A COMMON FIXED POINT THEOREM FOR TWO SEQUENCES OF SELF-MAPPINGS

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ABSTRACT. In this paper a common fixed point theorem for two sequences of self-mappings from a complete metric space $M$ to $M$ is proved. Our theorem is a generalization of Hadzic's fixed point theorem[1].

KEY WORDS AND PHRASES. A common fixed point, self-mappings and complete metric spaces.

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1. INTRODUCTION.

Banach's fixed point theorem has been generalized by many authors. Among such investigations there are several, interesting and important studies[2]. Particularly, K. Iseki[3] proved a fixed point theorem of a sequence of self-mappings from a complete metric space $M$ to $M$. We are interested in fixed point theorems of a sequence of self-mappings since they pertain to the problem of finding an equilibrium point of a difference equation $x_{n+1} = f(n, x_n) \quad (n = 1, 2, ...)$.

Recently O. Hadzic proved the existence of a common fixed point for the sequence of self-mappings $(A_j)(j = 1, 2, ...)$, $S$ and $T$ where $A_j$ commutes with $S$ and $T$. His result is as follows:

THEOREM 1. Let $(M,d)$ be a complete metric space, $S, T : M \to M$ be continuous, $A_j : M \to SM \cap TM (j = 1, 2, ...)$ so that $A_j$ commutes with $S$ and $T$ and for every $i, j (i \neq j, i, j = 1, 2, ...)$ and every $x, y \in M$:

$$d(A_x, A_y) \leq q d(Sx, Ty), \quad 0 < q < 1 \quad (1.1)$$

Using Theorem 1, he gave a generalization of Gohde's fixed point theorem and extended Krasnoseliski's fixed point theorem.

In this paper we shall present a generalization of Hadzic's fixed point theorem.
2. MAIN THEOREMS.

Let $\mathbb{N}$ denote the set of all positive integers. In this section we shall prove the following theorem.

**THEOREM A.** Let $(M,d)$ be a complete metric space and let $\{A_p\}, \{B_p\}(p,q = 1,2,\ldots)$, be two sequences of mappings from $M$ to $M$.

Suppose that the following conditions are satisfied; for all $m,n \in \mathbb{N}$ and all $x,y \in M$,

(a) there exists a constant $k$ ($0 < k < 1$) such that

$$d(A_{2n-1}x, A_{2n}y) \leq kd(B_{2n-1}x, B_{2n}y),$$

$$d(A_{2n}x, A_{2n+1}y) \leq kd(B_{2n}x, B_{2n+1}y),$$

for all $m \geq n \geq 1$,

(b) $A_{2n}B_{2m} = B_{2m}A_{2n}$ and $A_{2n-1}B_{2m-1} = B_{2m-1}A_{2n-1}$,

(c) $B_{2n}B_{2m} = B_{2m}B_{2n}$ and $B_{2m-1}B_{2n-1} = B_{2n-1}B_{2m-1}$,

(d) $A_{2n-1}(M) \subset B_{2n}(M)$ and $A_{2n}(M) \subset B_{2n+1}(M)$.

If each $B_q(q = 1,2,\ldots)$ is continuous, then there exists a unique fixed point for two sequences $\{A_p\}$ and $\{B_q\}(p,q = 1,2,\ldots)$.

**PROOF.** Let $x_0$ be an arbitrary point in $M$. By condition (d) there exists a point $x_1 \in M$ such that $A_1x_0 = B_2x_1$. Next we choose a point $x_2 \in M$ such that $A_2x_1 = B_3x_2$. Inductively, we can define by condition (d), the sequence $\{x_n\}$ such that

$$A_{2n-1}x_{2n-2} = B_{2n-1}x_{2n-2} \quad \text{and} \quad A_{2n}x_{2n-1} = B_{2n+1}x_{2n}, \quad n \in \mathbb{N}. \quad (2.1)$$

First of all we shall show that $\{B_nx_{n-1}\}$ is a Cauchy sequence. By (2.1) and condition (a), we obtain that for all $n \in \mathbb{N}$

$$d(B_{2n-1}x_{2n-2}, B_{2n}x_{2n-1}) = d(A_{2n-2}x_{2n-3}, A_{2n-1}x_{2n-2})$$

$$\leq kd(B_{2n-2}x_{2n-3}, B_{2n-1}x_{2n-2}) = kd(A_{2n-3}x_{2n-4}, A_{2n-2}x_{2n-3})$$

$$\leq k^2d(B_{2n-3}x_{2n-4}, B_{2n-2}x_{2n-3}) \leq \ldots \leq k^{2n}d(B_1x_0, B_2x_1)$$

and similarly that

$$d(B_{2n}x_{2n-1}, B_{2n+1}x_{2n}) = d(A_{2n-1}x_{2n-2}, A_{2n}x_{2n-1})$$

$$\leq kd(B_{2n-1}x_{2n-2}, B_{2n}x_{2n-1}) \leq \ldots \leq k^{2n-1}d(B_1x_0, B_2x_1).$$
Since $0 < k < 1$, this implies that the sequence $\{B_n x_{n-1}\}$ is a Cauchy sequence. Thus $\{B_n x_{n-1}\}$ converges to some point $v$ in $M$ because $M$ is complete. Now since each $B_q (q \in \mathbb{N})$ is continuous, we obtain that

$$B_{2m} v = B_{2m} \left( \lim_{n \to \infty} B_{2n+1} x_{2n+1} \right) = \lim_{n \to \infty} (B_{2m} B_{2n+1} x_{2n+1})$$

$$= \lim_{n \to \infty} (B_{2m} A_{2n+1} x_{2n+1}) = \lim_{n \to \infty} (A_{2n} B_{2m} x_{2n+1})$$

and similarly that $B_{2m+1} v = \lim_{n \to \infty} (A_{2n+1} B_{2m+1} x_{2n+1})$ and $B_{2m-1} v = \lim_{n \to \infty} (A_{2n-1} B_{2m-1} x_{2n-1})$. Hence by condition (c), we have

$$d(B_{2m} v, B_{2m+1} v) = \lim_{n \to \infty} d(A_{2m} B_{2m} x_{2n+1}, A_{2m+1} B_{2m+1} x_{2n+1})$$

$$\leq \lim_{n \to \infty} k d(B_{2m+1} B_{2m} x_{2n+1}, B_{2m+1} x_{2n+1})$$

$$= k d(B_{2m} v, B_{2m+1} v)$$

and $d(B_{2m} v, B_{2m-1} v) \leq k d(B_{2m} v, B_{2m-1} v)$ (m $\in \mathbb{N}$) in like manner, which implies that $B_m v = B_{m+1} v$ for all $m \geq 1$. Next we shall show that $A_n v = B_n v$ for all $n \leq 1$. By (2.1), conditions (b) and (c), we have

$$d(B_{2n+1} B_{2m+1} x_{2n+1}, A_{2n} v) = d(A_{2n+1} B_{2n+1} x_{2n+1}, A_{2n} v)$$

$$\leq k d(B_{2n} B_{2n+1} x_{2n+1}, A_{2n} v)$$

$$= k d(B_{2n} v, B_{2n+1} v)$$

Thus letting $m \to \infty$, we obtain that $d(B_{2n+1} v, A_{2n} v) \leq k d(B_{2n+1} v, B_{2n} v)$ from which it follows that $A_{2n} v = B_{2n+1} v$ for all $n \geq 1$. And since

$$d(A_{2n+1} v, A_{2n} v) \leq k d(B_{2n+1} v, B_{2n} v)$$

we obtain that $A_n v = A_{n+1} v = B_{n+1} v = B_n v$ for all $n \in \mathbb{N}$. Furthermore, for all $n \in \mathbb{N}$, we obtain

$$d(A_{2n} v, A_{2n-1} v) \leq k d(B_{2n} v, B_{2n-1} v)$$

and $d(A_{2n-1} v, A_{2n} v) \leq k d(B_{2n-1} v, B_{2n} v)$. Therefore we obtain $u = A_p(u) = B_p(u)$ for all $p \geq 1$ setting $u = A_n v$ because $0 < k < 1$. 

Now we shall prove that $u$ is a unique common fixed point of $\{A_p\}$ and $\{B_p\}$. If there exists another point $w$ such that $w = A_p w = B_p w$ for all $p > 1$, then

$$d(u, w) = d(A_{2m-1}u, A_{2m}w) \leq kd(B_{2m-1}u, B_{2m}w)$$

$$\leq kd(u, w),$$

which is a contradiction since $0 < k < 1$. Therefore $u$ is a unique common fixed point of two sequences of self-mappings $\{A_p\}$ and $\{B_p\}$. This completes the proof.

If $S = B_{2n-1}$ and $T = B_{2n}(n = 1, 2, ...)$, we obtain Theorem 1 as the corollary of Theorem A. Next we obtain the following theorem which is a generalization of Theorem 1 in [4].

**THEOREM B.** Let $(M, d)$ be a complete metric space and let $\{T_p\}$ $(p = 1, 2, ...)$ be a sequence of mappings from $M$ to $M$. Suppose that the following conditions as satisfied for all $m > n > 0$ and $x, y \in M$

(e) there exists a constant $h$ ($h > 1$) such that

$$d(T_{2n-1}x, T_{2n}y) \geq hd(x, y)$$

and

$$d(T_{2n}x, T_{2n+1}y) \geq hd(x, y),$$

(f) $T_p T_q = T_q T_p$ (p, q are even or odd respectively).

If every $T_n$ is continuous on $M$ and $T_n(M) = M(n = 1, 2, ...)$, then there exists a unique fixed point for $T_n$.

**PROOF.** Set $A_n = I$ ($I$ is the identity map from $M$ to $M$) in Theorem A. The proof is complete.

**REMARK 1.** We remark that the mapping $f: X \to X$ in Theorem 1 of [4] is continuous from the condition of the theorem.

**REFERENCES**

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