SUPER AND SUBSOLUTIONS FOR ELLIPTIC EQUATIONS ON ALL OF $\mathbb{R}^n$

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By construction sub and supersolutions for the following semilinear elliptic equation

$$-\Delta u(x) = \lambda g(x)f(u(x)), \quad x \in \mathbb{R}^n,$$

which arises in population genetics, we derive some results about the theory of existence of solutions as well as asymptotic properties of the solutions for every $n$ and for the function $g: \mathbb{R}^n \to \mathbb{R}$ such that $g$ is smooth and is negative at infinity.

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1. Introduction. In this paper, we discuss the existence and nonexistence of solutions as well as asymptotic properties of the solutions of the equation

$$-\Delta u(x) = \lambda g(x)f(u(x)), \quad x \in \mathbb{R}^n, \quad 0 \leq u(x) \leq 1 \quad (1.1)$$

which arises in population genetics (see [7, 11]). The unknown function $u$ corresponds to the relative frequency of an allele and is hence constrained to have values between 0 and 1. The real parameter $\lambda > 0$ corresponds to the reciprocal of a diffusion coefficient.

We assume throughout that $g: \mathbb{R}^n \to \mathbb{R}$ is smooth which changes sign on $\mathbb{R}^n$. Also we will assume throughout that $f$ satisfies the condition $f: [0, 1] \to \mathbb{R}$ is a smooth function such that $f(0) = f(1) = 0$, $f'(0) > 0$, $f'(1) < 0$, and $f(u) > 0$ for all $0 < u < 1$.

By the definition of $f$, it is clear that $(1.1)$ has the trivial solutions $u \equiv 0$ and $u \equiv 1$.

The existence of solutions for $(1.1)$ in the bounded region case with Dirichlet or Neumann boundary conditions is discussed in [7, 11], but in this case all of $\mathbb{R}^n$ is much more complicated (see [3, 6, 7, 8, 9, 12, 13]). The results obtained in [7] with the assumption that $g$ is negative at infinity show that the existence theory for solutions of $(1.1)$ is very different for the two cases $n = 1, 2$ and $n \geq 3$.

Some of the nontrivial solutions were bifurcating off the trivial solution $u \equiv 0$. In order to investigate these bifurcation phenomena, it was necessary to understand the eigenvalues and eigenfunctions of the corresponding linearized problem

$$-\Delta u(x) = \lambda g(x)f'(0)u(x), \quad x \in \mathbb{R}^n. \quad (1.2)$$

The existence of positive principal eigenfunctions of $(1.2)$ with the following conditions on $g$ was considered in [6]:

(i) $g$ is negative and bounded away from zero at infinity; or

(ii) $|g(x)| \leq k/(1 + |x|^2)^\alpha$, $n \geq 3$,

for some constants $k > 0$ and $\alpha > 1$, and these results for the case $g^+ \in L^{n/2}(\mathbb{R}^n)$, $n \geq 3$ where $g^+(x) = \max\{g(x), 0\}$ are extended in [3].
In this paper, we investigate the existence of solutions of (1.1) with the assumption that \( g \) or \( g^+ \) are small at infinity.

Our analysis is based on the construction of sub and supersolutions.

It is proved in [2] that the positive principal eigenvalue of the Dirichlet boundary value problem

\[
- \Delta u(x) = \lambda g(x)u(x), \quad x \in D,
\]

\[
u(x) = 0, \quad x \in \partial D,
\]

where \( D \) is a bounded domain with smooth boundary has the variational characterisation

\[
\lambda_1^+(D) = \inf \left\{ \int_D |\nabla u(x)|^2 \, dx : u \in H_0^1(D), \int_D gu^2 \, dx = 1 \right\}.
\]

Also, it is well known that the above infimum is attained and a minimizer \( \phi_1 > 0 \) is smooth, that is, \( c^2(D) \). Hence \( \phi_1 \) satisfies the Dirichlet boundary value problem (1.3), so \( \phi_1 \) is a principal eigenfunction corresponding to principal eigenvalue \( \lambda_1^+(D) \).

Suppose, however, that \( g = g^+ - g^- \) where \( g^+(x) = \max\{g(x), 0\} \) and \( g^-(x) = \min\{g(x), 0\} \).

If \( n \geq 3 \) and \( g^+ \in L^{n/2}(\mathbb{R}^n) \), then for all \( u \in H_0^1(D) \) such that \( \int_D gu^2 \, dx = 1 \) we have

\[
1 = \int_D gu^2 \, dx \leq \int_D g^+ u^2 \, dx \leq \|g^+\|_{L^{n/2}(D)} \|u\|^2_{L^2(D)} \leq c(n) \|g^+\|_{L^{n/2}(D)} \|\nabla u\|^2_{L^2(D)},
\]

where \( c(n) \) is the embedding constant of \( H_0^1(D) \) into \( L^{2n/(n-2)}(D) \) and is independent of \( D \) (see Brézis and Nirenberg [5, page 443]). Thus

\[
\lambda_1^+(D) \geq \|\nabla u\|^2_{L^2(D)} \geq \left\{ c(n) \|g^+\|_{L^{n/2}(D)} \right\}^{-1} > 0.
\]

Also, it is well known (see [1]) that if \( g^+ \in L^{n/2}(\mathbb{R}^n) \), then \( \lambda^* = \lim_{R \to \infty} \lambda_1^+(B_R(0)) \) exists and \( \lambda^* \) is the principal eigenvalue of the equation

\[
- \Delta u(x) = \lambda g(x)u(x), \quad x \in \mathbb{R}^n
\]

and there exists a corresponding principal eigenfunction \( \phi \) such that \( \phi(x) \to 0 \) as \( |x| \to \infty \). In addition, \( \lambda^* \) can be characterized as follows (see [1, Lemma 2.7])

\[
\lambda^* = \inf \left\{ \int_{\mathbb{R}^n} |\nabla u(x)|^2 \, dx : u \in c_0^\infty(\mathbb{R}^n), \int_{\mathbb{R}^n} gu^2 \, dx = 1 \right\}.
\]
**Theorem 1.1** (see [10]). If \( \lambda > \lambda^* \), then there exists \( u \geq 0 \ (u \neq 0) \) with compact support such that \( u \) is a subsolution of

\[
-\Delta u(x) = \lambda g(x) f(u(x)), \quad x \in B_R(0),
\]

\[
u(x) = 0, \quad x \in \partial B_R(0)
\]

for all \( R \) sufficiently large, also we can choose \( u \) sufficiently small.

2. **Sub and supersolutions for** \( n \geq 3 \). We assume \( D \subset \mathbb{R}^n \) is a bounded region with smooth boundary. We consider the following boundary value problem:

\[
-\Delta u(x) = \lambda g(x) f(u(x)), \quad x \in D,
\]

\[
u(x) = 0, \quad x \in \partial D.
\]

If \( \lambda > 0 \) be fixed, we can choose \( c > 0 \) such that for \( u, 0 \leq u \leq 1 \), the function \( u \rightarrow \lambda g(x) f(u) + cu \), for every \( x \in D \), is an increasing function.

Let \( h(x, u) = \lambda g(x) f(u) + cu \), then we have \( h(x, 0) \equiv 0 \) and \( h(x, 1) \equiv c \). We can write (2.1) as

\[
-\Delta u(x) + cu(x) = h(x, u(x)), \quad x \in D,
\]

\[
u(x) = 0, \quad x \in \partial D.
\]

It is well known that (2.2) has a unique solution \( u = Kf \) (see Amann [4]), where \( K \) is given by an integral operator whose kernel is the Green’s function for the problem, that is,

\[
(Kf)(x) = \int_D G(x, y) h(y, u(y)) \, dy.
\]

In (2.3), \( G(x, y) \) is the Green’s function of the operator \(-\Delta + c\) with Dirichlet boundary condition, also we can write (2.3) as \( u = KN(u) \) in where \( K : c(D) \to c^\alpha(D) \) is a compact linear integral operator with kernel \( G \) (see [4]) and \( N : c(D) \to c(D) \) is the Nemytskii operator corresponding to \( h \). Since \( h(x, \cdot) \) is increasing, it is easy to see that \( N \) is an increasing operator, that is, if \( u_1 \geq u_2 \), then \( Nu_1 \geq Nu_2 \).

We call \( u \in c^2(D) \) is a subsolution of (2.2) or equivalently (2.1) if we have

\[
-\Delta u(x) + cu(x) \leq h(x, u(x)), \quad x \in D,
\]

\[
u(x) \leq 0, \quad x \in \partial D,
\]

and \( u \in c(D) \) is a subsolution of (2.3) if

\[
u(x) \leq \int_D G(x, y) h(y, u(y)) \, dy, \quad x \in D,
\]

that is, \( u \leq KN(u) \). The definition of supersolution is quite similar.

It is well known that if \( v, w \) are sub and supersolutions of (2.2) (or for (2.3)), respectively, and \( v \leq w \), then there exists a solution \( u \) of (2.2) (of (2.3)) such that \( v \leq u \leq w \).
3. The case when \( n = 1, 2 \). In this section, we consider the problem

\[
-\triangle u(x) = \lambda g(x)f(u(x)), \quad x \in \mathbb{R}^n, \\
0 \leq u(x) \leq 1, \quad x \in \mathbb{R}^n,
\]

where \( g: \mathbb{R}^n \rightarrow \mathbb{R} \) is a continuous function which changes sign on \( \mathbb{R}^n \) and it has the following condition: (G) there exists \( R_0 > 0 \) such that \( g(x) < 0 \) for all \( x \in \mathbb{R}^n \) whenever \( |x| > R_0 \).

Also \( f \in C^1([0,1]) \) with the conditions

\[
f(0) = 0 = f'(1), \quad f''(0) > 0, \quad f'(1) < 0, \quad f(u) > 0, \quad 0 < u < 1.
\]

**Theorem 3.1** (see [7]). Let \( u \) be a nontrivial solution of (4.1). Then there exists a real constant \( k \) such that \( 0 < u(x) < k < 1 \) for all of \( x \) in \( \mathbb{R}^n \).

Now by using Theorem 3.1 and condition (G) on \( g \), we conclude that \( \Delta u(x) > 0 \) for all of \( x \in \mathbb{R}^n \) with \( |x| > R_0 \).

**Theorem 3.2.** Let \( u \) be a nontrivial solution of (4.1). Then \( u \) is nonconstant in out of the ball \( B_{R_0}(0) \).

**Proof.** Using assumption on \( g \), we have \( \Delta u(x) > 0 \) for all of \( x \in \mathbb{R}^n \) with \( |x| > R_0 \), so \( |\nabla u(x)| > 0 \) whenever \( |x| > R_0 \). Hence \( u \) is a nonconstant function in out of the ball \( B_{R_0}(0) \).

**Theorem 3.3.** Let \( n = 1 \) and \( u \) be a nontrivial solution of (4.1). Then \( u \) is a strictly decreasing function on \( (R_0, \infty) \) and increasing function on \( (-\infty, -R_0) \).

**Proof.** By using assumption on \( g \), we have \( u''(x) > 0 \) for all of \( x \in \mathbb{R}^n \) with \( |x| > R_0 \). So, \( u \) can have only one of the possibilities (a) and (b) in Figure 3.1. Figure 3.1(a) is impossible because we must have \( 0 \leq u(x) \leq 1 \) for all \( x \in \mathbb{R}^n \). So \( u \) satisfy in Figure 3.1(b), thus \( u \) is strictly decreasing in out of ball \( B_{R_0}(0) \).

**Theorem 3.4.** Let \( n = 2 \) and \( u \) be a solution of (4.1) which is radially symmetric, then \( u \) is a strictly monotone function in out of the ball \( B_{R_1}(0) \), where \( R_1 > R_0 \).

**Proof.** It is obvious by using maximum principle.

4. The case when \( n \geq 3 \). Let \( g \) satisfy condition (G). It is easy to see that

\[
\Pi(x) = \begin{cases} 
1, & |x| \leq R_0, \\
\left( \frac{R_0}{|x|} \right)^{(n-2)}, & |x| > R_0,
\end{cases}
\]

is a supersolution of (4.1), so we are ready to prove the following theorem.

**Theorem 4.1.** If \( \lambda > \lambda^* \), then there exists a nonconstant solution \( u \) of (4.1) such that

\[
\lim_{|x| \rightarrow \infty} u(x) = 0.
\]
Proof. We consider $\overline{u}$ as a supersolution of (4.1). Also there exists a subsolution $\underline{u}$ of (4.1) with compact support and sufficiently small (see [10]). So we can choose $\underline{u}$ such that $\underline{u} \leq \overline{u}$, so there exists a solution $u$ of (4.1) such that $\underline{u} \leq u \leq \overline{u}$. Also by using the definition of $\overline{u}$, we have $\lim_{|x| \to \infty} u(x) = 0$.

**Theorem 4.2.** Let $\alpha > 1$ and $\lambda > 0$ be arbitrary. Then there exists a supersolution $\overline{u}$ of (4.1) such that $|\overline{u}(x)| \leq c|x|^{-\beta}$ for a constant $c > 0$, and

$$\beta = \begin{cases} n - 2, & n < 2\alpha, \\ 2\alpha - 2, & n > 2\alpha. \end{cases}$$

(4.3)

Proof. Using condition (G) of the function $g$, we have

$$|g^+(x)| \leq \frac{k}{(1 + |x|^2)^{\alpha}},$$

(4.4)

where $k \geq M(1 + R_0^2)^\alpha, M = \max g^+(x)$. So using [10, Lemma 4.3], the proof is complete.

References


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