i=1}^{\infty} b_i^{-2} E[\|f_i\|^2 \mid F_{i-1}] < \infty,$$

it follows that $\lim S_n/b_n = 0$ (a.s.).

PROOF. The proof is essentially the same as for the real case. For details the reader is referred to [9], Theorem 2.18, pp. 35-36.

We now obtain an important inequality. Let $F_1, F_2, \dots F_q$ be any finite sequence of integrable multivalued functions whose values are weakly compact subsets of a separable Banach space Y, and let $H_1, H_2, \dots, H_q$ denote an arbitrary sequence of expanding σ-fields in Σ′.

LEMMA 2. For every $\varepsilon > 0$, there exists a sequence $f_1, f_2, \dots, f_q$ such that f_i is an integrable selector of F_i, and such that for each i, f_i is H_i-measurable, and

$$\Delta(.\sum_{i=1}^{q} F_i, .\sum_{i=1}^{q} E[F_i|H_{i-1}]) \le \int \| \sum_{i=1}^{q}(f_i - E[f_i|H_{i-1}]) \| dP + 2\varepsilon.$$

PROOF. Let $\Omega_1 = \{\omega / \operatorname{Sup}_s d(s, .\sum_{i=1}^{q} F_i) \ge \operatorname{Sup}_t d(t, .\sum_{i=1}^{q} E[F_i|H_{i-1}])\}$, and let $\Omega_2 = \Omega - \Omega_1$. Here s ranges over $.\sum_{i=1}^{q} E[F_i|H_{i-1}]$, and t ranges over $.\sum_{i=1}^{q} F_i$. Since $.\sum_{i=1}^{q} E[F_i|H_{i-1}]$ is a measurable multivalued function, it has a sequence of selectors $\{v_m\}$ such that

$$\text{cl}\ \{v_m(\omega)\} = .\sum_{i=1}^{q} E[F_i|H_{i-1}](\omega) \quad \text{a.s.}$$

The sequence $\{v_m\}$ is Σ-measurable. Now there exists functions $v_{m(\cdot)}(\cdot)$, which we continue to denote by v_m, such that

$$d(v_m, .\sum_{i=1}^{q} F_i) \ge \operatorname{Sup}_s d(s, .\sum_{i=1}^{q} F_i) - \varepsilon .$$

Thus

$$\int_{\Omega_1} \delta(.\sum_{i=1}^{q} F_i, .\sum_{i=1}^{q} E[F_i|H_{i-1}])dP \le \int_{\Omega_1} d(v_m, .\sum_{i=1}^{q} F_i)dP + \varepsilon .$$

By Theorem 5.1 of [10] there exists H_i measurable functions g_{mi} which are selectors of F_i such that the right hand side of the above inequality is dominated by

$$\int_{\Omega_1} \| v_m - \sum_{i=1}^{q} E[g_{mi}|H_{i-1}] \| dP + \int_{\Omega_1} d(\sum_{i=1}^{q} E[g_{mi}|H_{i-1}], .\sum_{i=1}^{q} F_i)dP + \varepsilon ,$$

and where, moreover,

$$\|v_m - \sum_{i=1}^{q} E[g_{mi}|H_{i-1}] \|_1 < \varepsilon .$$

Hence

$$\int_{\Omega_1} \delta(.\sum_{i=1}^{q} F_i, \sum_{i=1}^{q} E[F_i|H_{i-1}])dP \le \int_{\Omega_1} \| \sum_{i=1}^{q} (E[g_{mi}|H_{i-1}] - g_{mi}\| dP + 2\varepsilon .$$

Now if $\omega \in \Omega_2$, since $\Sigma_{i=1}^{q} F_i$ is Σ-measurable, we can pick a sequence $u_{m(\cdot)}(\cdot)$, which we denote by u_m, of selectors of $\Sigma_{i=1}^{q} F_i$ such that

$$d(u_m, \Sigma_{i=1}^{q} E[F_i|H_{i-1}]) \geq \sup_t d(t, \Sigma_{i=1}^{q} E[F_i|H_{i-1}]) - \varepsilon .$$

Hence

$$\int_{\Omega_2} \delta(\Sigma_{i=1}^{q} F_i, \Sigma_{i=1}^{q} E[F_i|H_{i-1}])dP \leq \int_{\Omega_2} d(u_m, \Sigma_{i=1}^{q} E[F_i|H_{i-1}]dP + \varepsilon$$

$$\leq \int_{\Omega_2} \| \Sigma_{i=1}^{q} (u_{mi} - E[u_{mi}|H_{i-1}]) \| dP + \varepsilon ,$$

where u_{mi} are H_i-measurable selectors of F_i.

Thus the lemma is proved by picking $f_i = g_{mi}$ on Ω_1, and $f_i = u_{mi}$ on Ω_2.

We are now ready to prove the main result.

THEOREM. Let F_n be a *-mixing sequence with $F_n \in L^2_{wk}(Y)$ where Y is a separable and reflexive Banach space. Assume

(i) $\Sigma_{n=1}^{\infty} b_n^{-2} E|F_n|^2 < \infty$,

(ii) $\sup_n b_n^{-1} \Sigma_{i=1}^{n} E|F_i| < \infty$,

where b_n is a sequence of positive constants increasing to infinity. Then

$$\Delta(b_n^{-1} \Sigma_{i=1}^{n} F_i, b_n^{-1} \Sigma_{i=1}^{n} E(F_i)) \to 0 .$$

PROOF. By the *-mixing property, there exists N such that the *-mixing inequality (2.2) holds for $n \geq N$ and $m \geq 1$. Given $\varepsilon > 0$, pick $n_0 \geq N$ large enough so that $\phi(n_0) < \varepsilon$. Thus since Y is reflexive for all positive integers i and j we have by theorem 5.4 of [10],

$$\Delta(E(F_{in_0+j} \mid F_{n_0+j},\dots F_{(i-1)n_0+j}), E(F_{in_0+j})) \tag{3.1}$$

$$= \Delta\{E[E(F_{in_0+j} \mid \mathcal{F}_{(i-1)n_0+j} \mid G_{(i-1)n_0+j}], E[E(F_{in_0+j} \mid G_{(i-1)n_0+j})]\}$$

where $\mathcal{F}_{(i-1)n_0+j}$ is the σ-field generated by $F_1, F_2, \dots , F_{(i-1)n_0+j}$ and $G_{(i-1)n_0+j}$ is generated by $F_{n_0+j}, F_{2n_0+j}, \dots , F_{(i-1)n_0+j}$.

By theorem 5.2 of [10] the right hand side of (3.1) is dominated by

$$\Delta[E(F_{in_0+j} \mid \mathcal{F}_{(i-1)n_0+j}), E(F_{in_0+j})] \leq \phi(n_0) E \mid F_{in_0+j} \mid . \tag{3.2}$$

The last inequality is a consequence of the *-mixing inequality. Now

$$\Delta(b_n^{-1} \Sigma_{i=1}^{n} F_i, b_n^{-1} \Sigma_{i=1}^{n} E(F_i))$$

$$\leq \Delta(b_n^{-1} \Sigma_{i=1}^{n_0} F_i, b_n^{-1} \Sigma_{i=1}^{n_0} E(F_i)) + \Delta(b_n^{-1} \Sigma_{i=1}^{q-1} \Sigma_{j=1}^{n_0-1} F_{in_0+j},$$

$$b_n^{-1} \cdot \sum_{i=1}^{q-1} \cdot \sum_{j=1}^{n_0-1} E(F_{in_0+j}))$$

$$+ \Delta(b_n^{-1} \cdot \sum_{j=1}^{r} F_{qn_0+j},\ b_n^{-1} \cdot \sum_{j=1}^{r} E(F_{qn_0+j})), \tag{3.3}$$

where for any $n \geq n_0$, $n = qn_0 + r$ where q and r are positive integers such that $0 \leq r \leq n_0 - 1$. The inequality (3.3) holds because

$$\Delta(A \dot{+} B,\ C \dot{+} D) \leq \Delta(A,C) + \Delta(B,D).$$

(See p. 162 of [10])

The first term of the right hand side in (3.3) can be written as

$$b_n^{-1}\ \Delta(\cdot \sum_{i=1}^{n_0} F_i,\ \cdot \sum_{i=1}^{n_0} E(F_i))$$

and goes to zero when $n \to \infty$ since n_0 is fixed and $b_n \to \infty$. It remains to show that the second and third term of the right hand side of (3.3) goes to zero as $n \to \infty$. We give the proof for the second term, the third term is handled similarly.

By the triangular inequality the second term is dominated by

$$\Delta(b_n^{-1} \cdot \sum_{i=1}^{q-1} \cdot \sum_{j=1}^{n_0-1} F_{in_0+j},\ b_n^{-1} \cdot \sum_{i=1}^{q-1} \cdot \sum_{j=1}^{n_0-1} E(F_{in_0+j} \mid G_{(i-1)n_0+j}))$$

$$+ \Delta(b_n^{-1} \cdot \sum_{i=1}^{q-1} \cdot \sum_{j=1}^{n_0-1} E(F_{in_0+j} \mid G_{(i-1)n_0+j}),\ b_n^{-1} \cdot \sum_{i=1}^{q-1} \cdot \sum_{j=1}^{n_0-1} E(F_{in_0+j}))$$

$$\leq \sum_{j=1}^{n_0-1} b_n^{-1}\ \Delta(\cdot \sum_{i=1}^{q-1} F_{in_0+j},\ \cdot \sum_{i=1}^{q-1} E[F_{in_0+j} \mid G_{(i-1)n_0+j}])$$

$$+ b_n^{-1}\ \phi(n_0) \sum_{j=1}^{n_0-1} \sum_{i=1}^{q-1} E|F_{in_0+j}| . \tag{3.4}$$

The last inequaltiy follows from (3.1) and (3.2).

The second term on the right hand side of (3.4) can be made arbitrarily small by the *-mixing condition and condition (ii) of the theorem. It remains to show that the first term on the right hand side of (3.4) goes to 0 as r (and therefore q) goes to infinity.

By Lemma 2, this term is dominated by

$$\sum_{j=1}^{n_0-1} b_n^{-1} \int \| \sum_{i=1}^{q-1} (f_{in_0+j} - E[f_{in_0+j} \mid G_{(i-1)n_0+j}]) \|\ dP + 2\ \varepsilon$$

where f_{in_0+j} is a G_{in_0+j}-measurable integrable selector of F_{in_0+j}. Furthermore, it is easy to show that

$$S_{q-1} = \sum_{i=1}^{q-1} (f_{in_0+j} - E[f_{in_0+j} \mid G_{(i-1)n_0+j}]$$

is a martingale with respect to $\{G_{(q-1)n_0+j}\}$ $q \geq 1$.

Condition (i) of the theorem implies that the hypothesis of Lemma 1 holds, so that for each fixed n_0 and j,

$$b_n^{-1} \| \sum_{i=1}^{q-1} (f_{in_0+j} - E[f_{in_0+j} \mid G_{(i-1)n_0+j}]) \| \to 0 \quad \text{a.s.} \tag{3.5}$$

The left hand side of (3.5) is dominated by

$$b_{q-1}^{-1} \| \sum_{i=1}^{q-1} (f_{in_0+j} - E[f_{in_0+j} \mid G_{(i-1)n_0+j}]) \|$$

$$\leq b_{q-1}^{-1} \sum_{i=1}^{q-1} | F_{in_0+j} | + b_{q-1}^{-1} \sum_{i=1}^{q-1} |E[F_{in_0+j} \mid G_{(i-1)n_0+j}] | .$$

Since the map $F_{in_0+j} \to E[F_{in_0+j} \mid G_{(i-1)n_0+j}]$ is non-expansive, (see Th. 5.2 of [10]), it follows that the right hand side of the above inequality has L^1 norm less than or equal to

$$2b_{q-1}^{-1} \sum_{i=1}^{q-1} E \mid F_{in_0+j} | .$$

Thus

$$b_n^{-1} \int \| \sum_{i=1}^{q-1} (f_{in_0+j} - E[f_{in_0+j} \mid G_{(i-1)n_0+j}]) \| \, dP \to 0$$

by condition (ii) and the dominated convergence theorem. This completes the proof of the theorem.

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