NONLINEAR ERGODIC THEOREMS FOR A SEMITOPOLOGICAL SEMIGROUP OF NON-LIPSCHITZIAN MAPPINGS WITHOUT CONVEXITY

G. LI AND J. K. KIM

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Let $G$ be a semitopological semigroup, $C$ a nonempty subset of a real Hilbert space $H$, and $\mathcal{S} = \{T_t : t \in G\}$ a representation of $G$ as asymptotically nonexpansive type mappings of $C$ into itself. Let $L(x) = \{z \in H : \inf_{s \in G} \sup_{t \in G} \|T_{ts} x - z\| = \inf_{t \in G} \|T_t x - z\|\}$ for each $x \in C$ and $L(\mathcal{S}) = \cap_{x \in C} L(x)$. In this paper, we prove that $\cap_{s \in G} \text{conv}\{T_{ts} x : t \in G\} \cap L(\mathcal{S})$ is nonempty for each $x \in C$ if and only if there exists a unique nonexpansive retraction $P$ of $C$ into $L(\mathcal{S})$ such that $PT_s = P$ for all $s \in G$ and $P(x) \in \text{conv}\{T_s x : s \in G\}$ for every $x \in C$. Moreover, we prove the ergodic convergence theorem for a semitopological semigroup of non-Lipschitzian mappings without convexity.

1. Introduction and preliminaries

Let $H$ be a Hilbert space with norm $\|\cdot\|$ and inner product $(\cdot, \cdot)$. Let $G$ be a semitopological semigroup, that is, a semigroup with a Hausdorff topology such that for each $s \in G$ the mappings $s \mapsto s \cdot t$ and $s \mapsto t \cdot s$ of $G$ into itself are continuous. Let $C$ be a nonempty subset of $H$ and let $\mathcal{S} = \{T_t : t \in G\}$ be a semigroup on $C$, that is, $T_{st}(x) = T_s T_t(x)$ for all $s, t \in G$ and $x \in C$. Recall that a semigroup $\mathcal{S}$ is said to be

(a) nonexpansive if $\|T_t x - T_t y\| \leq \|x - y\|$ for $x, y \in C$ and $t \in G$.

(b) asymptotically nonexpansive [6] if there exists a function $k : G \mapsto [0, \infty)$ with $\inf_{s \in G} \sup_{t \in G} k_{ts} \leq 1$ such that $\|T_t x - T_t y\| \leq k_t \|x - y\|$ for $x, y \in C$ and $t \in G$.

(c) of asymptotically nonexpansive type [6] if for each $x$ in $C$, there is a function $r(\cdot, x) : G \mapsto [0, \infty)$ with $\inf_{s \in G} \sup_{t \in G} r(ts, x) = 0$ such that $\|T_t x - T_t y\| \leq \|x - y\| + r(t, x)$ for all $y \in C$ and $t \in G$.

It is easily seen that (a)$\Rightarrow$(b)$\Rightarrow$(c) and that both the inclusions are proper (cf. [6, page 112]).

Baillon [1] proved the first nonlinear mean ergodic theorem for nonexpansive mappings in a Hilbert space: let $C$ be a nonempty closed convex subset of a Hilbert space $H$ and $T$ a nonexpansive mapping of $C$ into itself. If the set $F(T)$ of fixed points of $T$
is nonempty, then for each \( x \in C \), the Cesáro means

\[
S_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} T^k x
\]

converge weakly as \( n \to \infty \) to a point of \( F(T) \). In this case, putting \( y = P x \) for each \( x \in C \), \( P \) is a nonexpansive retraction of \( C \) onto \( F(T) \) such that \( PT = TP = P \) and \( P x \in \text{conv} \{ T^n x : n = 0, 1, 2, \ldots \} \) for each \( x \in C \), where \( \text{conv} A \) is the closure of the convex hull of \( A \). The analogous results are given for nonexpansive semigroups on \( C \) by Baillon [2] and Brezis-Browder [3]. In [10], Mizoguchi-Takahashi proved a nonlinear ergodic retraction theorem for Lipschitzian semigroups by using the notion of submean. Recently, Li and Ma [8, 9] proved the nonlinear ergodic retraction theorems for non-Lipschitzian semigroups in a Banach space without using the notion of submean. Also, in 1992, Takahashi [13] proved the ergodic theorem for nonexpansive semigroups on condition that \( \bigcap_{s \in G} \text{conv} \{ T_{st} x : t \in G \} \subset C \) for some \( x \in C \).

In this paper, without using the concept of submean, we prove nonlinear ergodic theorem for semitopological semigroup of non-Lipschitzian mappings without convexity in a Hilbert space. We first prove that if \( C \) is a nonempty subset of a Hilbert space \( H, G \) a semitopological semigroup, and \( \mathcal{S} = \{ T_t : t \in G \} \) a representation of \( G \) as asymptotically nonexpansive type mappings of \( C \) into itself, then \( \bigcap_{s \in G} \text{conv} \{ T_{st} x : t \in G \} \bigcap L(\mathcal{S}) \) is nonempty for each \( x \in C \) if and only if there exists a unique nonexpansive retraction \( P \) of \( C \) into \( L(\mathcal{S}) \) such that \( PT_s = P \) for all \( s \in G \) and \( P x \) is in the closed convex hull of \( \{ T_s x : s \in G \} \), where \( L(x) = \{ z : \inf_{s \in G} \sup_{t \in G} \| T_{ts} x - T_t x - z \| = \inf_{t \in G} \| T_t x - z \| \} \) and \( L(\mathcal{S}) = \bigcap_{x \in C} L(x) \). By using this result, we also prove the ergodic convergence theorem for semitopological semigroup of non-Lipschitzian mapping without convexity. Our results are generalizations and improvements of the previously known results of Brézis-Browder [3], Hirano-Takahashi [4], Mizoguchi-Takahashi [10], Takahashi-Zhang [14], and Takahashi [11, 12, 13] in many directions. Further, it is safe to say that in the results [1, 2, 3, 4, 5, 7, 10, 11, 12, 13, 14], many key conditions are not necessary.

2. Ergodic convergence theorems

Throughout this paper, we assume that \( C \) is a nonempty subset of a real Hilbert space \( H, G \) a semitopological semigroup, and \( \mathcal{S} = \{ T_t : t \in G \} \) an asymptotically nonexpansive type semigroup on \( C \). For each \( x \in C \), define \( L(x) \) and \( L(\mathcal{S}) \) by

\[
L(x) = \left\{ z : \inf_{s \in G} \sup_{t \in G} \| T_{ts} x - z \| = \inf_{t \in G} \| T_t x - z \| \right\}, \quad L(\mathcal{S}) = \bigcap_{x \in C} L(x),
\]

respectively. We denote \( F(\mathcal{S}) \) by the set \( \{ x \in C : T_s(x) = x \text{ for all } s \in G \} \) of common fixed point of \( \mathcal{S} \). We begin with the following lemma.

**Lemma 2.1.** Let \( C \) be a nonempty subset of a Hilbert space \( H \) and \( \mathcal{S} = \{ T_t : t \in G \} \) an asymptotically nonexpansive type semigroup on \( C \). Then \( F(\mathcal{S}) \subset L(\mathcal{S}) \).
Proof. Let \( x \in C \) and \( f \in F(\mathcal{S}) \). Since \( \mathcal{S} \) is asymptotically nonexpansive type, for an arbitrary \( \varepsilon > 0 \), there exists \( s_0 \in G \) such that for all \( t \in G \)

\[
r(t s_0, f) < \varepsilon.
\]

(2.2)

Hence, for each \( a \in G \),

\[
\inf_{s \in G} \sup_{t \in G} \| T_{t s} x - f \| \leq \sup_{t \in G} \| T_{t s_0} x - f \| \leq \sup_{t \in G} \left( \| T_{a} x - f \| + r(t s_0, f) \right)
\]

\[
\leq \| T_{a} x - f \| + \varepsilon.
\]

(2.3)

Since \( \varepsilon > 0 \) is arbitrary, we have

\[
\inf_{s \in G} \sup_{t \in G} \| T_{t s} x - f \| \leq \inf_{t \in G} \| T_{t} x - f \|.
\]

This completes the proof. \( \square \)

Remark 2.2. It is not easy to prove that \( F(\mathcal{S}) \) is nonempty when \( C \) is not a convex subset. However, we can show that \( L(\mathcal{S}) \) is nonempty under some conditions and it is important for the ergodic convergence theorem.

The following proposition plays a crucial role in the proof of our main theorems in this paper.

Proposition 2.3. Let \( G \) be a semitopological semigroup, \( C \) a nonempty subset of a Hilbert space \( H \), and \( \mathcal{S} = \{ T_t : t \in G \} \) an asymptotically nonexpansive type semigroup on \( C \). Then, for every \( x \in C \), the set

\[
\bigcap_{s \in G} \text{conv} \{ T_{t s} x : t \in G \} \bigcap L(x),
\]

(2.4)

consists of at most one point.

Proof. Let \( u, v \in \bigcap_{s \in G} \text{conv} \{ T_{t s} x : t \in G \} \bigcap L(x) \), without loss of generality, we assume that

\[
\inf_{t \in G} \| T_t x - u \|^2 \leq \inf_{t \in G} \| T_t x - v \|^2.
\]

(2.5)

Now, for each \( t, s \in G \), since

\[
\| u - v \|^2 + 2 \langle T_{t s} x - u, u - v \rangle = \| T_{t s} x - v \|^2 - \| T_{t s} x - u \|^2,
\]

(2.6)

we have

\[
\| u - v \|^2 + 2 \inf_{t \in G} \langle T_{t s} x - u, u - v \rangle \geq \inf_{t \in G} \| T_{t s} x - v \|^2 - \sup_{t \in G} \| T_{t s} x - u \|^2,
\]

(2.7)

\[
\geq \inf_{t \in G} \| T_t x - v \|^2 - \sup_{t \in G} \| T_{t s} x - u \|^2.
\]

From \( u \in L(x) \), we have

\[
\| u - v \|^2 + 2 \sup_{s \in G} \inf_{t \in G} \langle T_{t s} x - u, u - v \rangle \geq \inf_{t \in G} \| T_t x - v \|^2 - \inf_{s \in G} \sup_{t \in G} \| T_{t s} x - u \|^2
\]

\[
= \inf_{t \in G} \| T_t x - v \|^2 - \inf_{t \in G} \| T_t x - u \|^2 \geq 0.
\]

(2.8)
Therefore, for \( \varepsilon > 0 \) there is an \( s_1 \in G \) such that
\[
\|u - v\|^2 + 2(T_{t s_1} x - u, u - v) > -\varepsilon \quad \forall t \in G.
\] (2.9)
From \( v \in \text{conv}\{T_{t s_1} x : t \in G\} \), we have
\[
\|u - v\|^2 + 2(v - u, u - v) \geq -\varepsilon.
\] (2.10)
This inequality implies that \( \|u - v\|^2 \leq \varepsilon \). Since \( \varepsilon > 0 \) is arbitrary, we have \( u = v \). This completes the proof. \( \Box \)

Remark 2.4. In the Takahashi-Zhang’s result [14], it is assumed that \( C \) is a closed convex subset, \( G \) a reversible semigroup, and \( \mathfrak{I} \) an asymptotically nonexpansive semigroup. Proposition 2.3 shows those key conditions are not necessary.

Let \( m(G) \) be the Banach space of all bounded real-valued functions on a semi-topological semigroup \( G \) with the supremum norm and let \( X \) be a subspace of \( m(G) \) containing constants. Then, an element \( \mu \) of \( X^\ast \) (the dual space of \( X \)) is called a mean on \( X \) if \( \|\mu\| = \mu(1) = 1 \). Let \( \mu \) be a mean on \( X \) and \( f \in X \). Then, according to time and circumstances, we use \( \mu_t (f(t)) \) instead of \( \mu(f) \).

For each \( s \in G \) and \( f \in m(G) \), we define elements \( l_s f \) and \( r_s f \) in \( m(G) \) given by \( (l_s f)(t) = f(st) \) and \( (r_s f)(t) = f(ts) \) for all \( t \in G \), respectively.

Throughout the rest of this section, let \( X \) be a subspace of \( m(G) \) containing constants invariant under \( l_s \) and \( r_s \) for each \( s \in G \). Furthermore, suppose that for each \( x \in C \) and \( y \in H \), a function \( f(t) = \|T_t x - y\|^2 \) is in \( X \). For \( \mu \in X^\ast \), we define the value \( \mu_t (T_t x, y) \) of \( \mu \) at this function. By Riesz theorem, there exists a unique element \( \mathfrak{I}_{\mu x} \) in \( X \) such that
\[
\mu_t (T_t x, y) = \langle \mathfrak{I}_{\mu x}, y \rangle \quad \forall y \in H.
\] (2.11)

Lemma 2.5. Suppose that \( X \) has an invariant mean \( \mu \). Then we have
\[
\bigcap_{s \in G} \text{conv}\{T_{t s} x : t \in G\} \bigcap L(x) = \{\mathfrak{I}_{\mu x}\} \quad \text{for every } x \in C.
\] (2.12)

Further, if \( T_t \) is continuous for each \( t \in G \) and \( \bigcap_{s \in G} \text{conv}\{T_{t s} x : t \in G\} \subset C \) for some \( x \in C \), then \( \mathfrak{I}_{\mu x} \in F(\mathfrak{I}) \).

Proof. Since \( \mu \) is an invariant mean, it is easy to show that \( \mathfrak{I}_{\mu x} \in \bigcap_{s \in G} \text{conv}\{T_{t s} x : t \in G\} \) for each \( x \in C \). By Proposition 2.3, it is enough to prove that \( \mathfrak{I}_{\mu x} \in L(x) \) for each \( x \in C \). To this end, let \( \varepsilon > 0 \), since \( \mathfrak{I} \) is an asymptotically nonexpansive type semigroup, for each \( t \in G \) there is an \( h_t \in G \) such that for each \( h \in G \),
\[
r(hh_t, T_t x) < \varepsilon.
\] (2.13)

Put \( M = \sup_{t, x \in G} \|T_t x - T_s x\| \), then we have
\[
\|T_{h_t x} - \mathfrak{I}_{\mu x}\|^2 - \|T_t x - \mathfrak{I}_{\mu x}\|^2 = \mu_s \left( \|T_{h_t t} x - T_t x\|^2 - \|T_t x - T_s x\|^2 \right)
\] (2.14)
\[
= \mu_s \left( \|T_{h_t t} x - T_{h_t s} x\|^2 - \|T_t x - T_s x\|^2 \right)
\]
\[
\leq 2M \varepsilon \quad \text{for each } h \in G.
\]
Hence, we have
\[
\inf_{s \in G} \sup_{h \in G} \| T_{hs} x - \mathcal{N}_{\mu} x \|^2 \leq \| T_t x - \mathcal{N}_{\mu} x \|^2 + 2M \varepsilon \quad \forall t \in G.
\] (2.15)

Since \( \varepsilon > 0 \) is arbitrary, we have \( \mathcal{N}_{\mu} x \in L(x) \). Finally, suppose that \( \bigcap_{s \in G} \text{conv} \{ T_{st} x : t \in G \} \subseteq C \) and each \( T_t \) is continuous from \( C \) into itself. Then, we can easily prove that \( \mathcal{N}_{\mu} x \in \bigcap_{s \in G} \text{conv} \{ T_{st} x : t \in G \} \) and hence we have \( \mathcal{N}_{\mu} x \in C \). For each \( h \in G \) and \( \varepsilon \in (0, 1) \), there exists \( 0 < \delta < \varepsilon \) such that \( \| T_t y - T_h \mathcal{N}_{\mu} x \| < \varepsilon \) whenever \( y \in C \) and \( \| y - \mathcal{N}_{\mu} x \| \leq \delta \). Since \( \mathcal{N} \) is an asymptotically nonexpansive type semigroup, there is \( s_0 \in G \) such that
\[
r(t, \mathcal{N}_{\mu} x) < \frac{1}{2(M_1 + 1)} \delta^2 \quad \forall t \in G,
\] (2.16)

where \( M_1 = \sup_{t \in G} \| T_t x - \mathcal{N}_{\mu} x \| \). Then for each \( t, s \in G \), we have
\[
\| T_{ss_0} \mathcal{N}_{\mu} x - \mathcal{N}_{\mu} x \|^2 + 2(T_t x - \mathcal{N}_{\mu} x, T_{ss_0} \mathcal{N}_{\mu} x - T_{ss_0} \mathcal{N}_{\mu} x)
\]
\[
= \| T_t x - T_{ss_0} \mathcal{N}_{\mu} x \|^2 - \| T_t x - \mathcal{N}_{\mu} x \|^2
\]
\[
= \| T_{ss_0} x - T_{ss_0} \mathcal{N}_{\mu} x \|^2 - \| T_t x - \mathcal{N}_{\mu} x \|^2 - \| T_{ss_0} x - T_{ss_0} \mathcal{N}_{\mu} x \|^2 + \| T_t x - T_{ss_0} \mathcal{N}_{\mu} x \|^2
\]
\[
\leq \delta^2 - \| T_{ss_0} x - T_{ss_0} \mathcal{N}_{\mu} x \|^2 + \| T_t x - T_{ss_0} \mathcal{N}_{\mu} x \|^2.
\] (2.17)

It follows that
\[
\| T_{ss_0} \mathcal{N}_{\mu} x - \mathcal{N}_{\mu} x \| \leq \delta \quad \forall s \in G.
\] (2.18)

This implies that
\[
\| T_h \mathcal{N}_{\mu} x - \mathcal{N}_{\mu} x \| \leq \| T_h \mathcal{N}_{\mu} x - T_h T_{ss_0} \mathcal{N}_{\mu} x \|^2 + \| T_{ss_0} \mathcal{N}_{\mu} x - \mathcal{N}_{\mu} x \| < 2\varepsilon.
\] (2.19)

Since \( \varepsilon > 0 \) is arbitrary, we have \( T_h \mathcal{N}_{\mu} x = \mathcal{N}_{\mu} x \). This completes the proof. \( \square \)

Now, we prove a nonlinear ergodic theorem for asymptotically nonexpansive type semigroups without convexity. Before doing this, we give a definition concerning means. Let \( \{ \mu_\alpha : \alpha \in A \} \) be a net of means on \( X \), where \( A \) is a directed set. Then \( \{ \mu_\alpha : \alpha \in A \} \) is said to be asymptotically invariant if for each \( f \in X \) and \( s \in G \),
\[
\mu_\alpha (f) - \mu_\alpha (l_s f) \longrightarrow 0, \quad \mu_\alpha (f) - \mu_\alpha (r_s f) \longrightarrow 0.
\] (2.20)

**Theorem 2.6.** Let \( C \) be a nonempty subset of a Hilbert space \( H \), \( X \) an invariant subspace of \( m(G) \) containing constants, and \( \mathcal{N} = \{ T_t : t \in G \} \) an asymptotically nonexpansive type semigroup on \( C \). If for each \( x \in C \) and \( y \in H \), the function \( f \) on \( G \) defined by \( f(t) = \| T_t x - y \|^2 \) belong to \( X \), then for an asymptotically invariant net \( \{ \mu_\alpha : \alpha \in A \} \) on \( X \), the net \( \{ \mathcal{N}_{\mu} x \}_\alpha \) converges weakly to an element \( x_0 \in L(x) \).
Further, if $T_t$ is continuous for each $t \in G$ and $\bigcap_{s \in G} \text{conv}\{T_{st}x : t \in G\} \subset C$, then $x_0 \in F(\mathcal{I})$.

**Proof.** Let $W$ be the set of all weak limit points of subnet of the net $\{\mathcal{I}_{\mu_\alpha}x : \alpha \in \Lambda\}$. By Proposition 2.3, it is enough to prove that

$$W \subset \bigcap_{s \in G} \text{conv}\{T_{ts}x : t \in G\} \cap L(x). \quad (2.21)$$

To show this, let $z \in W$ and let $\{\mathcal{I}_{\mu_{\alpha\beta}}x\}$ be a subnet of $\{\mathcal{I}_{\mu_\alpha}x\}$ such that $\{\mathcal{I}_{\mu_{\alpha\beta}}x\}$ converges weakly to $z$. Now, without loss of generality, we can suppose that $\{\mathcal{I}_{\mu_{\alpha\beta}}x\}$ converges weakly* to $\mu \in X^*$. It is easily seen that $\mu$ is an invariant mean on $X$ and then Lemma 2.5 implies that $z = \mathcal{I}_\mu x \in \bigcap_{s \in G} \text{conv}\{T_{ts}x : t \in G\} \cap L(x)$. This completes the proof. \[\square\]

Let $C(G)$ be the Banach space of all bounded continuous real-valued functions on $G$ and let $RUC(G)$ be the space of all bounded right uniformly continuous functions on $G$, that is, all $f \in C(G)$ such that the mapping $s \mapsto r_s f$ is continuous. Then $RUC(G)$ is a closed subalgebra of $C(G)$ containing constants and invariant under $l_s$ and $r_s$.

As a direct consequence of Theorem 2.6, we obtain the following corollary.

**Corollary 2.7 (see [13]).** Let $C$ be a nonempty subset of a Hilbert space $H$ and let $G$ be a semitopological semigroup such that $RUC(G)$ has an invariant mean. Let $\mathcal{I} = \{T_t : t \in G\}$ be a nonexpansive semigroup on $C$ such that $T_t x : t \in G$ is bounded and $\bigcap_{s \in G} \text{conv}\{T_{ts}x : t \in G\} \subset C$ for some $x \in C$. Then, $F(\mathcal{I}) \neq \emptyset$. Further, for an asymptotically invariant net $\{\mu_\alpha\}_{\alpha \in \Lambda}$ of means on $RUC(G)$, the net $\{\mathcal{I}_{\mu_\alpha}x\}_{\alpha \in \Lambda}$ converges weakly to an element $x_0 \in F(\mathcal{I})$.

**Remark 2.8.** For the proof of Corollary 2.7, Takahashi [13] used the condition $\bigcap_{s \in G} \text{conv}\{T_{ts}x : t \in G\} \subset C$. But, from Theorem 2.6, we can prove the result without this condition except proving the fact that the weak limit of $\{\mathcal{I}_{\mu_\alpha}x\}$ is in $F(\mathcal{I})$.

### 3. Nonexpansive retractions

In this section, we prove an ergodic retraction theorem for a semitopological semigroup of asymptotically nonexpansive type mappings without convexity.

**Theorem 3.1.** Let $C$ be a nonempty subset of a Hilbert space $H$ and let $\mathcal{I} = \{T_t : t \in G\}$ be a semitopological semigroup of asymptotically nonexpansive type mappings on $C$ such that $L(\mathcal{I}) \neq \emptyset$. Then the following statements are equivalent:

(a) $\bigcap_{s \in G} \text{conv}\{T_{ts}x : t \in G\} \cap L(\mathcal{I}) \neq \emptyset$ for each $x \in C$.

(b) There is a unique nonexpansive retraction $P$ of $C$ into $L(\mathcal{I})$ such that $PT_t = P$ for every $t \in G$ and $Px \in \text{conv}\{T_{t}x : t \in G\}$ for every $x \in C$.

**Proof.** (b) $\Rightarrow$ (a). Let $x \in C$, then $Px \in L(\mathcal{I})$. Also $Px \in \bigcap_{s \in G} \text{conv}\{T_{ts}x : t \in G\}$. In fact, for each $s \in G$, $Px = PT_s x \in \text{conv}\{T_{ts}x : t \in G\} = \text{conv}\{T_{ts}x : t \in G\}$. 


(a)⇒(b). Let \( x \in C \). Then by Proposition 2.3, \( \cap_{s \in G} \text{conv}\{ T_{ts} x : t \in G \} \cap L(\mathfrak{F}) \) contains exactly one point \( P x \). For each \( a \in G \), we have

\[
\{ P T_a x \} = \cap_{s \in G} \text{conv}\{ T_{tsa} x : t \in G \} \cap L(\mathfrak{F}) \supseteq \cap_{s \in G} \text{conv}\{ T_{ts} x : t \in G \} \cap L(\mathfrak{F}) = \{ P x \}
\]  

(3.1)

and hence we have \( P T_a = P \) for every \( a \in G \).

Finally, we have to show that \( P \) is nonexpansive. Let \( x, y \in C \) and \( 0 < \lambda < 1 \). Then for any \( \varepsilon > 0 \), there exists \( s_1 \in G \) such that

\[
\sup_{t \in G} \| T_{ts_1} x - Py \| \leq \inf_{t \in G} \| T_t x - Py \| + \varepsilon,
\]

(3.2)

from \( Py \in L(\mathfrak{F}) \). Hence, we have

\[
\| \lambda T_{ts_1} x + (1 - \lambda) Px - Py \|^2 \\
= \| \lambda (T_{ts_1} x - Py) + (1 - \lambda) (Px - Py) \|^2 \\
= \lambda \| T_{ts_1} x - Py \|^2 + (1 - \lambda) \| Px - Py \|^2 - \lambda (1 - \lambda) \| T_{ts_1} x - Px \|^2 \\
\leq \lambda (\| T_{ab} x - Py \|^2 + \varepsilon)^2 + (1 - \lambda) \| Px - Py \|^2 - \lambda (1 - \lambda) \inf_{t \in G} \| T_t x - Px \|^2.
\]

(3.3)

for each \( t, s, a, b \in G \). Since \( \varepsilon > 0 \) is arbitrary, this implies

\[
\inf_{s \in G} \sup_{t \in G} \| \lambda T_{ts} x + (1 - \lambda) Px - Py \|^2 \\
\leq \lambda \| T_{ab} x - Py \|^2 + (1 - \lambda) \| Px - Py \|^2 - \lambda (1 - \lambda) \inf_{t \in G} \| T_t x - Px \|^2 \\
= \| \lambda T_{ab} x + (1 - \lambda) Px - Py \|^2 + \lambda (1 - \lambda) \| T_{ab} x - Px \|^2 - \lambda (1 - \lambda) \inf_{t \in G} \| T_t x - Px \|^2.
\]

(3.4)

Then it is easily seen that

\[
\inf_{s \in G} \sup_{t \in G} \| \lambda T_{ts} x + (1 - \lambda) Px - Py \|^2 - \lambda (1 - \lambda) \inf_{t \in G} \inf_{b \in G} \sup_{a \in G} \| T_{ab} x - Px \|^2 \\
\leq \sup_{b \in G} \sup_{a \in G} \| \lambda T_{ab} x + (1 - \lambda) Px - Py \|^2 - \lambda (1 - \lambda) \inf_{t \in G} \| T_t x - Px \|^2.
\]

(3.5)

Since \( Px \in L(\mathfrak{F}) \), we have

\[
\inf_{s \in G} \sup_{t \in G} \| \lambda T_{ts} x + (1 - \lambda) Px - Py \|^2 \leq \inf_{s \in G} \sup_{t \in G} \| \lambda T_{ts} x + (1 - \lambda) Px - Py \|^2.
\]

(3.6)

Let

\[
h(\lambda) = \inf_{s \in G} \sup_{t \in G} \| \lambda T_{ts} x + (1 - \lambda) Px - Py \|^2.
\]

(3.7)
Then for any $\varepsilon > 0$, there exists $s_2 \in G$ such that for all $t \in G$,
\[
\|\lambda T_{ts_2}x + (1 - \lambda)Px - Py\|^2 \leq h(\lambda) + \varepsilon
\]
(3.8)
and hence
\[
(\lambda T_{ts_2}x + (1 - \lambda)Px - Py, Px - Py) \leq (h(\lambda) + \varepsilon)^{1/2}\|Px - Py\| \quad \forall t \in G.
\]
(3.9)
From $Px \in \text{conv}\{T_{ts_2}x : t \in G\}$, we have
\[
(\lambda Px + (1 - \lambda)Px - Py, Px - Py) \leq (h(\lambda) + \varepsilon)^{1/2}\|Px - Py\|.
\]
(3.10)
Since $\varepsilon > 0$ is arbitrary, this yields that
\[
\|Px - Py\|^2 \leq h(\lambda).
\]
(3.11)
That is,
\[
\|Px - Py\|^2 \leq \inf_{s \in G} \sup_{t \in G} \|\lambda T_{ts}x + (1 - \lambda)Px - Py\|^2.
\]
(3.12)
Now, one can choose an $s_3 \in G$ such that $\|T_{ts_3}x - Px\| \leq M$ for all $t \in G$, where $M = 1 + \inf_{t \in G} \|T_t x - Px\|$. Then, we have
\[
\|\lambda T_{ts_3}x + (1 - \lambda)Px - Py\|^2
\]
\[
= \|\lambda (T_{ts_3}x - Px) + (Px - Py)\|^2
\]
\[
= \lambda^2 \|T_{ts_3}x - Px\|^2 + \|Px - Py\|^2 + 2\lambda (T_{ts_3}x - Px, Px - Py)
\]
\[
\leq M^2\lambda^2 + \|Px - Py\|^2 + 2\lambda (T_{ts_3}x - Px, Px - Py).
\]
(3.13)
It then follows from (3.6) and (3.12) that
\[
2\lambda \sup_{s \in G} \inf_{t \in G} (T_{ts}x - Px, Px - Py)
\]
\[
\geq 2\lambda \sup_{s \in G} \inf_{t \in G} (T_{ts_3}x - Px, Px - Py)
\]
\[
\geq \sup_{s \in G} \inf_{t \in G} \|\lambda T_{ts}x + (1 - \lambda)Px - Py\|^2 - \|Px - Py\|^2 - M^2\lambda^2
\]
\[
= \sup_{s \in G} \inf_{t \in G} \|\lambda T_{ts}x + (1 - \lambda)Px - Py\|^2 - \|Px - Py\|^2 - M^2\lambda^2
\]
\[
\geq \|PT_{s_3}x - Py\|^2 - \|Px - Py\|^2 - M^2\lambda^2
\]
\[
= -M^2\lambda^2.
\]
(3.14)
Hence, we have
\[
\sup_{s \in G} \inf_{t \in G} (T_{ts}x - Px, Px - Py) \geq -\frac{1}{2}M^2\lambda.
\]
(3.15)
Letting $\lambda \to 0$, then we have
\[
\sup_{s \in G} \inf_{t \in G} (T_{ts}x - Px, Px - Py) \geq 0.
\]
(3.16)
Let $\varepsilon > 0$, then there is $s_4 \in G$ such that
\[
\forall t \in G.
\]
For such an $s_4 \in G$, from (3.16), we have
\[
\sup_{s \in G} \inf_{t \in G} \left( T_{ts}T_{s_4}x - PT_{s_4}x, PT_{s_4}x - Py \right) \geq 0
\]
and hence there is $s_5 \in G$ such that
\[
\inf_{t \in G} \left( T_{ts}T_{s_4}x - PT_{s_4}x, PT_{s_4}x - Py \right) > -\varepsilon.
\]
Then, from $PT_{s_4}x = Px$, we have
\[
\inf_{t \in G} \left( T_{ts}T_{s_5}T_{s_4}x - Px, Px - Py \right) > -\varepsilon.
\]
Similarly, from (3.16), we also have
\[
\sup_{s \in G} \inf_{t \in G} \left( T_{ts}T_{s_5}T_{s_4}y - PT_{s_5}T_{s_4}y, PT_{s_5}T_{s_4}y - Px \right) \geq 0,
\]
and there exists $s_6 \in G$ such that
\[
\inf_{t \in G} \left( T_{ts}T_{s_5}T_{s_4}y - PT_{s_5}T_{s_4}y, PT_{s_5}T_{s_4}y - Px \right) \geq -\varepsilon,
\]
that is,
\[
\inf_{t \in G} \left( Py - T_{ts_6}T_{s_5}T_{s_4}y, Px - Py \right) \geq -\varepsilon.
\]
On the other hand, from (3.20)
\[
\inf_{t \in G} \left( T_{ts_6}T_{s_5}T_{s_4}x - Px, Px - Py \right) > -\varepsilon.
\]
Combining (3.23) and (3.24), we have
\[
-2\varepsilon < \left( T_{ts_6}T_{s_5}T_{s_4}x - T_{ts_6}T_{s_5}T_{s_4}y, Px - Py \right) - \| Px - Py \|^2 \\
\leq \| T_{ts_6}T_{s_5}T_{s_4}x - T_{ts_6}T_{s_5}T_{s_4}y \| \cdot \| Px - Py \| - \| Px - Py \|^2 \\
\leq \left( r \left( ts_6T_{s_5}T_{s_4}, x \right) + \| x - y \| \right) \cdot \| Px - Py \| - \| Px - Py \|^2 \\
\leq \left( \varepsilon + \| x - y \| \right) \cdot \| Px - Py \| - \| Px - Py \|^2.
\]
Since $\varepsilon > 0$ is arbitrary, this implies $\| Px - Py \| \leq \| x - y \|$. The proof is completed. 

Using Lemma 2.1, we have the following ergodic retraction theorem for asymptotically nonexpansive type semigroups.

**Theorem 3.2.** Let $C$ be a nonempty subset of a real Hilbert space $H$ and let $\mathcal{S} = \{ T_t : t \in G \}$ be a semitopological semigroup of asymptotically nonexpansive type mappings on $C$ such that $F(\mathcal{S}) \neq \emptyset$. Then the following statements are equivalent:

(a) $\bigcap_{t \in G} \text{conv}(T_{ts}x : t \in G) \cap F(\mathcal{S}) \neq \emptyset$ for each $x \in C$.

(b) There is a unique nonexpansive retraction $P$ of $C$ onto $F(\mathcal{S})$ such that $PT_t = T_tP = P$ for every $t \in G$ and $P \in \text{conv}(T_{ts}x : t \in G)$ for every $x \in C$. 
We denote by $B(G)$ the Banach space of all bounded real-valued functions on $G$ with supremum norm. Let $X$ be a subspace of $B(G)$ containing constants. Then, according to Mizoguchi-Takahashi [10], a real-valued function $\mu$ on $X$ is called a submean on $X$ if the following conditions are satisfied:

1. $\mu(f + g) \leq \mu(f) + \mu(g)$ for every $f, g \in X$;
2. $\mu(\alpha f) = \alpha \mu(f)$ for every $f \in X$ and $\alpha \geq 0$;
3. for $f, g \in X$, $f \leq g$ implies $\mu(f) \leq \mu(g)$;
4. $\mu(c) = c$ for every constant $c$.

The following corollaries are immediately deduced from Theorem 3.2.

**Corollary 3.3** (see [10]). Let $C$ be a closed convex subset of a Hilbert space $H$ and let $X$ be an $r_s$-invariant subspace of $B(G)$ containing constants which has a right invariant submean. Let $\mathcal{S} = \{T_t : t \in G\}$ be a Lipschitzian semigroup on $C$ with $\inf_s \sup_t k_{ts}^2 \leq 1$ and $F(\mathcal{S}) \neq \emptyset$, where $k_t$ is the Lipschitzian constants. If for each $x, y \in C$, the function $f$ on $G$ defined by

$$f(t) = \|T_t x - y\|^2 \quad \forall t \in G \quad (3.26)$$

and the function $g$ on $G$ defined by

$$g(t) = k_t^2 \quad \forall t \in G \quad (3.27)$$

belong to $X$, then the following statements are equivalent:

(a) $\bigcap_{t \in G} \text{conv} \{T_{ts} x : t \in G\} \cap F(\mathcal{S}) \neq \emptyset$ for each $x \in C$.

(b) There is a nonexpansive retraction $P$ of $C$ onto $F(\mathcal{S})$ such that $P T_t = T_t P = P$ for every $t \in G$ and $P x \in \text{conv} \{T_t x : t \in G\}$ for every $x \in C$.

**Corollary 3.4** (see [7]). Let $C$ be a nonempty closed convex subset of a Hilbert space $H$ and let $\mathcal{S} = \{T_t : t \in G\}$ be a continuous representation of a semitopological semigroup as nonexpansive mappings from $C$ into itself. If for each $x \in C$, the set $\bigcap_{t \in G} \text{conv} \{T_{ts} x : t \in G\} \cap F(\mathcal{S}) \neq \emptyset$, then there exists a nonexpansive retraction $P$ of $C$ onto $F(\mathcal{S})$ such that $P T_t = T_t P = P$ for every $t \in G$ and $P x \in \text{conv} \{T_t x : t \in G\}$ for every $x \in C$.

**Remark 3.5.** By Theorem 3.2, many key conditions, in Corollaries 3.3 and 3.4, such as $C$ is convex closed subset and $\mathcal{S}$ is continuous Lipschitzian semigroup, are not necessary.

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References


G. Li: DEPARTMENT OF MATHEMATICS, YANGZHOU UNIVERSITY, YANGZHOU 225002, CHINA

E-mail address: ligang@cims1.yzu.edu.cn

J. K. Kim: DEPARTMENT OF MATHEMATICS, KYUNGNAM UNIVERSITY, MASAN, KYUNGNAM 631-701, KOREA

E-mail address: jongkyuk@kyungnam.ac.kr
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Edson Denis Leonel, Department of Statistics, Applied Mathematics and Computing, Institute of Geosciences and Exact Sciences, State University of São Paulo at Rio Claro, Avenida 24A, 1515 Bela Vista, 13506-700 Rio Claro, SP, Brazil; edleonel@rc.unesp.br

Alexander Loskutov, Physics Faculty, Moscow State University, Vorob’evy Gory, Moscow 119992, Russia; loskutov@chaos.phys.msu.ru

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