ON A FUNCTIONAL EQUATION RELATED TO A GENERALIZATION OF FLETT'S MEAN VALUE THEOREM

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Abstract. In this paper, we characterize all the functions that attain their Flett mean value at a particular point between the endpoints of the interval under consideration. These functions turn out to be cubic polynomials and thus, we also characterize these.

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1. Introduction. In [5], Sahoo and Riedel gave a generalization of Flett’s mean value theorem [2] as follows.

Theorem 1.1. Let \( f \) be a real valued function which is differentiable in \([a,b]\), then there is a point \( c \in (a,b) \) such that

\[
f(c) - f(a) = (c - a) f'(c) - \frac{1}{2} \frac{f'(b) - f'(a)}{b - a} (c - a)^2.
\]

It is easy to see that if \( f'(b) = f'(a) \), then this reduces to Flett’s mean value theorem.

Aczél [1] and Haruki [3] used the Lagrange mean value theorem to ask the question of which functions attained their mean value at a prescribed point \( c \in (a,b) \), in particular, at the midpoint \( c = (a + b)/2 \). The answer is that only quadratic polynomials have the property that the mean value on any interval is attained at the midpoint of that interval. A natural question to ask is this same question for the above mean value theorem. It turns out that quadratic polynomials satisfy (1.1) for any \( c \), but, more interestingly, cubic polynomials satisfy it for \( c = (a + 3b)/4 \). Thus, the main question becomes whether cubic polynomials are the only functions having this property.

Following the approach in [1], we pexiderize (1.1) to obtain

\[
f(c) - f(a) = (c - a) h(c) - \frac{1}{2} \frac{h(b) - h(a)}{b - a} (c - a)^2,
\]

and now setting \( c = (a + 3b)/4 \) yields

\[
f\left(\frac{a + 3b}{4}\right) - f(a) = \frac{3}{4} (b - a) h\left(\frac{a + 3b}{4}\right) - \frac{9}{32} (b - a) (h(b) - h(a))
\]

or

\[
f\left(\frac{a + 3b}{4}\right) - f(a) = \frac{3}{4} (b - a) \left[ h\left(\frac{a + 3b}{4}\right) - \frac{3}{8} (h(b) - h(a)) \right].
\]
More generally, setting \( c = sa + tb \) with \( s + t = 1 \) and \( 0 < s, t < 1 \), we obtain

\[
f(sa + tb) - f(a) = (sa + tb - a)h(sa + tb) - \frac{1}{2} \frac{h(b) - h(a)}{b - a} (sa + tb - a)^2. \tag{1.5}
\]

The question we answer, in this paper, is: What are the functions \( f, h \) that satisfy the functional equations (1.4) and (1.5) for all \( a, b \in \mathbb{R} \)? In solving this functional equation, we do not assume any regularity conditions on \( f \) or \( h \).

2. Solution of the functional equation. The main work in solving this functional equation is to reduce (1.4) and (1.5) to a form where we can apply the following result by Székelyhidi [6, Thm. 9.5] and Wilson [7].

**Theorem 2.1.** Let \( G, S \) be commutative groups, \( n \) a nonnegative integer, \( \varphi_i, \psi_i \) additive functions from \( G \) into \( G \) and let \( \text{Ran}(\varphi_i) \subseteq \text{Ran}(\psi_i) (i = 1, \ldots, n + 1) \). Then if \( h, h_i, \varphi_i, \psi_i (i = 1, \ldots, n + 1) \) satisfy

\[
h(x) + \sum_{i=1}^{n+1} h_i(\varphi_i(x) + \psi_i(t)) = 0, \tag{2.1}
\]

then \( h \) is a generalized polynomial of degree at most \( n \).

Thus, we are able to prove our main result (Theorem 2.2).

**Theorem 2.2.** The real valued functions \( f \) and \( h \) are solutions of the functional equation (1.5) if and only if

\[
f(x) = \begin{cases} Ax^3 + Bx^2 + Cx + D & \text{if } s = \frac{1}{4}, t = \frac{3}{4}, \\ Bx^2 + Cx + D & \text{if } s \neq \frac{1}{4}, t = \frac{3}{4}, \end{cases} \tag{2.2}
\]

\[
h(x) = \begin{cases} 3Ax^2 + 2Bx + C & \text{if } s = \frac{1}{4}, t = \frac{3}{4}, \\ 2Bx + C & \text{if } s \neq \frac{1}{4}, t = \frac{3}{4}. \end{cases} \tag{2.3}
\]

**Proof.** It is easy to check that the functions \( f \) and \( h \), given above, do satisfy the functional equation (1.5).

To show that these are the only solutions, we start by rewriting (1.5) using \( s + t = 1 \) as follows:

\[
f(a + t(b - a)) - f(a) = t(b - a) \left[ h(a + t(b - a)) - \frac{t}{2} [h(b) - h(a)] \right]. \tag{2.4}
\]

Now, letting \( u = (b - a)/3 \), we obtain

\[
f(a + 3tu) - f(a) = 3tu \left[ h(a + 3tu) - \frac{t}{2} [h(3u + a) - h(a)] \right]. \tag{2.5}
\]

Now, we replace \( a \) by \( a - tu \) in (2.5) and get

\[
f(a + 2tu) - f(a - tu) = 3tu \left[ h(a + 2tu) - \frac{t}{2} [h((3-t)u + a) - h(a - tu)] \right]. \tag{2.6}
\]
Similarly, using $a = a - 2tu$ in (2.5), we get

$$f(a + tu) - f(a - 2tu) = 3tu \left[ h(a + tu) - \frac{t}{2} \left[ h((3 - 2t)u + a) - h(a - 2tu) \right] \right].$$  

(2.7)

Interchanging $u$ with $-u$ in (2.7) gives

$$f(a - tu) - f(a + 2tu) = 3tu \left[ h(a - tu) - \frac{t}{2} \left[ h((3 - 2t)u + a) - h(a + 2tu) \right] \right].$$  

(2.8)

Comparing (2.8) and (2.6) gives, for $a, u \in \mathbb{R}$,

$$\left[ h(a - tu) - \frac{t}{2} \left[ h((3 - 2t)u + a) - h(a + 2tu) \right] \right] = \left[ h(a + 2tu) - \frac{t}{2} \left[ h((3 - t)u + a) - h(a - tu) \right] \right],$$  

(2.9)

which simplifies to

$$t \left[ h((3 - t)u + a) - h((-3 + 2t)u + a) - (h(a - tu) - h(a + 2tu)) \right] = -2 \left[ h(a - tu) - h(a + 2tu) \right].$$  

(2.10)

Collecting the terms of $h$ that have the same argument, we obtain

$$(2 - t)h(a + 2tu) - (2 - t)h(a - tu) - th((3 - t)u + a) + th((-3 + 2t)u + a) = 0.$$  

(2.11)

Writing $x = a + 2tu$ and dividing (2.11) by $(2 - t)$ yields

$$h(x) - h(x - 3tu) - \frac{t}{2 - t} h(x + 3(1 - t)u) + \frac{t}{2 - t} h(x - 3u) = 0.$$  

(2.12)

Thus, since $t \neq 0$ is fixed, (2.12) is of the form of equation (2.1) and hence, $h(x)$ is a generalized polynomial of degree at most 2,

$$h(x) = \beta(x, x) + \alpha(x) + C,$$  

(2.13)

where $\beta$ is a symmetric, biadditive function and $\alpha$ is an additive function and $C$ is an arbitrary real constant.

Setting $a = 0$ in (2.5), we get

$$f(x) = x \left[ h(x) - \frac{t}{2} \left[ h\left(\frac{x}{t}\right) - h(0) \right] \right] + D,$$  

(2.14)

and substituting from (2.13), we obtain

$$f(x) = x \beta(x, x) + x \alpha(x) + Cx - x \frac{t}{2} \beta\left(\frac{x}{t}, \frac{x}{t}\right) - x \frac{t}{2} \alpha\left(\frac{x}{t}\right) + D.$$  

(2.15)

To prove the continuity of $f$ and $h$, let us substitute the solutions given in (2.15) into (2.5). We see that both the left- and the right-hand side of (2.5) are polynomial functions in $a$ and $u$. The equality of the two sides implies, therefore, the equality
of terms which are of the same degree with respect to \( a \) and \( u \). First, comparing the terms of degree 1 with respect to each variable, we get

\[
3a \left[ \alpha(3tu) - \frac{t}{2} \alpha(3u) \right] + 3tu \left[ \alpha(a) - \frac{t}{2} \alpha \left( \frac{a}{t} \right) \right] = 3tu \alpha(a),
\]

(2.16)

whence, substituting \( ta \) instead of \( a \) and dividing by \( t/2 \), we get

\[
tu \alpha(a) = 2a \alpha(tu) - ta \alpha(u).
\]

(2.17)

Dividing both sides by \( tua \), we obtain

\[
\frac{\alpha(a)}{a} = 2 \frac{\alpha(tu)}{tu} - \frac{\alpha(u)}{u} \quad \forall a \neq 0 \neq u.
\]

(2.18)

In particular, \( \alpha(a)/a \) does not depend on \( a \) and, therefore, \( \alpha(a) = 2Ba \) for some constant \( B \).

Now, let us compare the terms of degree 2 with respect to \( a \) and those of degree 1 with respect to \( u \). We get

\[
6a \left[ \beta(a, tu) - \frac{t}{2} \beta \left( \frac{a}{t}, u \right) \right] + 3tu \left[ \beta(a, a) - \frac{t}{2} \beta \left( \frac{a}{t}, \frac{a}{t} \right) \right] = 3tu \beta(a, a).
\]

(2.19)

Rearranging and simplifying, we get

\[
6a \left[ \beta(a, tu) - \frac{t}{2} \beta \left( \frac{a}{t}, u \right) \right] = 3t^2 \frac{\beta(a, a)}{2} - \frac{\beta \left( \frac{a}{t}, \frac{a}{t} \right)}{2}.
\]

(2.20)

or, after substituting \( ta \) instead of \( a \) and dividing by \( 3t/2 \),

\[
4a \left[ \beta(ta, tu) - \frac{t}{2} \beta(a, u) \right] = tu \beta(a, a).
\]

(2.21)

Dividing (2.21) by \( a^2 u \), we obtain

\[
4 \left[ \frac{\beta(ta, tu)}{au} - \frac{t}{2} \frac{\beta(a, u)}{au} \right] = \frac{t \beta(a, a)}{a^2} \quad \text{for } a \neq 0 \neq u.
\]

(2.22)

Using the symmetry of \( \beta \), we infer that

\[
\frac{\beta(a, a)}{a^2} = \frac{\beta(u, u)}{u^2} \quad \forall u \neq 0 \neq a,
\]

(2.23)

whence, it follows that \( \beta(a, a) = 3Aa^2 \) for some constant \( A \). Comparing this with formulae for \( f \) and \( h \), we see that

\[
f(x) = 3A \left( 1 - \frac{1}{2t} \right)x^3 + Bx^2 + Cx + D,
\]

\[
h(x) = 3Ax^2 + 2Bx + C.
\]

(2.24)

Inserting (2.24) into (1.5), we get, after simplifying,

\[
27t \left( 1 - \frac{1}{2t} \right) Aa^2 u + 81t^2 \left( 1 - \frac{1}{2t} \right) Aau^2 = 9t Aa^2 u + 27t^2 Aau^2 \quad \forall a, u \in \mathbb{R},
\]

(2.25)

whence, it follows that \( A = 0 \) provided \( t \neq 3/4 \). Note that, for \( t = 3/4 \), we have \( 3A \left( 1 - \left( 1/2t \right) \right) = A \) and the assertion follows from (2.24). \( \square \)
REFERENCES


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