

## ON $n$ -NORMED SPACES

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**ABSTRACT.** Given an  $n$ -normed space with  $n \geq 2$ , we offer a simple way to derive an  $(n-1)$ -norm from the  $n$ -norm and realize that any  $n$ -normed space is an  $(n-1)$ -normed space. We also show that, in certain cases, the  $(n-1)$ -norm can be derived from the  $n$ -norm in such a way that the convergence and completeness in the  $n$ -norm is equivalent to those in the derived  $(n-1)$ -norm. Using this fact, we prove a fixed point theorem for some  $n$ -Banach spaces.

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**1. Introduction.** Let  $n \in \mathbb{N}$  and  $X$  be a real vector space of dimension  $d \geq n$ . (Here we allow  $d$  to be infinite.) A real-valued function  $\|\cdot, \dots, \cdot\|$  on  $X^n$  satisfying the following four properties

- (1)  $\|x_1, \dots, x_n\| = 0$  if and only if  $x_1, \dots, x_n$  are linearly dependent;
- (2)  $\|x_1, \dots, x_n\|$  is invariant under permutation;
- (3)  $\|x_1, \dots, x_{n-1}, \alpha x_n\| = |\alpha| \|x_1, \dots, x_{n-1}, x_n\|$  for any  $\alpha \in \mathbb{R}$ ;
- (4)  $\|x_1, \dots, x_{n-1}, y + z\| \leq \|x_1, \dots, x_{n-1}, y\| + \|x_1, \dots, x_{n-1}, z\|$ ,

is called an  $n$ -norm on  $X$  and the pair  $(X, \|\cdot, \dots, \cdot\|)$  is called an  $n$ -normed space.

A trivial example of an  $n$ -normed space is  $X = \mathbb{R}^n$  equipped with the following  $n$ -norm:

$$\|x_1, \dots, x_n\|_E := \text{abs} \left( \begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{vmatrix} \right), \quad (1.1)$$

where  $x_i = (x_{i1}, \dots, x_{in}) \in \mathbb{R}^n$  for each  $i = 1, \dots, n$ . (The subscript  $E$  is for Euclidean.)

Note that in an  $n$ -normed space  $(X, \|\cdot, \dots, \cdot\|)$ , we have, for instance,  $\|x_1, \dots, x_n\| \geq 0$  and  $\|x_1, \dots, x_{n-1}, x_n\| = \|x_1, \dots, x_{n-1}, x_n + \alpha_1 x_1 + \cdots + \alpha_{n-1} x_{n-1}\|$  for all  $x_1, \dots, x_n \in X$  and  $\alpha_1, \dots, \alpha_{n-1} \in \mathbb{R}$ .

The theory of 2-normed spaces was first developed by Gähler [3] in the mid 1960's, while that of  $n$ -normed spaces can be found in [11]. Recent results can be found, for example, in [9, 10]. Related works on  $n$ -metric spaces and  $n$ -inner product spaces may be found, for example, in [1, 2, 4, 5, 7, 6, 12].

In this note, we will show that every  $n$ -normed space with  $n \geq 2$  is an  $(n-1)$ -normed space and hence, by induction, an  $(n-r)$ -normed space for all  $r = 1, \dots, n-1$ . In particular, given an  $n$ -normed space, we offer a simple way to derive an  $(n-1)$ -norm from the  $n$ -norm, different from that in [5].

We will also apply our result to study convergence and completeness in  $n$ -normed spaces, which will be defined later. This enables us to prove a fixed point theorem for some  $n$ -normed spaces.

The case  $n = 2$  was previously studied in [8].

**2. Preliminary results.** Suppose hereafter that  $n \geq 2$  and  $(X, \|\cdot, \dots, \cdot\|)$  is an  $n$ -normed space of dimension  $d \geq n$ . Take a linearly independent set  $\{a_1, \dots, a_n\}$  in  $X$ . With respect to  $\{a_1, \dots, a_n\}$ , define the following function  $\|\cdot, \dots, \cdot\|_\infty$  on  $X^{n-1}$  by

$$\|x_1, \dots, x_{n-1}\|_\infty := \max \{\|x_1, \dots, x_{n-1}, a_i\| : i = 1, \dots, n\}. \tag{2.1}$$

Then we have the following result.

**THEOREM 2.1.** *The function  $\|\cdot, \dots, \cdot\|_\infty$  defines an  $(n - 1)$ -norm on  $X$ .*

**PROOF.** We will verify that  $\|\cdot, \dots, \cdot\|_\infty$  satisfies the four properties of an  $(n - 1)$ -norm.

(1) If  $x_1, \dots, x_{n-1}$  are linearly dependent, then  $\|x_1, \dots, x_{n-1}\| = 0$  for each  $i = 1, \dots, n$ , and hence  $\|x_1, \dots, x_{n-1}\|_\infty = 0$ . Conversely, if  $\|x_1, \dots, x_{n-1}\|_\infty = 0$ , then  $\|x_1, \dots, x_{n-1}, a_i\| = 0$  and accordingly  $x_1, \dots, x_{n-1}, a_i$  are linearly dependent for each  $i = 1, \dots, n$ . But this can only happen when  $x_1, \dots, x_{n-1}$  are linearly dependent.

(2) Since  $\|x_1, \dots, x_{n-1}, a_i\|$  is invariant under any permutation of  $\{x_1, \dots, x_{n-1}\}$ , we find that  $\|x_1, \dots, x_{n-1}\|_\infty$  is also invariant under any permutation.

(3) Observe that

$$\begin{aligned} \|x_1, \dots, x_{n-2}, \alpha x_{n-1}\|_\infty &= \max \{\|x_1, \dots, x_{n-2}, \alpha x_{n-1}, a_i\| : i = 1, \dots, n\} \\ &= |\alpha| \max \{\|x_1, \dots, x_{n-2}, x_{n-1}, a_i\| : i = 1, \dots, n\} \\ &= |\alpha| \|x_1, \dots, x_{n-2}, x_{n-1}\|_\infty. \end{aligned} \tag{2.2}$$

(4) Observe that

$$\begin{aligned} \|x_1, \dots, x_{n-2}, y + z\|_\infty &= \max \{\|x_1, \dots, x_{n-2}, y + z, a_i\| : i = 1, \dots, n\} \\ &\leq \max \{\|x_1, \dots, x_{n-2}, y, a_i\| : i = 1, \dots, n\} \\ &\quad + \max \{\|x_1, \dots, x_{n-2}, z, a_i\| : i = 1, \dots, n\} \\ &= \|x_1, \dots, x_{n-2}, y\|_\infty + \|x_1, \dots, x_{n-2}, z\|_\infty. \end{aligned} \tag{2.3}$$

Therefore  $\|\cdot, \dots, \cdot\|_\infty$  defines an  $(n - 1)$ -norm on  $X$ . □

**COROLLARY 2.2.** *Every  $n$ -normed space is an  $(n - r)$ -normed space for all  $r = 1, \dots, n - 1$ . In particular, every  $n$ -normed space is a normed space.*

**REMARK 2.3.** Note that in general the function  $\|x_1, \dots, x_{n-1}\|_p := \{\sum_{i=1}^n \|x_1, \dots, x_{n-1}, a_i\|^p\}^{1/p}$ , where  $1 \leq p \leq \infty$ , also defines an  $(n - 1)$ -norm on  $X$ . These  $(n - 1)$ -norms, however, are equivalent to  $\|\cdot, \dots, \cdot\|_\infty$ , as long as we use the same set of  $n$  vectors  $a_1, \dots, a_n$ . In certain cases, it is possible to get equivalent  $(n - 1)$ -norms even if we use different sets of  $n$  vectors.

**2.1. The standard case.** Take a look at a standard example. Let  $X$  be a real inner product space of dimension  $d \geq n$ . Equip  $X$  with the standard  $n$ -norm

$$\|x_1, \dots, x_n\|_S := \left| \begin{array}{ccc} \langle x_1, x_1 \rangle & \cdots & \langle x_1, x_n \rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, x_1 \rangle & \cdots & \langle x_n, x_n \rangle \end{array} \right|^{1/2}, \tag{2.4}$$

where  $\langle \cdot, \cdot \rangle$  denotes the inner product on  $X$ . (If  $X = \mathbb{R}^n$ , then this  $n$ -norm is exactly the same as the Euclidean  $n$ -norm  $\|\cdot, \dots, \cdot\|_E$  mentioned earlier.)

Notice that for  $n = 1$ , the above  $n$ -norm is the usual norm  $\|x_1\|_S = \langle x_1, x_1 \rangle^{1/2}$ , which gives the length of  $x_1$ , while for  $n = 2$ , it defines the standard 2-norm  $\|x_1, x_2\|_S = \{\|x_1\|_S^2 \|x_2\|_S^2 - \langle x_1, x_2 \rangle^2\}^{1/2}$ , which represents the area of the parallelogram spanned by  $x_1$  and  $x_2$ . Further, if  $X = \mathbb{R}^3$ , then  $\|x_1, x_2, x_3\|_S = \|x_1, x_2, x_3\|_E$  is nothing but the volume of the parallelepipeds spanned by  $x_1, x_2$ , and  $x_3$ . In general,  $\|x_1, \dots, x_n\|_S$  represents the volume of the  $n$ -dimensional parallelepiped spanned by  $x_1, \dots, x_n$  in  $X$ .

Now let  $\{e_1, \dots, e_n\}$  be an orthonormal set in  $X$ . Then, by [Theorem 2.1](#), the following function

$$\|x_1, \dots, x_{n-1}\|_\infty := \max \{\|x_1, \dots, x_{n-1}, e_i\|_S : i = 1, \dots, n\} \tag{2.5}$$

defines an  $(n - 1)$ -norm on  $X$ . Further, we have the following fact.

**FACT 2.4.** *On a standard  $n$ -normed space  $X$ , the derived  $(n - 1)$ -norm  $\|\cdot, \dots, \cdot\|_\infty$ , defined with respect to  $\{e_1, \dots, e_n\}$ , is equivalent to the standard  $(n - 1)$ -norm  $\|\cdot, \dots, \cdot\|_S$ . Precisely, we have*

$$\|x_1, \dots, x_{n-1}\|_\infty \leq \|x_1, \dots, x_{n-1}\|_S \leq \sqrt{n} \|x_1, \dots, x_{n-1}\|_\infty \tag{2.6}$$

for all  $x_1, \dots, x_{n-1} \in X$ .

**PROOF.** Assume that  $x_1, \dots, x_{n-1}$  are linearly independent. For each  $i = 1, \dots, n$ , write  $e_i = e_i^\circ + e_i^\perp$  where  $e_i^\circ \in \text{span}\{x_1, \dots, x_{n-1}\}$  and  $e_i^\perp \perp \text{span}\{x_1, \dots, x_{n-1}\}$ . Then we have

$$\begin{aligned} \|x_1, \dots, x_{n-1}, e_i\|_S &= \|x_1, \dots, x_{n-1}, e_i^\perp\|_S \\ &= \left| \begin{array}{cccc} \langle x_1, x_1 \rangle & \cdots & \langle x_1, x_{n-1} \rangle & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \langle x_{n-1}, x_1 \rangle & \cdots & \langle x_{n-1}, x_{n-1} \rangle & 0 \\ 0 & \cdots & 0 & \langle e_i^\perp, e_i^\perp \rangle \end{array} \right|^{1/2} \\ &\leq \left| \begin{array}{ccc} \langle x_1, x_1 \rangle & \cdots & \langle x_1, x_{n-1} \rangle \\ \vdots & \ddots & \vdots \\ \langle x_{n-1}, x_1 \rangle & \cdots & \langle x_{n-1}, x_{n-1} \rangle \end{array} \right|^{1/2} \\ &= \|x_1, \dots, x_{n-1}\|_S. \end{aligned} \tag{2.7}$$

Hence we get  $\|x_1, \dots, x_{n-1}\|_\infty \leq \|x_1, \dots, x_{n-1}\|_S$ .

Next, take a unit vector  $e = \alpha_1 e_1 + \dots + \alpha_n e_n$  such that  $e \perp \text{span}\{x_1, \dots, x_{n-1}\}$ . (Here we are still assuming that  $x_1, \dots, x_{n-1}$  are linearly independent.) Then, by properties (3) and (4) of the  $n$ -norm, we have

$$\begin{aligned} \|x_1, \dots, x_{n-1}\|_S &= \|x_1, \dots, x_{n-1}, e\|_S \\ &\leq |\alpha_1| \|x_1, \dots, x_{n-1}, e_1\|_S + \dots + |\alpha_n| \|x_1, \dots, x_{n-1}, e_n\|_S \\ &\leq (|\alpha_1| + \dots + |\alpha_n|) \|x_1, \dots, x_{n-1}\|_\infty. \end{aligned} \tag{2.8}$$

But, by the Cauchy-Schwarz inequality, we have

$$\sum_{i=1}^n |\alpha_i| \leq \left\{ \sum_{i=1}^n 1^2 \right\}^{1/2} \left\{ \sum_{i=1}^n |\alpha_i|^2 \right\}^{1/2} = \sqrt{n}. \tag{2.9}$$

Hence we obtain

$$\|x_1, \dots, x_{n-1}\|_S \leq \sqrt{n} \|x_1, \dots, x_{n-1}\|_\infty, \tag{2.10}$$

and this completes the proof. □

**2.2. The finite-dimensional case.** For finite-dimensional  $n$ -normed space  $(X, \|\cdot, \dots, \cdot\|)$ , we can in general derive an  $(n - 1)$ -norm from the  $n$ -norm in the following way. Take a linearly independent set  $\{a_1, \dots, a_m\}$  in  $X$ , with  $n \leq m \leq d$ . With respect to  $\{a_1, \dots, a_m\}$ , define the following function  $\|\cdot, \dots, \cdot\|_\infty$  on  $X^{n-1}$  by

$$\|x_1, \dots, x_{n-1}\|_\infty := \max \{ \|x_1, \dots, x_{n-1}, a_i\| : i = 1, \dots, m \}. \tag{2.11}$$

Then, as in [Theorem 2.1](#), the function  $\|\cdot, \dots, \cdot\|_\infty$  defines an  $(n - 1)$ -norm on  $X$ .

As we will see later, we can obtain a better  $(n - 1)$ -norm by using a set of  $d$ , rather than just  $n$ , linearly independent vectors in  $X$  (that is, by using a basis for  $X$ ).

**3. Applications and further results.** Recall that a sequence  $x(k)$  in an  $n$ -normed space  $(X, \|\cdot, \dots, \cdot\|)$  is said to *converge* to an  $x \in X$  (in the  $n$ -norm) whenever

$$\lim_{k \rightarrow \infty} \|x_1, \dots, x_{n-1}, x(k) - x\| = 0 \tag{3.1}$$

for every  $x_1, \dots, x_{n-1} \in X$ .

The following proposition says that the convergence in the  $n$ -norm implies the convergence in the derived  $(n - 1)$ -norm  $\|\cdot, \dots, \cdot\|_\infty$ , defined with respect to an arbitrary linearly independent set  $\{a_1, \dots, a_n\}$  in  $X$ .

**PROPOSITION 3.1.** *If  $x(k)$  converges to an  $x \in X$  in the  $n$ -norm, then  $x(k)$  also converges to  $x$  in the derived  $(n - 1)$ -norm  $\|\cdot, \dots, \cdot\|_\infty$ , that is,*

$$\lim_{k \rightarrow \infty} \|x_1, \dots, x_{n-2}, x(k) - x\|_\infty = 0 \tag{3.2}$$

for every  $x_1, \dots, x_{n-2} \in X$ .

**PROOF.** If  $x(k)$  converges to an  $x \in X$  in the  $n$ -norm, then

$$\lim_{k \rightarrow \infty} \|x_1, \dots, x_{n-2}, x(k) - x, a_i\| = 0 \tag{3.3}$$

for every  $x_1, \dots, x_{n-2} \in X$  and  $i = 1, \dots, n$ , and hence

$$\lim_{k \rightarrow \infty} \|x_1, \dots, x_{n-2}, x(k) - x\|_\infty = 0 \tag{3.4}$$

for every  $x_1, \dots, x_{n-2} \in X$ , that is,  $x(k)$  converges to  $x$  in the derived  $(n - 1)$ -norm  $\|\cdot, \dots, \cdot\|_\infty$ .  $\square$

**3.1. The standard case.** In a standard  $n$ -normed space  $(X, \|\cdot, \dots, \cdot\|_S)$ , the converse of Proposition 3.1 is also true, especially when the derived  $(n - 1)$ -norm  $\|\cdot, \dots, \cdot\|_\infty$  is defined with respect to an orthonormal set  $\{e_1, \dots, e_n\}$  in  $X$  as in Section 2.1.

**FACT 3.2.** *A sequence in a standard  $n$ -normed space  $X$  is convergent in the  $n$ -norm if and only if it is convergent in the derived  $(n - 1)$ -norm  $\|\cdot, \dots, \cdot\|_\infty$ .*

**PROOF.** Suppose that  $x(k)$  converges to an  $x \in X$  in the derived  $(n - 1)$ -norm  $\|\cdot, \dots, \cdot\|_\infty$ . We want to show that  $x(k)$  also converges to  $x$  in the  $n$ -norm. Take  $x_1, \dots, x_{n-1} \in X$ . Then one may observe that

$$\|x_1, \dots, x_{n-2}, x_{n-1}, x(k) - x\|_S \leq \|x_1, \dots, x_{n-2}, x(k) - x\|_S \|x_{n-1}\|_S, \tag{3.5}$$

where  $\|\cdot, \dots, \cdot\|_S$  and  $\|\cdot\|_S$  on the right-hand side denote the standard  $(n - 1)$ -norm and the usual norm on  $X$ , respectively. By Fact 2.4, we have

$$\|x_1, \dots, x_{n-2}, x_{n-1}, x(k) - x\|_S \leq \sqrt{n} \|x_1, \dots, x_{n-2}, x(k) - x\|_\infty \|x_{n-1}\|_S. \tag{3.6}$$

But  $\lim_{k \rightarrow \infty} \|x_1, \dots, x_{n-2}, x(k) - x\|_\infty = 0$ , and so we conclude that

$$\lim_{k \rightarrow \infty} \|x_1, \dots, x_{n-1}, x(k) - x\|_S = 0, \tag{3.7}$$

that is,  $x(k)$  converges to  $x$  in the  $n$ -norm.  $\square$

**COROLLARY 3.3.** *A sequence in a standard  $n$ -normed space is convergent in the  $n$ -norm if and only if it is convergent in the standard  $(n - 1)$ -norm and, by induction, in the standard  $(n - r)$ -norm for all  $r = 1, \dots, n - 1$ . In particular, a sequence in a standard  $n$ -normed space is convergent in the  $n$ -norm if and only if it is convergent in the usual norm  $\|\cdot\|_S := \langle \cdot, \cdot \rangle^{1/2}$ .*

**3.2. The finite-dimensional case.** We also have a similar result for finite-dimensional  $n$ -normed space  $(X, \|\cdot, \dots, \cdot\|)$ . Let  $\{b_1, \dots, b_d\}$  be a basis for  $X$ . With respect to  $\{b_1, \dots, b_d\}$ , define the following function  $\|\cdot, \dots, \cdot\|_\boxtimes$  on  $X^{n-1}$  by

$$\|x_1, \dots, x_{n-1}\|_\boxtimes := \max \{ \|x_1, \dots, x_{n-1}, b_i\| : i = 1, \dots, d \}. \tag{3.8}$$

Then, as mentioned before, the function  $\|\cdot, \dots, \cdot\|_\boxtimes$  defines an  $(n - 1)$ -norm on  $X$ .

With this derived  $(n - 1)$ -norm, we have the following result.

**PROPOSITION 3.4.** *A sequence in the finite-dimensional  $n$ -normed space  $X$  is convergent in the  $n$ -norm if and only if it is convergent in the derived  $(n - 1)$ -norm  $\|\cdot, \dots, \cdot\|_\boxtimes$ .*

**PROOF.** If a sequence in  $X$  is convergent in the  $n$ -norm, then it will certainly be convergent in the  $(n - 1)$ -norm  $\|\cdot, \dots, \cdot\|_{\boxtimes}$ . Conversely, suppose that  $x(k)$  converges to an  $x \in X$  in  $\|\cdot, \dots, \cdot\|_{\boxtimes}$ . Take  $x_1, \dots, x_{n-1} \in X$ . Writing  $x_{n-1} = \alpha_1 b_1 + \dots + \alpha_d b_d$ , we get

$$\begin{aligned} \|\|x_1, \dots, x_{n-2}, x_{n-1}, x(k) - x\| &\leq |\alpha_1| \|\|x_1, \dots, x_{n-2}, x(k) - x, b_1\| \\ &\quad + \dots + |\alpha_d| \|\|x_1, \dots, x_{n-2}, x(k) - x, b_d\| \\ &\leq (|\alpha_1| + \dots + |\alpha_d|) \|\|x_1, \dots, x_{n-2}, x(k) - x\|_{\boxtimes}. \end{aligned} \tag{3.9}$$

But  $\lim_{k \rightarrow \infty} \|\|x_1, \dots, x_{n-2}, x(k) - x\|_{\boxtimes} = 0$ , and so we obtain

$$\lim_{k \rightarrow \infty} \|\|x_1, \dots, x_{n-1}, x(k) - x\| = 0, \tag{3.10}$$

that is,  $x(k)$  converges to  $x$  in the  $n$ -norm. □

**3.3. The standard, separable case.** We go back to the standard case, where  $X$  is a real inner product space of dimension  $d \geq n$  equipped with the standard  $n$ -norm  $\|\cdot, \dots, \cdot\|_S$  as in Section 2.1. But suppose now that  $X$  is separable and that  $\{e_i : i \in I_d\}$ , where  $I_d := \{1, \dots, d\}$  (if  $d < \infty$ ) or  $\mathbb{N}$  (if  $d = \infty$ ), is an orthonormal basis for  $X$ . For every  $x_1, \dots, x_{n-1} \in X$  and every basis vector  $e_i$  ( $i \in I_d$ ), we have

$$\|\|x_1, \dots, x_{n-1}, e_i\|_S \leq \|\|x_1, \dots, x_{n-1}\|_S, \tag{3.11}$$

where  $\|\cdot, \dots, \cdot\|_S$  on the right-hand side denotes the standard  $(n - 1)$ -norm on  $X$ . Hence, with respect to  $\{e_i : i \in I_d\}$ , we may define the function  $\|\cdot, \dots, \cdot\|_{\boxtimes}$  on  $X^{n-1}$  by

$$\|\|x_1, \dots, x_{n-1}\|_{\boxtimes} := \sup \{\|\|x_1, \dots, x_{n-1}, e_i\|_S : i \in I_d\} \tag{3.12}$$

and check that it also defines an  $(n - 1)$ -norm on  $X$ . Moreover, we have the following relation between the two derived  $(n - 1)$ -norms  $\|\cdot, \dots, \cdot\|_{\boxtimes}$  and  $\|\cdot, \dots, \cdot\|_{\infty}$  (the latter being defined with respect to  $\{e_1, \dots, e_n\}$  only):

$$\|\|x_1, \dots, x_{n-1}\|_{\infty} \leq \|\|x_1, \dots, x_{n-1}\|_{\boxtimes} \leq \|\|x_1, \dots, x_{n-1}\|_S \leq \sqrt{n} \|\|x_1, \dots, x_{n-1}\|_{\infty} \tag{3.13}$$

for every  $x_1, \dots, x_{n-1} \in X$ . Hence we conclude the following fact.

**FACT 3.5.** *On a standard  $n$ -normed space  $X$ , the two derived  $(n - 1)$ -norms  $\|\cdot, \dots, \cdot\|_{\infty}$  and  $\|\cdot, \dots, \cdot\|_{\boxtimes}$  and the standard  $(n - 1)$ -norm  $\|\cdot, \dots, \cdot\|_S$  are equivalent. Accordingly, a sequence in a standard  $n$ -normed space  $X$  is convergent in the  $n$ -norm if and only if it is convergent in one of the three  $(n - 1)$ -norms.*

**3.4. Cauchy sequences, completeness and fixed point theorem.** Recall that a sequence  $x(k)$  in an  $n$ -normed space  $(X, \|\cdot, \dots, \cdot\|)$  is called *Cauchy* (with respect to the  $n$ -norm) if

$$\lim_{k, l \rightarrow \infty} \|\|x_1, \dots, x_{n-1}, x(k) - x(l)\| = 0 \tag{3.14}$$

for every  $x_1, \dots, x_{n-1} \in X$ . If every Cauchy sequence in  $X$  converges to an  $x \in X$ , then  $X$  is said to be *complete* (with respect to the  $n$ -norm). A complete  $n$ -normed space is then called an  *$n$ -Banach space*.

By replacing the phrases “ $x(k)$  converges to  $x$ ” with “ $x(k)$  is Cauchy” and “ $x(k) - x$ ” with “ $x(k) - x(l)$ ,” we see that the analogues of [Proposition 3.1](#), [Fact 3.2](#), [Corollary 3.3](#), [Proposition 3.4](#), and [Fact 3.5](#) hold for Cauchy sequences.

Hence, for the standard or finite-dimensional case, we have the following result.

**PROPOSITION 3.6.** (a) *A standard  $n$ -normed space is complete if and only if it is complete with respect to one of the three  $(n - 1)$ -norms  $\|\cdot, \dots, \cdot\|_\infty$ ,  $\|\cdot, \dots, \cdot\|_\boxtimes$ , or  $\|\cdot, \dots, \cdot\|_S$ . By induction, a standard  $n$ -normed space is complete if and only if it is complete with respect to the usual norm  $\|\cdot\|_S := \langle \cdot, \cdot \rangle^{1/2}$ .*

(b) *A finite-dimensional  $n$ -normed space is complete if and only if it is complete with respect to the derived  $(n - 1)$ -norm  $\|\cdot, \dots, \cdot\|_\boxtimes$ .*

Consequently, we have the following result.

**COROLLARY 3.7** (fixed point theorem). *Let  $(X, \|\cdot, \dots, \cdot\|)$  be a standard or finite-dimensional  $n$ -Banach space, and  $T$  a contractive mapping of  $X$  into itself, that is, there exists a constant  $C \in (0, 1)$  such that*

$$\|x_1, \dots, x_{n-1}, Ty - Tz\| \leq C \|x_1, \dots, x_{n-1}, y - z\| \tag{3.15}$$

for all  $x_1, \dots, x_{n-1}, y, z$  in  $X$ . Then  $T$  has a unique fixed point in  $X$ .

**PROOF.** First consider the case  $n = 2$  (see [8]). By [Proposition 3.6](#), we know that  $X$  is a Banach space with respect to the derived norm  $\|\cdot\|_\infty$  (for standard case) or  $\|\cdot\|_\boxtimes$  (for finite-dimensional case). Since the mapping  $T$  is also contractive with respect to  $\|\cdot\|_\infty$  or  $\|\cdot\|_\boxtimes$ , we conclude by the fixed point theorem for Banach spaces that  $T$  has a unique fixed point in  $X$ . For  $n > 2$ , the result follows by induction.  $\square$

**REMARK 3.8.** In the finite-dimensional case, it is actually enough to assume that  $X$  is an  $n$ -normed space because we know that all finite-dimensional normed spaces are complete and, by [Proposition 3.6\(b\)](#), so are all finite-dimensional  $n$ -normed spaces.

**4. Concluding remark.** We have shown that an  $n$ -normed space with  $n \geq 2$  is an  $(n - 1)$ -normed space and that, for the standard or finite-dimensional case, the  $(n - 1)$ -norm can be derived from the  $n$ -norm in such a way that the convergence and completeness in the  $n$ -norm is equivalent to those in the derived  $(n - 1)$ -norm.

Below is an example of a non-standard, infinite-dimensional 2-normed space for which we can derive a norm from the 2-norm such that the convergence and completeness in the 2-norm is equivalent to those in the derived norm.

Let  $X = l^\infty$ , the space of bounded sequences of real numbers. Equip  $X$  with the following 2-norm

$$\|x, y\| := \sup_{i \in \mathbb{N}} \sup_{j \in \mathbb{N}} |x_i y_j - x_j y_i|, \tag{4.1}$$

where  $x = (x_1, x_2, x_3, \dots)$  and  $y = (y_1, y_2, y_3, \dots)$ . Let  $a_1 = (1, 0, 0, \dots)$  and  $a_2 = (0, 1, 0, \dots)$ .

With respect to  $\{a_1, a_2\}$ , we derive the norm  $\|\cdot\|_\infty$  via

$$\|x\|_\infty := \max\{\|x, a_1\|, \|x, a_2\|\}. \quad (4.2)$$

But  $\|x, a_1\| = \sup_{i \in \mathbb{N} \setminus \{1\}} |x_i|$  and  $\|x, a_2\| = \sup_{i \in \mathbb{N} \setminus \{2\}} |x_i|$ , and so we obtain

$$\|x\|_\infty = \sup_{i \in \mathbb{N}} |x_i|, \quad (4.3)$$

the usual norm on  $l^\infty$ .

Now suppose that  $x(k)$  is a sequence in  $X$  that converges to  $x$  in the derived norm  $\|\cdot\|_\infty$ . For every  $y \in X$ , we have

$$\begin{aligned} \|x(k) - x, y\| &= \sup_{i \in \mathbb{N}} \sup_{j \in \mathbb{N}} |(x_i(k) - x_i)y_j - (x_j(k) - x_j)y_i| \\ &\leq \sup_{i \in \mathbb{N}} \sup_{j \in \mathbb{N}} |x_i(k) - x_i| |y_j| + |x_j(k) - x_j| |y_i| \\ &\leq 2\|x(k) - x\|_\infty \|y\|_\infty, \end{aligned} \quad (4.4)$$

whence  $\lim_{k \rightarrow \infty} \|x(k) - x, y\| = 0$ . Hence  $x(k)$  converges to  $x$  in the 2-norm  $\|\cdot, \cdot\|$ .

Thus, for this particular example, we see that the convergence in the 2-norm is equivalent to that in the derived norm. By similar arguments, we can also verify that the completeness in the 2-norm is equivalent to that in the derived norm.

For general non-standard, infinite-dimensional  $n$ -normed spaces, however, it is unknown whether we can always derive an  $(n-1)$ -norm from the  $n$ -norm such that the convergence and completeness in the  $n$ -norm is equivalent to those in the derived  $(n-1)$ -norm.

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