Let $X$, $X'$ be two locally finite, preordered sets and let $R$ be any indecomposable commutative ring. The incidence algebra $I(X, R)$, in a sense, represents $X$, because of the well-known result that if the rings $I(X, R)$ and $I(X', R)$ are isomorphic, then $X$ and $X'$ are isomorphic. In this paper, we consider a preordered set $X$ that need not be locally finite but has the property that each of its equivalence classes of equivalent elements is finite. Define $I^*(X, R)$ to be the set of all those functions $f : X \times X \to R$ such that $f(x, y) = 0$, whenever $x \not\leq y$ and the set $S_f$ of ordered pairs $(x, y)$ with $x < y$ and $f(x, y) \neq 0$ is finite. For any $f, g \in I^*(X, R)$, $r \in R$, define $f + g$, $fg$, and $rf$ in $I^*(X, R)$ such that $(f + g)(x, y) = f(x, y) + g(x, y)$, $fg(x, y) = \sum_{x \leq z \leq y} f(x, z)g(z, y)$, $rf(x, y) = r \cdot f(x, y)$. This makes $I^*(X, R)$ an $R$-algebra, called the weak incidence algebra of $X$ over $R$. In the first part of the paper it is shown that indeed $I^*(X, R)$ represents $X$. After this all the essential one-sided ideals of $I^*(X, R)$ are determined and the maximal right (left) ring of quotients of $I^*(X, R)$ is discussed. It is shown that the results proved can give a large class of rings whose maximal right ring of quotients need not be isomorphic to its maximal left ring of quotients.

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1. Introduction. Let $X$ and $X'$ be two locally finite, preordered sets, and let $R$ be a commutative ring. Under what conditions are incidence rings $I(X, R)$ and $I(X', R)$ isomorphic? In particular, under what conditions on $R$ can one conclude that $X$ and $X'$ are isomorphic, when the two incidence rings $I(X, R)$ and $I(X', R)$ are isomorphic? The latter question has been discussed by many authors. One of the earliest results in this direction is by Stanley [9], who proved that if $R$ is a field, then the two incidence rings are isomorphic if and only if $X$ and $X'$ are isomorphic. Froelich [4] extended this result to the case of an indecomposable ring $R$. Similar questions have been examined in [1, 3, 10] in case $R$ need not be commutative.

Now consider any preordered set $X$ that need not be locally finite. Two elements $x, y \in X$ are said to be equivalent, $x \sim y$, if $x \leq y \leq x$. In Section 3, the isomorphism problem for weak incidence algebras is discussed. Let $X$ and $X'$ be two preordered sets in each of which every equivalence class is finite, and let $R, R'$ be two commutative rings such that the weak incidence algebras $I^*(X, R)$ and $I^*(X', R')$ are isomorphic as rings. In case $R$ and $R'$ are indecomposable, Theorem 3.10 shows that $X$, $X'$ are isomorphic and $R$, $R'$ are isomorphic. The main aim of Section 4 is to prove some results that can help in studying the maximal ring of quotients of an $I^*(X, R)$. Similar work has been done in a recent paper [2] for certain classes of incidence algebras. In [7], Spiegel determines some essential ideals of an incidence algebra of a locally finite, partially
ordered set. Here we are in a position to determine all the essential one-sided ideals of an $S = I^*(X,R)$ whenever $R$ is indecomposable. A particular essential right ideal $T$ is isolated and the ring $Q = \text{Hom}_T(T,T)$ is discussed in Theorems 4.8, 4.9, and 4.10. This ring $Q$ is used to give some results on maximal right (left) ring of quotients of $S$.

2. Preliminaries. All rings considered here are with identity $1 \neq 0$. As the various concepts discussed here for weak incidence algebras are similar to those for incidence algebras, for details on incidence algebras one may consult [8]. We now collect some results on rings and modules.

**Lemma 2.1.** For any commutative ring $R$ and any positive integer $n$, if $M_R = R^{(n)}$ is isomorphic to its summand $N$, then $M = N$.

**Proof.** Now $M = N \oplus K$. For any maximal ideal $P$ of $R$, the localization $M_P = N_P \oplus K_P$. As the ranks of the free $R_P$-modules $M_P$ and $N_P$ are the same and finite, $K_P = 0$. Hence $K = 0$.

**Lemma 2.2.** Let $R$ be a commutative ring and let $K$ be any ring such that $M_n(R) \cong M_m(K)$. Then $m$ divides $n$. If $n = m$, then $R \cong K$.

**Proof.** The first part follows from Wedderburn’s structure theorem for simple artinian algebras, and the second part is in [6].

**Lemma 2.3.** Let $T$ be any ring and let $e, e', f, f'$ be any four idempotents in $T$ such that $eT \equiv e'T$, $fT \equiv f'T$. Then $eTf \neq 0$ if and only if $e'Tf' \neq 0$.

**Proof.** The hypothesis gives that $\text{Hom}_T(fT,eT) \cong \text{Hom}_T(f'T,e'T)$, $eTf \equiv e'Tf'$, as abelian groups. This proves the result.

3. Isomorphism. Let $X$ be any preordered set (i.e., $X$ is a set with a relation $\leq$ that is reflexive and transitive). For any $x, y \in X$, set $x \prec y$, if $x \leq y \leq x$. Then $\prec$ is an equivalence relation. A preordered set $X$ is said to be a class finite, preordered set if, for any $x \in X$, the equivalence class $[x] = \{ y \in X : x \leq y \leq x \}$ is finite. Henceforth we take $X$ to be a class finite, preordered set and $R$ a commutative ring. The set $K^*(X,R) = \{ f \in I^*(X,R) : f(x,y) = 0 \text{ whenever } x \prec y \}$ is a nil ideal. Indeed, given $f \in K^*(X,R)$, $f^{m+1} = 0$, for $m = |S_f|$. Indeed, one can see that each member of $K^*(X,R)$ is strongly nilpotent, as defined in [8, page 176], so $K^*(X,R)$ is contained in the lower nil radical of $I^*(X,R)$. Let $Y$ be a representative partially ordered subset of $X$. For any $x \in X$, let $[x] = n_x$. For each $x \in X$, the set $B_x = \{ f \in I^*(X,R) : f(u,v) = 0 \text{ whenever } u \prec x \text{ or } v \prec x \}$, is a ring with $\delta_x$ as identity, where $\delta_x(u,v) = 0$, whenever $u \prec x$, $v \prec x$, or $u \neq v$, and $\delta_x(u,u) = 1$ whenever $u \prec x$. Let $\delta$ denote the identity element of $I^*(X,R)$. For any $x, y \in X$, with $x \leq y$, let $e_{xy} \in I^*(X,R)$ be such that $e_{xy}(u,v) = 0$, for $(u,v) \neq (x,y)$, and $e_{xy}(x,y) = 1$. Each of $e_{xy}$ is called a matrix unit of $I^*(X,R)$. We write $e_x = e_{xx}$. Then $B_x$ is the $n_x \times n_x$ full matrix ring over $R$ with $e_{uv} : u \prec x$, $v \prec x$ as its set of matrix units. Let $M_n(R)$ denote the $n \times n$ full matrix ring over $R$. Further, $D^*(X,R) = \{ f \in I^*(X,R) : f(u,v) = 0 \text{ whenever } u \prec v \}$ is a subring of $I^*(X,R)$, each $B_x$ is an ideal of $D^*(X,R)$. Set $S = I^*(X,R)$, $K = K^*(X,R)$, $D = D^*(X,R)$. For any subset $Z$ of $X$, let $E_Z \subset S$ be such that $E_Z(u,u) = 1$ for $u \in Z$, and $E_Z(x,y) = 0$ otherwise. For any
f \in S$, support of $f$, denoted by suppt($f$), equals \{(x, y) : f(x, y) \neq 0\}, the cardinality of suppt($f$) is called the weight of $f$ and we denote it by wt($f$). Let $X'$ be another class finite, preordered set. Let $R'$ be another commutative ring. We use the same symbols for the matrix units of $I^*(X, R)$ or $I^*(X', R')$ and so on, but $S' = I^*(X', R')$, $K' = K^*(X', R')$, and $D' = D^*(X', R')$. Let $Y$ and $Y'$ be fixed representative partially ordered subsets of $X$ and $X'$, respectively. For any two distinct members $y, z$ of $Y$, $\delta_y, \delta_z$ are orthogonal idempotents. Any $f \in S$ will be sometimes denoted by the formal sum $\sum_{x,y} f(x,y)e_{xy}$ (or by the matrix $[f(x,y)]$ indexed by $X$). The following is obvious.

**Lemma 3.1.** (i) $I^*(X, R) = D^*(X, R) \oplus K^*(X, R)$ as abelian groups.
(ii) $D^*(X, R) = \Pi_{y \in Y} B_y$, where $Y$ is any representative partially ordered subset of $X$.
(iii) $I^*(X, R)/K^*(X, R) \cong \Pi_{y \in Y} M_{n_y}(R) \cong D^*(X, R)$, where $Y$ is any representative partially ordered subset of $X$.
(iv) For any $f, e_{xy} \in I^*(X, R)$, wt($f e_{xy}$) is finite, that is, $f e_{xy} = \sum_{u \leq y} a_{uy} e_{uy}$, with finitely many $a_{ux} \neq 0$.

It follows from (ii) that $K^*(X, R)$ does not equal the Jacobson radical of $S$, unless the Jacobson radical of $R$ is zero. For any $f \in S$, we write $f = f_D + f_K$ with $f_D \in D$ and $f_K \in K$; $f_D$ is called the diagonal of $f$. The following is obvious.

**Lemma 3.2.** For any nonempty subset $Z$ of $X$, $E_Z S E_Z \cong I^*(Z, R)$.

**Lemma 3.3.** For any two idempotents $f, g \in S$, $f S g = 0$ if and only if $f_D S g_D = 0$.

**Proof.** In $\overline{S} = S/K, f + K = f_D + K$. As $K$ is nil, we get $f S = f_D S$. After this, Lemma 2.3 completes the proof.

**Lemma 3.4.** Let $0 \neq e = e^2 \in S$.
(i) $e_D$ is a nonzero idempotent and $e_D \delta_y = \delta_y e_D$ for any $y \in Y$.
(ii) There exists $y \in Y$ such that $e_D \delta_y = \delta_y e_D \neq 0$.
(iii) For any $y \in Y, e' = ee_D \delta_y e$ is an idempotent such that $e'(u, v) = \sum e(u, w_1) e(w_1, w_2) e(w_2, v)$, where the summation runs over $w_1, w_2$ in $[y] \cap [u, v]$. Further, $e - e'$, $e'$ are orthogonal idempotents. If $e_D \delta_y \neq 0$, then $e' \neq 0$.

**Proof.** (i) is obvious. Now $S/K = D = \Pi_{y \in Y} B_y \cong D, \overline{S} = \Pi \delta_y$, and $\overline{e} = \overline{e_D}$. It follows that for some $y \in Y$, $\overline{e} \delta_y = \overline{e_D \delta_y} = 0$. This proves (ii). Consider any $y \in Y$ and $e' = ee_D \delta_y e$. The definition of the product of two members of $S$ gives that $e'(u, v) = \sum e(u, w_1) e(w_1, w_2) e(w_2, v)$, where the summation runs over all $w_1, w_2$ in $[y] \cap [u, v]$. Then we have $(e')^2(u, v) = \sum_{u \leq w \leq v} e'(u, w)e'(w, v) = \sum e(u, w_1) e(w_1, w_2) e(w_2, w) e(w, w_3) e(w_3, w_4) e(w_4, v)$, where summation runs over all $w_1, w_2, w_3, w_4$ in $[y] \cap [u, v]$ such that $w_2 \leq w \leq w_3$. Thus $(e')^2(u, v) = \sum e(u, w_1) e(w_1, w_4) e(w_4, v) = e'(u, v)$. Hence $e'$ is an idempotent. As $ee' = e' = e e'$, it follows that $e - e'$ is an idempotent orthogonal to $e'$. If $e_D \delta_y \neq 0$, as obviously $\overline{e} = \overline{e_D \delta_y e}$ in $S/K$, we get $e' \neq 0$.

**Lemma 3.5.** (i) If $e \in S$ is an indecomposable idempotent, then there exists a unique $y \in Y$ such that $e = ee_D \delta_y e$.
(ii) Let $e \in S$ be a nonzero idempotent such that $e_D \in B_y$ for some $y \in Y$. Then $e = ee_D \delta_y e$; this $y$ is uniquely determined by $e$. 
**Proof.** (i) In $\mathcal{S} = S/K$, $\overline{\sigma} = \overline{\sigma y}$ is an indecomposable idempotent. So there exists a unique $y \in Y$ such that $\overline{\sigma y} = \overline{\sigma y} y$. By Lemma 3.4(iii), $e' = ee_D y e$ is a nonzero idempotent. As $e - e'$ is orthogonal to $e'$ and $e$ is indecomposable, $e = e'$. 

(ii) The hypothesis gives $\overline{\sigma} = \overline{ee_D y e}$. Then Lemma 3.4(iii) gives $e = ee_D y e$. \hfill $\Box$

**Theorem 3.6.** Let $R$ be any indecomposable commutative ring and $X$ any class finite, preordered set. Then for any automorphism $\sigma$ of $S = I^*(X, R)$, $\sigma (K) = K$.

**Proof.** Consider any $f \in S \backslash K$. For some $x - y$, $f(x, y) \neq 0$. Then $g = e_x f e_y$ is such that $g(x, x) \neq 0$ and $g = e_x g e_x$. So $\sigma(g) = e g e \sigma(e)$, where $e = \sigma(e_x)$ is an indecomposable idempotent. Let $Y$ be a representative partially ordered subset of $X$. By Lemma 3.5, there exists unique $z \in Y$ such that $e = ee_D z e$, $e_D \in B_z$. Thus $\sigma(g) = ee_D z e \sigma(g) ee_D z e \neq 0$, $\delta e e \sigma(g) ee_D z \neq 0$, so for some $u, v \in [z]$, $\sigma(g)(u, v) \neq 0$. Hence $\sigma(e) \notin K$. Consequently, $\sigma(f) \notin K$. This proves the result. \hfill $\Box$

**Lemma 3.7.** For some $y, y' \in Y$, let there exist idempotents $e \in B_y, f \in B_{y'}$ such that $e S f \neq 0$. Then $e_y S e_{y'} \neq 0$.

**Proof.** The hypothesis gives that $\delta y S y' \neq 0$, so there exist $u \in [y], v \in [y']$ such that $e_u S e_v \neq 0$. After this, Lemma 2.3 completes the proof. \hfill $\Box$

**Lemma 3.8.** If for some idempotent $f \in S$, $f S \equiv \delta y S$ for some $y \in Y$, then $f_D = \delta y$.

**Proof.** We have $f_D S \equiv \delta y S$. In $\mathcal{S} = S/K, f_D S \equiv \delta y S$, so $f_D \in B_y$ and $f_D B_y \equiv B_y$. By Lemma 2.1, $f_D = \delta y$. \hfill $\Box$

**Lemma 3.9.** Let $R, R'$ be indecomposable and $\sigma : S \rightarrow S'$ an isomorphism. 
There exists a one-to-one mapping $\eta$ of $Y$ onto $Y'$ such that $\sigma(\delta y) = \delta_{\eta(y)} + g_{\eta(y)}$ for some $g_{\eta(y)} \in K', |\{y\}| = |\{\eta(y)\}|$, and $R \equiv R'$.

**Proof.** The hypothesis gives that for any $x \in X$, $e_x$ is an indecomposable idempotent in $S$. Now $\sigma(\delta y) S' = \oplus \sum u - y \sigma(e_u) S'$ As these $\sigma(e_u) S'$ are indecomposable and isomorphic right ideals, there exist unique $\eta(y) \in Y'$ such that each $\sigma(e_u) D \in B_{\eta(y)}$. Consequently, $\sigma(\delta y) D \in B_{\eta(y)}$, and $\sigma(\delta y) D \delta_{\eta(y)} = \delta_{\eta(y)} \sigma(\delta y) D$. By Lemma 3.5(iii), $\sigma(\delta y) = \sigma(\delta y) D \delta_{\eta(y)} \sigma(\delta y)$. Similarly,

$$\sigma^{-1}(\delta_{\eta(y)}) = \sigma^{-1}(\delta_{\eta(y)}) (\sigma^{-1}(\delta_{\eta(y)})) D \delta z \sigma^{-1}(\delta_{\eta(y)})$$

(3.1)

for some $z \in Y$. So, $\delta_{\eta(y)} = \delta_{\eta(y)} \sigma((\sigma^{-1}(\delta_{\eta(y)})) D) \sigma(\delta z) \delta_{\eta(y)}$. Thus, in $\mathcal{S'} = S'/K'$,

$$\sigma(\delta y) = \sigma(\delta y) \sigma(\delta y) D \delta_{\eta(y)} \sigma((\sigma^{-1}(\delta_{\eta(y)})) D) \sigma(\delta z) \delta_{\eta(y)} \sigma(\delta y).$$

(3.2)

In $\mathcal{S'}$, $\overline{\delta_{\eta(y)}}$ is a central idempotent. Thus

$$\overline{\sigma(\delta y)} = \overline{\sigma(\delta y)} \overline{\sigma((\sigma^{-1}(\delta_{\eta(y)})) D) \sigma(\delta z) \delta_{\eta(y)} \sigma(\delta y)},$$

(3.3)
which equals zero, if \( z \neq y \). Hence \( z = y \) and \( \eta \) is a bijection from \( Y \) onto \( Y' \). We get \( \sigma(\delta_y) = \delta_{\eta(y)} \sigma((\sigma^{-1}(\delta_{\eta(y)}))D \delta_y) \) and \( \delta_{\eta(y)} = \delta_{\eta(y)} \sigma((\sigma^{-1}(\delta_{\eta(y)}))D \delta_y) \). Hence \( \sigma(\delta_y) = \delta_{\eta(y)} \). This shows that \( \sigma(\delta_y) = \delta_{\eta(y)} + g_{\eta(y)} \) for some \( g_{\eta(y)} \in K' \). Now \( \delta_y \delta_y = B_y \). As \( \sigma(\delta_y)S' \equiv \delta_{\eta(y)}S' \), it follows that \( B_y \equiv B_{\eta(y)} \). By Lemma 3.3, \( |[y]| = |[\eta(y)]| \) and \( R \equiv R' \).

**Theorem 3.10.** Let \( X \) and \( X' \) be two class finite, preordered sets. Let \( R \) and \( R' \) be any two indecomposable commutative rings. If there exists an isomorphism of \( I^*(X,R) \) onto \( I^*(X',R') \), then \( X \) and \( X' \) are isomorphic and the rings \( R \) and \( R' \) are isomorphic.

**Proof.** We use the terminology developed before Theorem 3.10. Consider any \( u, v \in Y \) such that \( u \leq v \). Then \( e_UE_v \neq 0 \), \( \sigma(e_u)S' \sigma(e_v) \neq 0 \). It follows from Lemma 3.9 that \( \sigma(e_u)_D \in B'_{\eta(u)} \), \( \sigma(e_v)_D \in B'_{\eta(v)} \). By Lemma 3.3, \( \sigma(e_u)_DS' \sigma(e_v)_D \neq 0 \), \( e_{\eta(u)}S'e_{\eta(v)} \neq 0 \), hence \( \eta(u) \leq \eta(v) \). Thus \( \eta \) is an isomorphism of \( Y \) onto \( Y' \). Also by Lemma 3.9, \( |[y]| = |[\eta(y)]| \), hence it follows that \( X \) and \( X' \) are isomorphic. By Lemma 3.9, \( R \) and \( R' \) are isomorphic.

**Lemma 3.11.** For any commutative ring \( T \) and any class finite, preordered set \( X \), the following hold.

(i) A central idempotent \( e \in I^*(X,T) \) is centrally indecomposable if and only if \( e = gE_Z \) for some indecomposable idempotent \( g \in T \) and a connected component \( Z \) of \( X \).

(ii) Let \( g \) and \( h \) be two indecomposable idempotents in \( T \) and let \( Z, Z' \) be two connected components of \( X \); the rings \( gE_ZI^*(X,T) \), \( hE_ZI^*(X,T) \) are isomorphic if and only if the rings \( gT, hT \) are isomorphic and \( Z, Z' \) are isomorphic.

**Proof.** (i) Consider any central idempotent \( e \in I^*(X,T) \). On the same lines as for incidence algebras, it can be easily seen that \( e(x,y) = 0 \), whenever \( x \neq y \). For any connected component \( Z \) of \( X \), there exists an idempotent \( gE_Z \) in \( T \) such that \( e(x,x) = gE_Z \) for every \( x \in Z \). Using this, (i) follows. (ii) As \( gE_ZI^*(X,T) \equiv I^*(Z,gT) \) and \( hE_ZI^*(X,T) \equiv I^*(Z',hT) \), the result follows from Theorem 3.10.

Let \( T \) be any ring. Let \( \ln(T) \) be the set of all centrally indecomposable central idempotents of \( T \). Two central idempotents \( g, h \) of \( T \) are said to be equivalent if the rings \( gT \) and \( hT \) are isomorphic. For any central idempotent \( g \in T \), \( [g] \) denotes the set of central idempotents in \( T \) equivalent to \( g \).

**Theorem 3.12.** Let \( R \) and \( R' \) be any two commutative rings and let \( X \), \( X' \) be two class finite, preordered sets. Let \( \sigma : I^*(X,R) \to I^*(X',R') \) be a ring isomorphism. Let \( g \in \ln(R) \) and let \( Z \) be a connected component of \( X \).

(i) There exist unique \( g' \in \ln(R') \) and unique connected component \( Z' \) of \( X' \) such that \( \sigma(gE_Z) = g'E_{Z'} \); further, \( Z \equiv Z' \), \( |[g]| = |[g']||[Z]| = |[gE_Z]| = |[g'E_{Z'}]| = |[g']||[Z']| \).

(ii) If the cardinalities of \( |[g]| \) and \( |[g']| \) are finite and equal, then \( X \) and \( X' \) are isomorphic.

**Proof.** (i) The first part follows from Lemma 3.11(i); the second part follows from Lemma 3.11(ii). (ii) If \( |[g]| = |[g']| \) and they are finite, if follows from (i) that, given any
connected component $Z$ of $X$, there exists a connected component $Z'$ of $X'$ isomorphic to $Z$, and $[Z], [Z']$ have the same cardinalities. Consequently, $X$ and $X'$ are isomorphic.

The following is immediate from Theorem 3.12.

**Corollary 3.13.** Let $R$ be any commutative ring such that $R$ admits an indecomposable idempotent $g$ for which the equivalence class $[g]$ is finite. Let $X$ and $X'$ be any two class finite, preordered sets. If the rings $I^+(X,R)$ and $I^+(Y,R)$ are isomorphic, then $X$ and $X'$ are isomorphic.

**4. Essential right ideals and maximal ring of quotients.** Throughout $S = I^+(X,R)$, where $X$ is a class finite, preordered set and $R$ is a commutative ring in which 1 is indecomposable. Any $x \in X$ is said to be a maximal element if the equivalence class $[x]$ is maximal in the partially ordered set of the equivalence classes in $X$. For any $x, y \in X$, we say $x < y$, if $x \leq y$ but $[x] \neq [y]$. Set $X_0 = \{x \in X : x$ is maximal$\}$, $Y_0 = \{(x, y) \in X \times X_0 : x \leq y\}$, $Y_1 = \{(x, y) : x < y$ and there does not exist any $z \in X_0$ such that $y \leq z\}$, $Y_2 = \{(x, y) : x < y$ and there exists a $z \in X_0$ such that $y < z\}$, and $Y_3 = \{(x, y) \in X_0 \times X_0: [x] = [y]\}$. Further, $K = K^+(X,R)$. Now $L = \sum_{(x,y) \in Y_3} e_{xy} R$ is a right ideal of $S$. In [2], maximal rings of quotients of certain incidence algebras have been discussed. Here we intend to prove some results that can help in studying the maximal rings of quotients of $S$. Spiegel [7] has determined certain classes of essential ideals of an incidence algebra of a locally finite, preordered set. Here we determine all essential one-sided ideals of $S$. For the definitions of an essential submodule, dense submodule, and singular submodule of a module, one may refer to [5]. Let $M$ be any module, then $N \subseteq e M$ $(N \subseteq d M)$ denotes that $N$ is an essential (dense) submodule of $M$, and $Z(M)$ denotes the singular submodule of $M$. The concept of the maximal right ring of quotients of a ring is discussed in [5, Section 13].

**Lemma 4.1.** Let $K_1 = K + L$. Then $K_1$ is an essential right ideal of $S$ and $l \cdot \text{ann}(K_1) = 0$. Indeed for any $0 \neq f \in S$, there exists $e_{xw} \in K_1$ such that $0 \neq f e_{xw} \subseteq K_1$.

**Proof.** Let $0 \neq f \in S$. Then $f(u, v) \neq 0$ for some $u \leq v$. Suppose $f K_1 = 0$. If $v$ is not maximal in $X$, there exists $e_{uz} \in K$, and $f e_{uz} \neq 0$, which is a contradiction. Hence $v$ is maximal. Then $e_v \in K_1$ with $f e_v \neq 0$, which is again a contradiction. Hence $l \cdot \text{ann}(K_1) = 0$. In any case there exists $e_{xy} \in K_1$ such that $f e_{xy} \neq 0$. By applying induction on $wt(f e_{xy})$, we prove that for some $g \in S$, $0 \neq f e_{xy} g \subseteq K_1$, which will prove that $K_1 \subseteq e S$. Suppose $wt(f e_{xy}) = 1$. Then $f e_{xy} = a e_{uy}$, for some $0 \neq a \in R$. If $y$ is not maximal, for any $z > y$, $f e_{xy} e_{yz} = a e_{uz} \subseteq K_1$. If $y$ is maximal, then $e_y \in K_1$, so $f e_{xy} e_y = a e_{uy} \subseteq K_1$. To apply induction, suppose that $wt(f e_{xy}) = n > 1$, and for any $h \in S$, if for some $e_{uv} \in K_1$, $wt(h e_{uv}) < n$ and $h e_{uv} \neq 0$, then for some $e_{uz} \in S$, $0 \neq h e_{uv} e_{uz} \subseteq K_1$. We can write $f e_{xy} = a e_{uy} + h$, where $wt(h) = n - 1$ and $h(u, y) = 0$. For some $e_{ys} \in K_1$, $a e_{uy} e_{ys} = a e_{us} \subseteq K_1$. Then $f e_{xs} = a e_{us} + h e_{ys}$, with $wt(h e_{ys}) = n - 1$. By the induction hypothesis, there exists $e_{sw} \in K_1$ such that $0 \neq h e_{ys} e_{sw} \subseteq K_1$. Then $0 \neq f e_{xw} \subseteq K_1$. Hence $K_1 \subseteq e S$. □
We call a subset \( B \) of \( R \) an essential subset of \( R \) if, for each \( 0 \neq r \in R \), there exists an \( s \in R \) such that \( 0 \neq rs \in B \). Clearly the ideal of \( R \) generated by an essential subset is an essential ideal.

**Lemma 4.2.** Let \( E \subseteq S \). For any \( x \leq y \) in \( X \), let \( A_{xy} = \{ r \in R : re_{xy} \in E \} \), \( B_{xy} = \cup_{y \leq z} A_{xz} \).

(i) \( A_{xy} \subseteq A_{xw} \) whenever \( x \leq y \leq w \).

(ii) \( B_{xy} \) is an essential subset of \( R \).

**Proof.** (i) is trivial. Let \( 0 \neq r \in R \). Then for some \( g \in S \), \( 0 \neq re_{xy}g \in E \). For some \( y \leq w \), \( rg(y, w) \neq 0 \). This gives \( re_{xy}g \in E \). Suppose \( re_{xy}g \in E \). Therefore, \( B_{xy} \) is an essential subset of \( R \).

**Lemma 4.3.** Let \( \{ A_{xy} : \text{either } x < y, \text{ or } x \leq y \text{ and } y \text{ is maximal in } X \} \) be a family of ideals in \( R \) such that (i) \( A_{xy} \subseteq A_{xw} \) whenever \( y \leq z \), and (ii) for any \( x \leq y \) in \( X \), \( B_{xy} = \cup_{y \leq z} A_{xz} \) is an essential subset of \( R \). Then \( E = \sum_{x,y} A_{xy}e_{xy} \) is an essential right ideal of \( S \) and \( E \subseteq K_1 \).

**Proof.** It is easy to verify that \( E \) is a right ideal of \( S \) contained in \( K_1 \). Let \( 0 \neq f \in K_1 \).

By induction on \( wt(f) \), we prove that \( 0 \neq f e_{xy} \in E \) for some \( e_{xy} \in K_1 \), \( r \in R \), which will prove that \( E \subseteq S \). Suppose \( f = ae_{xy} \). As \( a \neq 0 \), there exists a \( z \geq y \) and an \( r \in R \) such that \( 0 \neq ar \in A_{xz} \). Then \( 0 \neq f e_{yz} = a e_{xz} \in E \). Here, if \( y \) is not maximal, choose \( z > y \); if \( y \) is maximal, choose \( y = z \); in any case \( e_{xz} \in K_1 \). Thus the result holds for \( wt(f) = 1 \). To apply induction, let \( wt(f) = n > 1 \), and let the result hold for any positive integer less than \( n \). We write \( f = ae_{xy} + h \), with \( 0 \neq a \in R \), \( e_{xy} \in K_1 \), \( wt(h) = n - 1 \), and \( h(x, y) = 0 \). There exists an \( e_{yz} \in K_1 \) such that \( 0 \neq a e_{xy} e_{yz} = a e_{xz} \in E \). Then \( 0 \neq f e_{xz} = a e_{xz} + h e_{yz} \). If \( h e_{yz} = 0 \), \( f e_{xz} = a e_{xz} \in E \), and we finish. Suppose \( h e_{yz} \neq 0 \). By the induction hypothesis, there exists \( b e_{zw} \in K_1 \), with \( b \in R \), such that \( 0 \neq h e_{yz} b e_{zw} \in E \).

Let Minness(\( S \)) be the set of all essential right ideals of the form given in Lemma 4.3.

**Lemma 4.4.** \( Z(S) = \{ f \in S : fE = 0 \text{ for some } E \in \text{Minness}(S) \} \).

**Proof.** Let \( f \in Z(S) \). For some \( E \subseteq S \), \( fE = 0 \). By Lemmas 4.2 and 4.3, there exists an \( E' \in \text{Minness}(S) \) such that \( E' \subseteq E \). Then \( fE' = 0 \). This proves the result.

**Theorem 4.5.** \( Z(S) \neq 0 \) if and only if \( Z(R) = 0 \).

**Proof.** Let \( Z(R) \neq 0 \). For some \( r \neq 0 \) and an essential ideal \( A \) of \( R \), \( rA = 0 \). In Lemma 4.3, by taking every \( A_{xy} = A \), we get an \( E \subseteq S \) such that \( rE = 0 \). Thus \( Z(S) \neq 0 \). Conversely, let \( Z(S) \neq 0 \). Consider any \( 0 \neq f \in Z(S) \). For some \( E \in \text{Minness}(S) \), \( fE = 0 \). Now \( f(u, v) \neq 0 \) for some \( u \leq v \). Then \( 0 \neq e_u f \in Z(S) \). Suppose there exists a maximal \( z \geq v \). As \( z \) is maximal, it follows from Lemma 4.3(i) that \( B_{uz} = A_{uz} \), so \( e_v f e_v A_{uz} = 0 \), \( f(u, v)A_{uz} = 0 \), \( f(u, v) \in Z(R) \). Hence \( Z(R) \neq 0 \).

**Proposition 4.6.** For any \( (x, y) \in Y_0 \), set \( A_{xy} = R \), for \( (x, y) \in Y_1 \), set \( A_{xy} = R \), and for \( (x, y) \in Y_2 \), set \( A_{xy} = 0 \). Let \( T = \sum_{x,y} e_{xy} A_{xy} \).

(i) Then \( T \) is an ideal of \( S \), \( T \subseteq S \), and \( l \cdot \text{ann}(T) = 0 \).

(ii) \( S \) embeds in the ring \( Q = \text{Hom}(T_S, T_S) \) such that \( S \) is dense in \( Q_S \).
Proof. That $T$ is an essential right ideal in $S$ follows from Lemma 4.3. Suppose that $0 \neq f \in l \cdot \text{ann}(T)$. Then $f(u, v) \neq 0$ for some $u \leq v$. Suppose there exists no maximal $z \geq v$. Choose any $w > v$. Then $e_{v, w} \in T$ but $f e_{v, w} \neq 0$, which is a contradiction. Hence there exists a maximal $z \geq v$. Then $e_{v, z} \in T$ and $f e_{v, z} \neq 0$, which is also a contradiction. Hence $l \cdot \text{ann}(T) = 0$. Consider any $e_{x, y} \in T$. By Lemma 3.1, $\sigma(e_{x, y})$ is finite, so $f e_{x, y} = \sum_{u \leq y} a_{u, y} e_{u, y}$, a finite sum. By definition, the following two cases arise.

Case 1. $y$ is maximal. Then every $e_{u, y} \in T$, so $f e_{x, y} \in T$.

Case 2. There does not exist any maximal $z \geq y$. Then $u < y$, $A_{u, y} = R$, $e_{u, y} \in T$, hence $f e_{x, y} \in T$.

This proves that $T$ is an ideal in $S$. For each $f \in S$, let $\lambda(f)$ be the left multiplication on $T$ by $f$. Then $\lambda$ is an embedding of $S$ in $Q$. Consider any $\sigma, \eta \in Q$, with $\sigma \neq 0$. Then for some $f \in T$, $\sigma(f) \neq 0$. We see that $\sigma \cdot \lambda(f) = \lambda(\sigma(f)) \neq 0$ and $\eta \cdot \lambda(f) = \lambda(\eta(f)) \in \lambda(S)$. Hence $S_{\lambda}$ is dense in $Q_{\sigma}$.

For each $x_0 \in X_0$, set $T_{[x_0]} = \sum \{ e_{x, y} R : (x, y) \in Y_3 \text{ and } [x_0] = [y] \}$, and set $T' = \sum \{ e_{x, y} R : (x, y) \in Y_1 \}$. Observe that $T_{[x_0]} = T_{[x_1]}$ if and only if $[x_0] = [x_1]$. Each of $T_{[x_0]}$, $T'$ is a right ideal of $S$ contained in $T$, and $T$ is a direct sum of these right ideals. Let $Z_0$ be the set of equivalence classes in $X$ given by the members of $X_0$. For any ring $P$, let $\hat{P}$ be the maximal right ring of quotients of $P$ [5, Section 13]. The following result can be easily deduced from various results and exercises given in [5, Sections 8 and 13].

**Theorem 4.7.** (i) For any family of rings $\{P_{\alpha} : \alpha \in \Lambda\}$, $P = \prod_{\alpha \in \Lambda} P_{\alpha}$, $\hat{P} = \prod_{\alpha \in \Lambda} \hat{P}_{\alpha}$.

(ii) For any two subrings $A, B$ of a ring $P$, if $A_4 \subseteq B_4$, $B_4 \subseteq A_4$, then $\hat{A} = \hat{B}$.

(iii) For any positive integer $n$ and any ring $P$, $M_n(P) = M_n(\hat{P})$.

**Theorem 4.8.** (i) $Q = \text{Hom}(T_S, T_S) = (\prod \{ \text{Hom}_S(T_{[x_0]}, T_{[x_0]}) : [x_0] \in Z_0 \}) \times \text{Hom}_S(T', T')$.

(ii) Maximal right rings of quotients of $S$ and $Q$ are the same.

(iii) Let $P_{[x_0]} = \text{Hom}_S(T_{[x_0]}, T_{[x_0]})$ and $P' = \text{Hom}(T', T')$. Then $\hat{S} \cong (\prod \{ \hat{P}_{[x_0]} : [x_0] \in Z_0 \}) \times \hat{P'}$.

**Proof.** To prove (i) it is enough to prove that $\text{Hom}_S(T_{[x_0]}, T_{[x_1]}) = 0$ whenever $[x_0] \neq [x_1]$, $\text{Hom}_S(T_{[x_0]}, T') = 0 = \text{Hom}_S(T', T_{[x_0]})$. Consider $\sigma \in \text{Hom}_S(T_{[x_0]}, T_{[x_1]})$. For any $e_{x, y} \in T_{[x_0]}$, $[x_0] = [y]$, so $e_{u, y} \notin T_{[x_1]}$, but $\sigma(e_{x, y}) = \sum_{u \leq y} a_{u, y} e_{u, y}$, $a_{u, y} \in R$. Thus $\sigma(e_{x, y}) = 0$, $\sigma = 0$. Similarly, we can prove that the others are also zero. As $S_{\lambda}$ is dense in $Q_{\sigma}$, $\hat{S} = \hat{Q}$. Because of (ii) and Theorem 4.7, we get $\hat{S} \cong (\prod \{ \hat{P}_{[x_0]} : [x_0] \in Z_0 \}) \times \hat{P'}$.

We now discuss matrix representations of $\text{Hom}_S(T_{[x_0]}, T_{[x_1]})$ and $\text{Hom}_S(T', T')$.

**Theorem 4.9.** Let $x_0$ be a maximal member of $X$, $U_{x_0} = \{ x \in X : x \leq x_0 \}$. Then $\text{Hom}_R(T_{[x_0]}, T_{[x_1]})$ is isomorphic to the ring of column-finite matrices over $R$ indexed by $U_{x_0}$.

**Proof.** Let $\sigma \in \text{Hom}_S(T_{[x_0]}, T_{[x_1]})$. For $e_{x, y} \in T_{[x_0]}$, $y - x_0$. If $\sigma(e_{x, y}) = \sum_{u \leq y} a_{u, y} e_{u, y}$, then for any other $e_{x, z} \in T_{[x_0]}$, $\sigma(e_{z, x}) = \sum_{u \leq z} a_{u, z} e_{u, z} = \sigma(e_{x, y}) e_{y, z} = \sum_{u \leq y} a_{u, y} e_{u, z}$, $a_{u, y} = a_{u, z}$. Conversely, any $\sigma \in \text{Hom}_R(T_{[x_0]}, T_{[x_1]})$, such that if $\sigma(e_{x, y}) = \sum_{u \leq y} a_{u, y} e_{u, y}$,
then $\sigma(e_{xz}) = \sum_{u \leq y} a_{ux} e_{uy}$ for $y \leq z$, is in $\text{Hom}_S(T_{[x_0]}, T_{[x_0]})$. Now $V_{x_0} = \{e_{xy} : x \leq U_{x_0}, y \leq x_0\}$ is an $R$-basis of $T_{[x_0]}$. We write $\sigma(e_{xy}) = \sum_{u,v} a_{uvxy} e_{uv}, e_{uv} \in V_{x_0}$. Then $a_{uvxy} = 0$, for $v \neq y$, $a_{uvxy} = a_{uxz}$, whenever $y \not\leq z$. We write $b_{ux} = a_{uvxy}$ and $b_{ux} = 0$ otherwise. We get matrix $[b_{ux}]$ over $R$ indexed by $U_{x_0}$. This matrix is column finite; $\sigma \rightarrow [b_{ux}]$ gives the desired isomorphism.

\begin{theorem}
Let $X' = \{y \in X : \text{there exist no maximal } z \geq y\}$. Let $G$ be the set of arrays $[a_{uvxy}]$ over $R$ indexed by $X' \times X' \times X'$ such that it has following properties:

(i) $a_{xy} = 0$, whenever $x \not\leq y$, $v \leq y$, or $x < v < y$,

(ii) for any fixed pair $(x, y)$ with $x < y$, the number of $v$ for which $a_{vyx} \not= 0$ is finite,

(iii) for $y \not\leq z$, $a_{vyx} = a_{vxz}$ if $v < y$, and $a_{vyx} = 0$ if $v \not< y$ and $v < z$.

In $G$, define addition componentwise and the product by $[a_{uvxy}][b_{vx}] = [c_{vxy}]$ such that $c_{vxy} = \sum w a_{uvy} b_{vx}$. Then $\text{Hom}_S(T', T') \equiv G$.

In case $X'$ has the property that for every pair of elements $u, v$ in $X'$ there exists a $w \in X'$ such that $u \leq w, v \leq w$, then any array $[a_{uvxy}] \in G$ has the following additional properties:

(iv) if $u, v \in X'$ are not comparable, then $a_{uv} = 0$,

(v) for $x < y$, $x < z$, $a_{xy} = a_{xz}$.

Put $b_{ux} = a_{vx}$. Then $[b_{ux}]$ is a column finite matrix indexed by $X'$ with the property that $b_{ux} = 0$ if $v > x$, or there exists $y > x$ such that $v < y$. Set $b_{ux} = 0$ in all other cases. Let $B$ be the set of all such matrices. Then $B$ is a ring isomorphic to $\text{Hom}_S(T', T')$.

\begin{proof}
Let $\sigma \in \text{Hom}_S(T', T')$. For any $x < y \leq z \in X'$, we have $\sigma(e_{xy}) = \sum c_{uvxy} e_{uy}, c_{uvxy} \in R, (u, v) \leq y_1$, with $c_{uvxy} = 0$, whenever $v \not= y$. So we can write $\sigma(e_{xy}) = \sum_{v < y} c_{vxy} e_{vyx}$, a finite sum. For $y \leq z$, $\sigma(e_{xz}) = \sigma(e_{xy}) e_{yz}$ gives $a_{vyx} = a_{yx}$ for $v < y$ and $a_{vyx} = 0$ whenever $v \not< y$, $v < z$. Suppose we have some $x < y < y'$, by considering $\sigma(e_{xy}) = \sigma(e_{xy}) e_{yy'}$ it follows that $a_{vyx} = 0$. For any other $(v, x, y') \in X' \times X' \times X'$, set $a_{vyx} = 0$. We get an array $[a_{uvxy}]$ with the desired properties. Conversely, any such array gives an $S$-endomorphism of $T'$. This gives the desired isomorphism.

Suppose every pair of elements in $X'$ have a common upper bound. Consider any $u, v, w \in X'$ that are not comparable. By (i), $a_{vw} = 0$ for any $x$; this proves (iv). Suppose $x < y$, $x < z$. There exists $w \in X'$ such that $y < w, z < w$. Then $\sigma(e_{xw}) e_{zw} = \sigma(e_{xy}) e_{yw} = \sigma(e_{uw})$ gives (v). Set $b_{ux} = a_{vx}$. Because of (v), $b_{ux}$ is well defined. It gives a matrix $[b_{ux}]$ indexed by $X'$, which is column finite and has the property that $b_{ux} = 0$ if either $v > x$, or there exists $y > x$ such that $v < y$. Let $B$ be the set of all column-finite matrices $[b_{ux}]$ over $R$ indexed by $X' \times X'$ with $b_{ux} = 0$, whenever either $v > x$ or there exists a $y > x$ such that $v < y$. Then $\text{Hom}_S(T', T')$ is isomorphic to the ring $B$.

\begin{remark}
Let $X$ be any locally finite, preordered set and let $R$ be any indecomposable commutative ring. Obviously, $S = I^*(X, R)$ is a subring of $S' = I(X, R)$. But $S_S$ need not be dense or essential in $S'_S$. So the maximal right rings of quotients of $S$ and $S'$ need not be the same; in fact, they need not be isomorphic (see the example given below). In case $S_S$ is dense in $S'$, the two rings will have the same maximal right ring of quotients. In that case, $S'$ can help in studying $S'$.

\end{remark}
Theorem 4.12 [2]. Let $X$ be any partially ordered set such that for any $x \in X$, there exists a maximal element $z \geq x$ and $L_z = \{ y \in X : y \leq z \}$ is finite. Let $X_0$ be the set of maximal elements of $X$. For each $z \in X_0$, let $n_z$ be the number of elements $y \leq z$. For the ring $S = I(X,R), \hat{S} \cong \Pi\{M_{n_z}(\hat{R}) : z \text{ runs over representatives of equivalence classes in } X_0\}$.

Proof. Let $f, g \in S' = I(X,R)$ with $g \neq 0$. For some $u, v \in X$, $g(u,v) \neq 0$. Then $g e_v \neq 0$. At the same time the hypothesis on $X$ gives that the support of $f e_v$ is finite, so $f e_v \in S = I(X,R)$. Hence $S_S$ is dense in $S'$. After this, Theorems 4.7, 4.8, and 4.9 complete the proof.

Example 4.13. Let $X = \mathbb{N}$ be the set of natural numbers and let $R$ be any indecomposable commutative ring. Consider $S = I^*(\mathbb{N}, R)$ and $S' = I^*(\mathbb{N}_0, R)$. Let $0 \neq f \in S'$. For some $r \in \mathbb{N}, f e_r \neq 0$. Clearly, the support of $f e_r$ is finite. Hence $S_S$ is dense in $S'$. So the maximal right quotient rings of $S$ and $S'$ are the same. Consider $g \in S'$ for which $g(1,n) = 1$ for every $n$, and $g(n,m) = 0$ otherwise. Then for any $h \in S, h g = 0$ or $h g = k g$ for some $0 \neq k \in \mathbb{N}$, so $S_S$ is not dense in $S'$. Thus maximal left rings of quotients of $S$ and $S'$ are not the same. We now show that they need not be isomorphic. Consider $R = F$ a countable field. As $\mathbb{N}$ has no maximal element, $K = T = T', Q = \text{Hom}_K(T', T')$. By Theorem 4.10, $Q$ is isomorphic to $S'$. But $S'$, as a right $S'$-module, is dense in the ring $L$ of all column-finite matrices over $F$, indexed by $\mathbb{N}$. It is well known that the ring of all column-finite matrices over a field, indexed by any set, is right self-injective. Hence $L$ is the maximal right ring of quotients of $S$ and $S'$. Let $\mathbb{N}'$ has unique maximal element $0, \mathbb{N}' = T_0$, by Theorem 4.9, the corresponding $Q'$ is isomorphic to the ring of all column-finite matrices over $F$, indexed by $\mathbb{N}$. So $Q'$ is right self-injective. However $S = I^*(\mathbb{N}, F)$ is anti-isomorphic to $S_1 = I^*(\mathbb{N}', F)$. So $Q''$, the maximal left ring of quotients of $S$, is isomorphic to the ring of all row-finite matrices over $F$, indexed by $\mathbb{N}$. Now $e_{00} Q''$ is a countable set, and any minimal left ideal of $Q''$ is generated by an element of $e_{00} Q''$, so the left socle of $e_{00} Q''$ is of countable rank. For $S'$, the left socle is $e_{00} S'$, which is of uncountable rank. Also $S'$ is left nonsingular. So $L'$, the maximal left ring of quotients of $S'$, is such that its left socle is of uncountable rank. This proves that the maximal left rings of quotients of $S$ and $S'$ are not isomorphic.

References


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<thead>
<tr>
<th>Manuscript Due</th>
<th>December 1, 2008</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Round of Reviews</td>
<td>March 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>June 1, 2009</td>
</tr>
</tbody>
</table>

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