MATRIX TRANSFORMATIONS FROM ABSOLUTELY CONVERGENT SERIES TO CONVERGENT SEQUENCES AS GENERAL WEIGHTED MEAN SUMMABILITY METHODS

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Abstract. We prove the necessary and sufficient conditions for an infinity matrix to be a mapping, from absolutely convergent series to convergent sequences, which is treated as general weighted mean summability methods. The results include a classical result by Hardy and another by Moricz and Rhoades as particular cases.

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1. Introduction. A series

$$\sum_{k=0}^{\infty} x_k$$

of complex numbers is said to be summable \((C,1)\) if the sequence

$$\frac{1}{n+1} \sum_{k=0}^{n} \sum_{i=0}^{k} x_{i,n} \quad n = 0,1,2,\ldots$$

converges to a finite limit as \(n \to \infty\).

In [1] Hardy proved the following theorem.

**Theorem 1.1.** The series (1.1) is summable \((C,1)\) to a finite number \(L\) if and only if the series

$$\sum_{n=0}^{\infty} \sum_{k=n}^{\infty} \frac{x_k}{k+1}$$

converges to the same limit.

For a sequence of positive numbers \((p_n)\), let \(P_n := \sum_{k=0}^{n} p_n\). A weighted mean matrix \(\tilde{N}\) is an infinity lower triangular matrix with entries (see [2])

$$a_{nk} := \frac{p_k}{P_n}, \quad k = 0,1,2,\ldots, n, \quad n = 0,1,2,\ldots$$

(1.4)

The series (1.1) is said to be summable \(\tilde{N}\) if the following sequence:

$$\frac{1}{P_n} \sum_{k=0}^{n} p_k \sum_{i=0}^{k} x_{i,n} \quad n = 0,1,2,\ldots,$$

(1.5)

converges to a finite limit as \(n \to \infty\).
It is clear that summable \((C, 1)\) is a special case of summable \(\tilde{N}\), where
\[
p_k = 1, \quad k = 0, 1, 2, \ldots.
\] (1.6)

Based on the above idea, Moricz and Rhoades [2] established a result for a broad class of summability methods, which include the method of summability \((C, 1)\) as a particular case.

**Theorem 1.2.** Let \(\tilde{N}\) be a weighted mean matrix determined by a sequence \((p_n)\) of positive numbers such that the following conditions are satisfied:
\[
P_n \to \infty, \quad p_n \to 0 \quad \text{as} \quad n \to \infty,
\]
\[
\sup_{n \geq 0} \left\{ \frac{p_{n+1} p_{n-1}}{p_n} + p_n \sum_{k=n}^{\infty} \frac{1}{p_{n+1}} \left| \frac{p_{k+1} p_k}{p_k} - \frac{p_{k+2} p_k}{p_{k+1} p_{k+2}} \right| \right\} < \infty, \quad (1.7)
\]

\[
\sup_{n \geq 0} \left\{ \frac{p_n}{p_{n+1}} + \frac{1}{p_n} \sum_{k=0}^{n} \left| \frac{p_k P_k + 1}{p_k} - \frac{p_k P_k - 1}{p_k} \right| \right\} < \infty,
\]

with the agreement that
\[
p_{-1} = P_{-1} := 0. \quad (1.8)
\]

Then the series (1.1) is summable \(\tilde{N}\) to a finite number \(L\) if and only if the series
\[
\sum_{n=0}^{\infty} \sum_{k=n}^{\infty} \frac{p_n}{p_k} x_k \quad (1.9)
\]
converges to the same limit \(L\).

In this paper, we will study the matrix transformations from the space of absolutely convergent series of complex numbers, \(l_1\), to the space of convergent sequences of complex numbers, \(c\). Then we shall establish a more general result for a broader class of weighted mean methods, which includes the method of summable \(\tilde{N}\) as a particular case if the series (1.1) is absolutely convergent.

2. Matrix transformations from \(l_1\) to \(c\). Let \(A = (a_{nk})\) be an infinity matrix with complex entries and let \(l\) denote the linear space of complex number sequences. For a sequence \(x = (x_n) \in l\), \(Ax\) is in \(l\) and its entries are given by
\[
(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k, \quad n = 0, 1, 2, \ldots \quad (2.1)
\]
provided the series converges to a finite complex number.

**Proposition 2.1.** Let \(a = (a_k)\) be a sequence of complex numbers. If for every \(x = (x_n) \in l_1\), the series
\[
\sum_{k=0}^{\infty} a_k x_k \quad (2.2)
\]
converges to a finite complex number, then

$$\sup_{k \geq 0} \{ |a_k| \} < \infty.$$  \hfill (2.3)

From Proposition 2.1, we have the following interesting result.

**Proposition 2.2.** Let $a = (a_k)$ be a sequence of complex numbers. If for every $x = (x_n) \in l_1$, the series

$$\sum_{k=0}^{\infty} a_k x_k$$  \hfill (2.4)

converges to a finite complex number, then the linear functional $f_a$ defined on $l_1$ by

$$f_a(x) = \sum_{k=0}^{\infty} a_k x_k$$  \hfill (2.5)

is a continuous (bounded) linear functional on $l_1$, such that

$$\|f_a\| = \sup_{k \geq 0} \{ |a_k| \}.$$  \hfill (2.6)

From Proposition 2.1, we know that $A$ is well defined as a mapping from $l_1$ to $l$, if and only if

$$\sup_{k \geq 0} \{ |a_{nk}| \} < \infty, \quad \text{for } n = 0, 1, 2, \ldots.$$  \hfill (2.7)

The following result has been proved in [4] by using functional analysis techniques. It is also proved by summability methods. We list the following theorem without proof.

**Theorem 2.3.** Let $A = (a_{nk})$ be an infinity matrix with complex entries. Then $A$ is a mapping from $l_1$ to $c$, if and only if the following conditions are satisfied:

(i) for every fixed $k = 0, 1, 2, \ldots$, the sequence $(a_{nk})$ converges to a finite limit as $n \to \infty$,

(ii) $\sup_{n,k \geq 0} \{ |a_{nk}| \} < \infty$.

Furthermore, if $A = (a_{nk})$ satisfies conditions (i) and (ii), then for every $x = (x_n) \in l_1$, we have

$$\lim_{n \to \infty} (Ax)_n = \lim_{n \to \infty} \sum_{k=0}^{\infty} a_{nk} x_k = \sum_{k=0}^{\infty} \left( \lim_{n \to \infty} a_{nk} \right) x_k.$$  \hfill (2.8)

The following corollary follows from Theorem 2.3 and (2.8).

**Corollary 2.4.** Let $A = (a_{nk})$ be an infinity matrix with complex entries. If $A$ is a mapping from $l_1$ to $c$, then the linear operator $A$ is continuous (bounded) linear operator such that

$$\|A\| = \sup_{n,k \geq 0} \{ |a_{nk}| \}.$$  \hfill (2.9)
3. Applications to summable \( (C,1) \) and summable \( \bar{N} \). The following corollary comes immediately from Theorem 2.3, which describes an equivalent reformulation of summability by more general weighted mean methods which are matrix transformations.

**Corollary 3.1.** Let \( A = (a_{nk}) \), \( B = (b_{nk}) \) be two infinity matrices with complex entries. Suppose \( A, B \) are mapping from \( l_1 \) to \( c \), that is \( A, B \) satisfying conditions (i), (ii) of Theorem 2.3. Then for every \( x = (x_n) \in l_1 \),

\[
\lim_{n \to \infty} (Ax)_n = \lim_{n \to \infty} (Bx)_n \tag{3.1}
\]

if and only if for every fixed \( k = 0, 1, 2, \ldots \),

\[
\lim_{n \to \infty} a_{nk} = \lim_{n \to \infty} b_{nk}. \tag{3.2}
\]

**Proof.** Since \( A, B \) satisfy conditions (i), (ii) of Theorem 2.3, then from (2.8), we have

\[
\lim_{n \to \infty} (Ax)_n = \lim_{n \to \infty} \sum_{k=0}^{\infty} a_{nk} x_k = \sum_{k=0}^{\infty} \left( \lim_{n \to \infty} a_{nk} \right) x_k, \tag{3.3}
\]

\[
\lim_{n \to \infty} (Bx)_n = \lim_{n \to \infty} \sum_{k=0}^{\infty} b_{nk} x_k = \sum_{k=0}^{\infty} \left( \lim_{n \to \infty} b_{nk} \right) x_k, \tag{3.4}
\]

for any \( x = (x_n) \in l_1 \). From (2.8) and (3.4), we see that (3.2) implies (3.1). Now, for every fixed \( k = 0, 1, 2, \ldots \), define \( x = (x_i) \) by

\[
x_i = \begin{cases} 
1, & \text{if } i = k, \\
0, & \text{if } i \neq k.
\end{cases} \tag{3.5}
\]

It is clear that \( x \in l_1 \). Equations (2.8) and (3.4) imply

\[
\lim_{n \to \infty} (Ax)_n = \lim_{n \to \infty} a_{nk}, \quad \lim_{n \to \infty} (Bx)_n = \lim_{n \to \infty} b_{nk}. \tag{3.6}
\]

From (3.6), we see that (3.1) implies (3.2). \( \square \)

Recall that for a sequence of positive numbers \((p_n)\), \( P_n = \sum_{k=0}^{n} p_k \). The series (1.1) is said to be summable \( \bar{N} \) if the following sequence:

\[
\frac{1}{P_n} \sum_{k=n}^{n} p_k \sum_{i=0}^{k} x_i, \quad n = 0, 1, 2, \ldots \tag{3.7}
\]

converges to a finite limit as \( n \to \infty \).

To generalize Theorem 1.2, we shall construct two weighted mean matrices according to the summability (3.7) and the following summability method:

\[
\sum_{n=0}^{\infty} \sum_{k=n}^{\infty} \frac{p_n}{P_k} x_k. \tag{3.8}
\]
Based on the sequence of positive numbers \((p_n)\), define two infinity matrices \(A = (a_{nk})\) and \(B = (b_{nk})\), by

\[
a_{nk} = \begin{cases} 
0, & \text{if } k > n, \\
\frac{p_n - p_{k-1}}{p_n}, & \text{if } k \leq n,
\end{cases}
\]  

(3.9)

\[
b_{nk} = \begin{cases} 
\frac{p_n}{P_k}, & \text{if } k > n, \\
1, & \text{if } k \leq n,
\end{cases}
\]  

(3.10)

where we agree that \(P_{-1} = 0\).

It can be seen that any sequence of positive numbers \((p_n)\), \(B = (b_{nk})\) defined by (3.10), always satisfies the conditions (i) and (ii) of Theorem 2.3 and \(A = (a_{nk})\) defined by (3.9) always satisfies the conditions (ii) of Theorem 2.3. Furthermore, \(A = (a_{nk})\) will satisfies the conditions (i) of Theorem 2.3 if the sequence \((p_n)\) satisfies

\[P_n \rightarrow \infty \quad \text{as } n \rightarrow \infty.\]  

(3.11)

Hence we have the following corollary of Theorem 2.3.

**Corollary 3.2.** For any sequence of positive numbers \((p_n)\), \(B = (b_{nk})\) defined by (3.10) is always a mapping from \(l_1\) to \(c\). If \((p_n)\) satisfying (3.11), then \(A = (a_{nk})\) defined by (3.9) is a mapping from \(l_1\) to \(c\).

The following corollary will give the Moricz and Rhoades’s result, Theorem 1.2, if the series (1.1) is absolutely convergent.

**Corollary 3.3.** Let \((p_n)\) be a sequence of positive numbers satisfying (3.11). Let \(A = (a_{nk})\), \(B = (b_{nk})\) be defined by (3.9) and (3.10). Then for any \(x = (x_n) \in l_1\), we have

\[
\lim_{n \to \infty} (Ax)_n = \lim_{n \to \infty} (Bx)_n = \sum_{k=0}^{\infty} x_k.
\]  

(3.12)

**Proof.** Notice that under condition (3.11), we have that for every fixed \(k = 0, 1, 2, \ldots\),

\[
\lim_{n \to \infty} a_{nk} = \lim_{n \to \infty} b_{nk} = 1.
\]  

(3.13)

Then the proof of this corollary follows Corollary 3.2 and the equalities (2.8) and (3.4) immediately. \(\square\)

From the definitions (3.9) and (3.10), we see that for every fixed \(n = 0, 1, 2, \ldots\),

\[
(Ax)_n = \frac{1}{p_n} \sum_{k=0}^{n} p_k \sum_{i=0}^{k} x_i, \quad (Bx)_n = \sum_{m=0}^{n} \sum_{k=m}^{\infty} \frac{p_n}{P_k} x_k.
\]  

(3.14)

Corollary 3.3 shows that if the sequence of positive numbers \((p_n)\) satisfies condition (3.11), then for any \(x = (x_n) \in l_1\), we have

\[
\lim_{n \to \infty} \frac{1}{p_n} \sum_{k=0}^{n} p_k \sum_{i=0}^{k} x_i = \sum_{n=0}^{\infty} \sum_{k=n}^{\infty} \frac{p_n}{P_k} x_k = \sum_{k=0}^{\infty} x_k.
\]  

(3.15)
In a particular case, as mentioned by Moricz and Rhoades [2], taking $p_k = 1$, for $k = 0, 1, 2, \ldots$, we find the Hardy’s result, Theorem 1.1, if that the series (1.1) is absolutely convergent, that is, for any $x = (x_n) \in l_1$,

$$\lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} \sum_{i=0}^{k} x_i = \sum_{n=0}^{\infty} \sum_{k=n}^{\infty} \frac{x_k}{k+1} = \sum_{k=0}^{\infty} x_k. \quad (3.16)$$

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**REFERENCES**


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