OSCILLATION PROPERTIES OF NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS OF \(n\)TH ORDER

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We consider the nonlinear neutral functional differential equation

\[
[r(t)[\dot{x}(t) + \int_a^b p(t, \mu)x(\tau(t, \mu))d\mu]^{(n-1)}] + \delta \int_c^d q(t, \xi)f(x(\sigma(t, \xi)))d\xi = 0
\]

with continuous arguments. We will develop oscillatory and asymptotic properties of the solutions.

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1. Introduction. Recently, several authors [2, 3, 4, 5, 6, 7, 12, 13, 14] have studied the oscillation theory of second-order and higher-order neutral functional differential equations, in which the highest-order derivative of the unknown function is evaluated both at the present state and at one or more past or future states. For some related results, refer to [1, 8, 10, 11].

In this paper, we extend these results to \(n\)th-order nonlinear neutral equations with continuous arguments

\[
\left[r(t)\left[x(t) + \int_a^b p(t, \mu)x(\tau(t, \mu))d\mu\right]^{(n-1)}\right] + \delta \int_c^d q(t, \xi)f(x(\sigma(t, \xi)))d\xi = 0, \tag{1.1}
\]

where \(\delta = \pm 1\), \(t \geq 0\), and establish some new oscillatory criteria. Suppose that the following conditions hold:

(a) \(r(t) \in C([t_0, \infty), \mathbb{R})\), \(r(t) \in C^1\), \(r(t) > 0\), and \(\int_0^\infty (dt/r(t)) = \infty\);
(b) \(p(t, \mu) \in C([t_0, \infty) \times [a,b], \mathbb{R})\), \(0 \leq p(t, \mu)\);
(c) \(\tau(t, \mu) \in C([t_0, \infty) \times [a,b], \mathbb{R})\), \(\tau(t, \mu) \leq t\) and \(\lim_{t \to \infty} \min_{\mu \in [a,b]} \tau(t, \mu) = \infty\);
(d) \(q(t, \xi) \in C([t_0, \infty) \times [c,d], \mathbb{R})\) and \(q(t, \xi) > 0\); 
(e) \(f(x) \in C(\mathbb{R}, \mathbb{R})\) and \(xf(x) > 0\) for \(x \neq 0\);
(f) \(\sigma(t, \xi) \in C([t_0, \infty) \times [c,d], \mathbb{R})\), and

\[
\lim_{t \to \infty} \min_{\xi \in [c,d]} \sigma(t, \xi) = \infty. \tag{1.2}
\]

A solution \(x(t) \in C[t_0, \infty)\) of (1.1) is called oscillatory if \(x(t)\) has arbitrarily large zeros in \([t_0, \infty), t_0 > 0\). Otherwise, \(x(t)\) is called nonoscillatory.
2. Main results. We will prove the following lemma to be used in Theorem 2.2.

**Lemma 2.1.** Let \( x(t) \) be a nonoscillatory solution of (1.1) and let \( z(t) = x(t) + \int_a^b p(t,\mu)x(\tau(t,\mu))d\mu \). Then, the following results hold:

(i) there exists a \( T > 0 \) such that for \( \delta = 1 \),

\[
 z(t)z^{(n-1)}(t) > 0, \quad t \geq T, \tag{2.1}
\]

and for \( \delta = -1 \) either

\[
 z(t)z^{(n-1)}(t) < 0, \quad t \geq T, \quad \text{or} \quad \lim_{t \to \infty} z^{(n-2)}(t) = \infty, \tag{2.2}
\]

(ii) if \( r'(t) \geq 0 \), then there exists an integer \( l, l \in \{0, 1, \ldots, n\} \) with \( (-1)^{n-l-1}\delta = 1 \) such that

\[
 z^{(i)}(t) > 0 \quad \text{on} \quad [T, \infty) \quad \text{for} \quad i = 0, 1, 2, \ldots, l,
\]

\[
 (-1)^{i-l}z^{(i)}(t) > 0 \quad \text{on} \quad [T, \infty) \quad \text{for} \quad i = l, l+1, \ldots, n \tag{2.3}
\]

for some \( t \geq T \).

**Proof.** Let \( x(t) \) be an eventually positive solution of (1.1), say \( x(t) > 0 \) for \( t \geq t_0 \). Then, there exists a \( t_1 \geq t_0 \) such that \( x(\tau(t,\mu)) \) and \( x(\sigma(t,\xi)) \) are also eventually positive for \( t \geq t_1, \xi \in [c, d] \), and \( \mu \in [a, b] \). Since \( x(t) \) is eventually positive and \( p(t,\mu) \) is nonnegative, we have

\[
 z(t) = x(t) + \int_a^b p(t,\mu)x(\tau(t,\mu))d\mu > 0 \quad \text{for} \quad t \geq t_1. \tag{2.4}
\]

(i) From (1.1), we have

\[
 \delta[r(t)z^{(n-1)}(t)'] = -\int_c^d q(t,\xi)f(x(\sigma(t,\xi)))d\xi. \tag{2.5}
\]

Since \( q(t,\xi) > 0 \) and \( f \) is positive for \( t \geq t_1 \), we have \( \delta[r(t)z^{(n-1)}(t)'] < 0 \). For \( \delta = 1 \), \( r(t)z^{(n-1)}(t) \) is a decreasing function for \( t \geq t_1 \). Hence, we can have either

\[
 r(t)z^{(n-1)}(t) > 0 \quad \text{for} \quad t \geq t_1 \tag{2.6}
\]

or

\[
 r(t)z^{(n-1)}(t) < 0 \quad \text{for} \quad t \geq t_2 \geq t_1. \tag{2.7}
\]

We claim that (2.6) is satisfied for \( \delta = 1 \). Suppose this is not the case, then we have (2.7). Since \( r(t)z^{(n-1)}(t) \) is decreasing,

\[
 r(t)z^{(n-1)}(t) \leq r(t_2)z^{(n-1)}(t_2) < 0 \quad \text{for} \quad t \geq t_2. \tag{2.8}
\]

Divide both sides of the last inequality by \( r(t) \) and integrate from \( t_2 \) to \( t \), respectively, then we obtain

\[
 z^{(n-2)}(t) - z^{(n-2)}(t_2) \leq r(t_2)z^{(n-1)}(t_2) \int_{t_2}^t \frac{dt}{r(t)} < 0 \quad \text{for} \quad t \geq t_2. \tag{2.9}
\]
Now, taking condition (a) into account we can see that $z^{(n-2)}(t) - z^{(n-2)}(t_2) \rightarrow -\infty$ as $t \rightarrow \infty$. That implies $z(t) \rightarrow -\infty$, but this is a contradiction to $z(t) > 0$. Therefore, for $\delta = 1$,

$$r(t)z^{(n-1)}(t) > 0 \quad \text{for } t \geq t_1. \quad (2.10)$$

Since both $z(t)$ and $r(t)$ are positive, we conclude that

$$z(t)z^{(n-1)}(t) > 0. \quad (2.11)$$

For $\delta = -1$, $r(t)z^{(n-1)}(t)$ is increasing. Hence, either

$$r(t)z^{(n-1)}(t) < 0 \quad \text{for } t \geq t_1, \quad (2.12)$$

or

$$r(t)z^{(n-1)}(t) > 0 \quad \text{for } t \geq t_2 \geq t_1. \quad (2.13)$$

If (2.12) holds, we replace $z(t)$ for $r(t)$ to get

$$z(t)z^{(n-1)}(t) < 0. \quad (2.14)$$

If (2.13) holds, using the increasing nature of $r(t)z^{(n-1)}(t)$, we obtain

$$r(t)z^{(n-1)}(t) \geq r(t_2)z^{(n-1)}(t_2) > 0 \quad \text{for } t \geq t_2. \quad (2.15)$$

Divide both sides of (2.15) by $r(t)$ and integrate from $t_2$ to $t$, then we get

$$z^{(n-2)}(t) - z^{(n-2)}(t_2) \geq r(t_2)z^{(n-1)}(t_2) \int_{t_2}^{t} \frac{dt}{r(t)} > 0 \quad \text{for } t \geq t_2. \quad (2.16)$$

Taking condition (a) into account, it is not difficult to see that $z^{(n-2)}(t) \rightarrow -\infty$ as $t \rightarrow \infty$. Hence, for $\delta = -1$, either (2.14) holds or $\lim_{t \rightarrow \infty} z^{(n-2)}(t) = \infty$.

(ii) From (1.1), we can see that

$$\delta[r'(t)z^{(n-1)}(t) + r(t)z^{(n)}(t)] = -\int_{c}^{d} q(t,\xi)f(x(\sigma(t,\xi)))d\xi, \quad (2.17)$$

and then

$$\delta z^{(n)}(t) = -\frac{\delta r'(t)z^{(n-1)}(t)}{r(t)} - \int_{c}^{d} \frac{q(t,\xi)f(x(\sigma(t,\xi)))}{r(t)}d\xi. \quad (2.18)$$

Using (i) and (2.18), we obtain

$$\delta z^{(n)}(t) < 0. \quad (2.19)$$

Suppose that $\lim_{t \rightarrow \infty} z^{(n-2)}(t) \neq \infty$ when $\delta = -1$. Thus, because of the positive nature of $z(t)$ and (2.19), there exists an integer $l$, $l \in \{0,1,\ldots,n\}$ with $(-1)^{n-l-1}\delta = 1$ by
Kiguradze’s lemma [9] such that
\[
\begin{align*}
-1^i z^{(i)}(t) > 0 & \quad \text{on } [T, \infty) \quad \text{for } i = 0, 1, 2, \ldots, l, \\
-1^{i-l} z^{(i)}(t) > 0 & \quad \text{on } [T, \infty) \quad \text{for } i = l, l+1, \ldots, n
\end{align*}
\]  
(2.20)
for some \( t \geq T \).

If \( \lim_{t \to \infty} z^{(n-2)}(t) = \infty \) and \( \delta = -1 \), \( z^{(n-1)}(t) \) is eventually positive. Moreover, \( z^{(n)}(t) \) is also eventually positive by (2.19). But, this is the case \( l = n \) in (2.20). Thus, the proof is complete.

**Theorem 2.2.** Let \( P(t) = \int_a^b p(t,\mu) \, d\mu < 1 \). Suppose that \( f \) is increasing and for all constant \( k > 0 \),
\[
\int_{\infty}^{\infty} \int_{c}^{d} q(s,\xi) f((1-P(\sigma(s,\xi)))k) \, d\xi \, ds = \infty.
\]  
(2.21)

(i) If \( \delta = 1 \), then every solution \( x(t) \) of (1.1) is oscillatory when \( n \) is even, and every solution \( x(t) \) of (1.1) is either oscillatory or satisfies
\[
\lim_{t \to \infty} |x(t)| = 0
\]  
(2.22)
when \( n \) is odd.

(ii) If \( \delta = -1 \), then every solution \( x(t) \) of (1.1) is either oscillatory or else
\[
\lim_{t \to \infty} |x(t)| = \infty \quad \text{or} \quad \liminf_{t \to \infty} |x(t)| = 0
\]  
(2.23)
when \( n \) is even, and every solution \( x(t) \) of (1.1) is either oscillatory or else
\[
\lim_{t \to \infty} |x(t)| = \infty
\]  
(2.24)
when \( n \) is odd.

**Proof.** Let \( x(t) \) be a nonoscillatory solution of (1.1), say \( x(t) > 0 \) for \( t \geq t_0 \). Let \( z(t) \) be a function defined by
\[
z(t) = x(t) + \int_a^b p(t,\mu) x(\tau(t,\mu)) \, d\mu.
\]  
(2.25)
Recall from Lemma 2.1, if \( \delta = 1 \), then (2.1) holds and if \( \delta = -1 \), either \( z(t)z^{(n-1)}(t) < 0 \) for \( t \geq T \) or \( \lim_{t \to \infty} z^{(n-2)}(t) = \infty \).

Suppose that \( \lim_{t \to \infty} z^{(n-2)}(t) \neq \infty \) for \( \delta = -1 \). Thus, there exist a \( t_1 \geq T \) and an integer \( l \in \{0, 1, \ldots, n-1\} \) with \( (-1)^{n-l-1} \delta = 1 \) such that
\[
\begin{align*}
-1^i z^{(i)}(t) > 0 & \quad i = 0, 1, 2, \ldots, l, \\
-1^{i-l} z^{(i)}(t) > 0 & \quad i = l, l+1, \ldots, n, \quad t \geq t_1,
\end{align*}
\]  
(2.26)
by Kiguradze’s lemma [9].
Let $n$ be even and $\delta = 1$, or $n$ be odd and $\delta = -1$. Since $(-1)^{n-l-1}\delta = (-1)^{-l-1} = 1$, then $l$ is odd. Now, $z(t)$ is increasing by (2.26). Therefore, we have

$$z(t) = x(t) + \int_a^b p(t,\mu)x(\tau(t,\mu))d\mu \leq x(t) + \int_a^b p(t,\mu)z(\tau(t,\mu))d\mu,$$

since $x(t) \leq z(t)$. Since $z(t)$ is increasing and $\tau(t,\mu) < t$, this will imply that

$$z(t) \leq x(t) + Pf(t).$$

(2.27)

Thus, we have

$$(1 - Pf(t))z(t) \leq x(t).$$

(2.28)

On the other hand, we have $z(t)$ positive and increasing with $\lim_{t \to \infty} \min_{\xi \in [a,b]} \sigma(t,\xi) = \infty$. These imply that there exist a $k > 0$ and a $t_2 \geq t_1$ such that

$$z(\sigma(t,\xi)) \geq k \quad \text{for } t \geq t_2.$$  

(2.30)

Integrating (1.1) from $t_2$ to $t$, then we have

$$\delta r(t)z^{(n-1)}(t) - \delta r(t_2)z^{(n-1)}(t_2) + \int_{t_2}^t \int_{c}^d q(s,\xi)f(x(\sigma(s,\xi)))d\xi ds = 0.$$ 

(2.31)

By (2.29), (2.30), and increasing nature of $f$, we obtain

$$f(x(\sigma(t,\xi))) \geq f((1 - Pf(t))k) \quad \text{for } t \geq t_2.$$ 

(2.32)

Substituting (2.32) into (2.31), we get

$$\delta r(t)z^{(n-1)}(t) - \delta r(t_2)z^{(n-1)}(t_2) + \int_{t_2}^t \int_{c}^d q(s,\xi)f((1 - Pf(t))k)d\xi ds \leq 0.$$ 

(2.33)

From (2.21) and (2.33), we can conclude that $\delta r(t)z^{(n-1)}(t) \to -\infty$ as $t \to \infty$. This contradicts the following:

$$z^{(n-1)}(t) > 0 \quad \text{for } \delta = 1,$$

$$z^{(n-1)}(t) < 0 \quad \text{for } \delta = -1.$$ 

(2.34)

Thus, this proves that $x(t)$ is oscillatory when $\delta = 1$ and $n$ is even, or $x(t)$ is either oscillatory or $\lim_{t \to \infty} z^{(n-2)}(t) = \infty$ when $\delta = -1$ and $n$ is odd. Obviously, if $\lim_{t \to \infty} z^{(n-2)}(t) = \infty$, then $\lim_{t \to \infty} x(t) = \infty$.

Let $n$ be odd and $\delta = 1$, or $n$ be even and $\delta = -1$. If the integer $l > 0$, then we can find the same conclusion as above. Let $l = 0$. Since

$$\int_\infty^d q(s,\xi)d\xi ds = \infty,$$

$$\lim_{l \to \infty} \delta r(t)z^{(n-1)}(t) = L \geq 0,$$

(2.35)
and by using these two in (2.31), then it is easy to see that
\[
\lim_{t \to \infty} \inf f(x(t)) = 0 \quad \text{or} \quad \lim_{t \to \infty} x(t) = 0.
\] (2.36)

This completes the proof.

\[\Box\]

**Example 2.3.** Consider the following functional differential equation:
\[
\left[ e^{-t/2} \left[ x(t) + \int_1^2 (1 - e^{-t-\mu}) x(t-\mu) d\mu \right] \right]' - \int_3^5 \frac{(e^2 + e - 1)(e^{(t+\xi)/3})}{4e^{7/2}(e-1)} x \left( \frac{t+\xi}{6} \right) d\xi = 0
\] (2.37)

so that \(\delta = -1\), \(n = 3\), \(r(t) = e^{-t/2}\), \(p(t,\mu) = 1 - e^{-t-\mu}\), \(\tau(t,\mu) = t - \mu\), \(q(t,\xi) = (e^2 + e - 1)(e^{(t+\xi)/3})/4e^{7/2}(e-1), f(x) = x\), \(\sigma(t,\xi) = (t+\xi)/6\) in (1.1).

We can easily see that the conditions of Theorem 2.2 are satisfied. Then, all solutions of this problem are either oscillatory or tends to infinity as \(t\) goes to infinity. It is easy to verify that \(x(t) = e^t\) is a solution of this problem.

**Theorem 2.4.** Let \(P(t) = \int_a^b p(t,\mu) d\mu < 1\), and let \(f\) be increasing and \(r(t) = 1\). Suppose that
\[
\int_\infty \int_c^d s^{n-1} q(s,\xi) f((1-P(\sigma(s,\xi)))k) d\xi ds = \infty
\] (2.38)

for every constant \(k > 0\). Then, every bounded solution \(x(t)\) of (1.1) is oscillatory when \((-1)^n\delta = 1\).

**Proof.** Let \(x(t)\) be a nonoscillatory solution of (1.1). We may assume that \(x(t) > 0\) for \(t \geq t_0\). Then, obviously there exists a \(t_1 \geq t_0\) such that \(x(t), x(\tau(t,\mu)), x(\sigma(t,\xi))\) are positive for \(t \geq t_1\), \(\mu \in [a,b]\), and \(\xi \in [c,d]\). Let \(z(t) = x(t) + \int_a^b p(t,\mu) x(\tau(t,\mu)) d\mu\), then from (1.1), \(\delta z^{(n)}(t) < 0\) for \(t \geq t_1\). Hence, for \(\delta = 1\), \(z^{(n-1)}(t)\) is increasing and for \(\delta = -1\), \(z^{(n-1)}(t)\) is decreasing.

Since \(z^{(n)}(t) < 0\) for \(\delta = 1\), by Kiguradze’s lemma [9] there exists an integer \(l, 0 \leq l \leq n-1\) with \(n-l\) is odd and for \(t \geq t_1\) such that
\[
z^{(i)}(t) > 0, \quad i = 0, 1, \ldots, l,
\]
\[
(-1)^{n-l} z^{(i)}(t) > 0, \quad i = l, l+1, \ldots, n-1.
\] (2.39)

For \(\delta = -1\), \(z^{(n)}(t) > 0\), by Kiguradze’s lemma [9] either
\[
z^{(i)}(t) > 0, \quad i = 0, 1, \ldots, n-1,
\] (2.40)

or there exists an integer \(l, 0 \leq l \leq n-2\) with \(n-l\) is even and for \(t \geq t_1\) such that
\[
z^{(i)}(t) > 0, \quad i = 0, 1, \ldots, l,
\]
\[
(-1)^{n-l} z^{(i)}(t) > 0, \quad i = l, l+1, \ldots, n-1.
\] (2.41)

Since \(z(t)\) is bounded, \(l\) cannot be 2 for both cases. Then for \((-1)^n\delta = 1\), we have
\[
(-1)^{l-1} z^{(i)}(t) > 0, \quad i = 1, 2, \ldots, n-1.
\] (2.42)
This shows that
\[ \lim_{t \to \infty} z^{(i)}(t) = 0 \text{ for } i = 1, 2, \ldots, n-1. \]  
(2.43)

Using (2.43) and integrating (1.1) \( n \) times from \( t \) to \( \infty \) to find
\[ (-1)^n \delta [z(\infty) - z(t)] = \frac{1}{(n-1)!} \int_{t}^{\infty} \int_{c}^{d} (s-t)^{n-1} q(s, \xi) f(x(\sigma(s, \xi))) d\xi ds, \]  
(2.44)

where \( z(\infty) = \lim_{t \to \infty} z(t) \). On the other hand, from (2.42), \( z(t) \) is increasing for large \( t \) and \( z(t) \) is positive, so we have
\[ f(x(\sigma(t, \xi))) \geq (1 - P(\sigma(t, \xi))) k \text{ for } t \geq t_1, k > 0 \]  
(2.45)
as in the proof of Theorem 2.2. Thus, from (2.44) and (2.45), we have
\[ z(\infty) - z(t_1) \geq \frac{1}{(n-1)!} \int_{t_1}^{\infty} \int_{c}^{d} (s-t)^{n-1} q(s, \xi) f((1 - P(\sigma(s, \xi))) k) d\xi ds. \]  
(2.46)

By (2.38), the right-hand side of the above inequality is \( \infty \), therefore \( z(\infty) = \infty \) and this contradicts the boundedness of \( z(t) \). Thus, every bounded solution \( x(t) \) of (1.1) is oscillatory when \( (-1)^n \delta = 1 \).

**Example 2.5.** Consider the following functional differential equation:
\[ \left[ x(t) + \int_{\pi}^{2\pi} \frac{1}{4} (1 - e^{-t}) \mu \left( t - \frac{\mu}{2} \right) d\mu \right]'' + \int_{\pi}^{5\pi/2} \left( \frac{1}{2} - e^{-t} \right) x(t + \xi) d\xi = 0, \quad t > -\ln \left( \frac{1}{2} \right) \]  
(2.47)
so that \( \delta = 1, n = 2, r(t) = 1, p(t, \mu) = (1 - e^{-t}) / 4, \tau(t, \mu) = t - \mu / 2, q(t, \xi) = 1/2 - e^{-t}, f(x) = x, \sigma(t, \xi) = (t + \xi) \) in (1.1).

We can easily see that the conditions of Theorem 2.4 are satisfied. Then, all bounded solutions of this problem are oscillatory. It is easy to verify that \( x(t) = \sin t \) is a solution of this problem.

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Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from “Qualitative Theory of Differential Equations,” allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the Mathematical Problems in Engineering aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

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