

MEASURE OF NONHYPERCONVEXITY AND FIXED-POINT THEOREMS

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The aim of this paper is to work with the measure of nonhyperconvexity in a similar way as J. Cano (1990) does with the index of nonconvexity. We apply this measure to obtain different extensions of the famous Schauder fixed-point theorem in hyperconvex spaces.

1. Introduction

In this paper, we work with the notion of the measure of nonhyperconvexity introduced by Cianciaruso and De Pascale [6] in order to obtain new fixed-point theorems in hyperconvex metric spaces. This class of metric spaces was introduced by Aronszajn and Panitchpakdi [1] in 1956 to study problems on extension of uniformly continuous mappings. Several and interesting properties of these spaces were shown in Aronszajn and Panitchpakdi's original paper, some of these properties turned out to be crucial in the successful searching for fixed-point theorems in hyperconvex metric spaces. More precisely, this research began when Sine [13] and Soardi [14] independently proved in 1979 that bounded hyperconvex metric spaces have the fixed-point property for nonexpansive mappings. Since then, authors like J. B. Baillon, M. A. Khamsi, W. A. Kirk, S. Park, G. Yuan, and many others, including both authors of this paper, have been attracted by this subject and have contributed to a wide development of it (for a recent survey see [11, Chapter 13]).

Hyperconvex metric spaces can be defined as follows: given two metric spaces (Y, d) and (X, ρ) , we say that a mapping $T : Y \rightarrow X$ is *nonexpansive* if $\rho(Tx, Ty) \leq d(x, y)$ for any x and y in Y . The pair (Y, X) is said to have the *extension property for nonexpansive mappings*, if every nonexpansive mapping from an arbitrary subset S of Y into X can be extended as a nonexpansive mapping to the whole Y into X . A metric space X is said to be *hyperconvex* if the pair (Y, X)

enjoys the extension property for nonexpansive mappings for any metric space Y .

Hyperconvex spaces have a good number of properties that is, for instance, any hyperconvex metric space is complete. The simplest example of hyperconvex spaces are the finite-dimensional real spaces endowed with the maximum norm. Other simple examples of hyperconvex metric spaces are provided by closed balls and intersection of closed balls of the above spaces endowed with the induced metric. For a complete treatment of these and other results on hyperconvexity, as well as for a more general study of hyperconvex metric spaces, the reader may consult the recent overview on hyperconvexity and fixed-point theory developed in [11, Chapter 13].

Cano [5] stated a new version of the well-known Schauder fixed-point theorem relaxing the condition on the convexity of the set by using the so-called measure of nonconvexity introduced by Eisenfeld and Lakshmikantham [7] in 1976. The aim of this paper is to work with a similar concept which measures the lack of hyperconvexity of a metric space, in order to sharpen, among others, Schauder and Darbo-Sadovskii's fixed-point theorems in hyperconvex spaces. In Section 2, the authors give an equivalent definition for the measure of nonhyperconvexity, introduced by Cianciaruso and De Pascale in [6], which will turn out to be more convenient for the kind of problems we are concerned with. Section 3 is devoted to obtain new fixed-point theorems under certain hyperconvex hypothesis regarding different compactness conditions on the mapping which are closely related to recent results on hyperconvex spaces.

2. Measure of nonhyperconvexity

Hyperconvex hulls of metric spaces are needed to define the measure of nonhyperconvexity. Isbell introduced the concept of a hyperconvex hull in his very celebrated paper [10] in the following way. Let X be a metric space. The pair (E, e) , where E is a hyperconvex space and e is an isometric embedding of X in E , is called a *hyperconvex hull* of the metric space X if no hyperconvex proper subset of E contains $e(X)$.

In particular, Isbell proved that for any metric space X there exists a *natural hyperconvex hull* which we will denote by $(\varepsilon(X), e)$, and that, although a hyperconvex hull need not be uniquely determined, any two hyperconvex hulls are isometric. This concept has been of great importance in order to obtain topological fixed-point theorems in hyperconvex spaces (see, e.g., [4, 9, 8, 11]). The following definition was given by Cianciaruso and De Pascale in [6].

Definition 2.1. Let X be a metric space and $(\varepsilon(X), e)$ its natural hyperconvex hull. Then, the measure of nonhyperconvexity of X is given by

$$\mu(X) = H(e(X), \varepsilon(X)), \quad (2.1)$$

where $H(\cdot, \cdot)$ stands for the Hausdorff distance between subsets of a metric

space, that is, in this case $H(e(X), \varepsilon(X)) = \sup\{d(y, e(X)) : y \in \varepsilon(X)\}$, where $d(\cdot, \cdot)$ is a distance between a point and a set.

The goal of this section is to attract the attention to the fact that the measure of nonhyperconvexity of a metric space can be defined in a less rigid way. This is provided by the following theorem.

THEOREM 2.2. *Let X be a metric space such that (E, id) , where id stands for the identity map on X , is a hyperconvex hull of X , then*

$$\mu(X) = H(X, E). \quad (2.2)$$

Proof. We only need to prove that $H(X, E) = H(e(X), \varepsilon(X))$, where $(\varepsilon(X), e)$ is the natural hyperconvex hull of X . However, this immediately follows from the fact, proved by Isbell in [10], that there exists an isometry $i : E \rightarrow \varepsilon(X)$ such that $i(X) = e(X)$. \square

Now, if X is a metric space, then there exists a hyperconvex space M such that $X \subseteq M$. It is not hard to observe (see [11, Chapter 13] for details) that there exists such a hyperconvex hull $(h(X), \text{id})$ of X as in the statement of the theorem. From now on, given a metric space X and $A \subseteq X$, $h(A)$ will stand for a hyperconvex hull of A as above. Notice that, in this way we avoid to explicitly deal with the hyperconvex space M . Notice also that, in particular, $\mu(A) = H(A, h(A))$.

The following proposition states a very important property of the measure of nonhyperconvexity. The proof of this fact, as well as further properties of this measure, can be found in [6].

PROPOSITION 2.3. *Let X be a complete metric space. Then $\mu(X) = 0$ if and only if X is hyperconvex.*

3. Fixed-point theorems

In this section, we deal with topological fixed-point theorems (i.e., Schauder and Darbo-Sadovskii type theorems). Our aim is to relax some of the hypothesis given in some hyperconvex versions of these theorems by using the measure of nonhyperconvexity. As a beginning, we introduce some notions.

Definition 3.1. Let X be a metric space. Then a mapping $f : X \rightarrow X$ is said to be μ -contractive if

$$\liminf_{n \rightarrow \infty} \mu(f^n(X)) = 0, \quad (3.1)$$

where f^n stands for the n th iterate of f .

The following definitions have been widely studied, the reader may consult [2] or [11, Chapter 8] for recent treatments on them.

Definition 3.2. Let A be a subset of a metric space X . Then the Kuratowski measure of noncompactness of A , $\alpha(A)$, is defined as

$$\alpha(A) = \inf \left\{ \varepsilon : A \subset \bigcup_{i=1}^{n(\varepsilon)} A_i \text{ with } \text{diam} A_i \leq \varepsilon \right\}. \quad (3.2)$$

Definition 3.3. Let X be a metric space. A mapping $f : X \rightarrow X$ is α -contractive if

$$\inf \{ \alpha(f^n(X)) : n \in \mathbb{N} \} = 0. \quad (3.3)$$

Theorem 3.4 is a version of the Schauder fixed-point theorem under hyperconvex hypothesis.

THEOREM 3.4. *Let C be a nonempty compact metric space and let $f : C \rightarrow C$ be a continuous and μ -contractive mapping. Then f has a fixed-point in C .*

Proof. Consider the set $A = \bigcap_{n=1}^{\infty} f^n(C)$. Obviously, A is a nonempty and compact subset of C such that $f(A) \subset A$. Furthermore, we have $H(A, h(A)) \leq H(A, h(f^n(C)))$ for any $n \in \mathbb{N}$. But $H(A, h(f^n(C))) \leq H(A, f^n(C)) + H(f^n(C), h(f^n(C)))$ for every n , so, passing to a subsequence if necessary and applying the fact that $f^n(C)$ is compact for every n , we deduce that $H(A, h(A)) = 0$. Since A is closed, it is hyperconvex. Hence, f has a fixed-point in A . \square

In connection with this theorem, we also have the following one.

THEOREM 3.5. *Let C be a nonempty compact metric space and let $f : C \rightarrow C$ be a continuous mapping such that $\mu(f^{n_0}(C)) = 0$ for a certain $n_0 \in \mathbb{N}$. Then f has a fixed-point in C .*

Proof. If for some $n_0 \in \mathbb{N}$, $\mu(f^{n_0}(C)) = 0$, then, by **Proposition 2.3**, $f^{n_0}(C)$ is hyperconvex as well as, in this case, compact. Hence, by the Schauder fixed-point in hyperconvex spaces, f has a fixed-point in $f^{n_0}(C)$. \square

The following lemma leads to an extension of **Theorem 3.4**.

LEMMA 3.6. *Let $\{K_i\}_{i=1}^{\infty}$ be a sequence of nonempty bounded and closed subsets of a metric space X such that $K_{i+1} \subset K_i$ for $i \in \mathbb{N}$ and $\liminf_{i \rightarrow \infty} \mu(K_i) = 0$. Then $\bigcap_{i=1}^{\infty} K_i$ is hyperconvex and*

$$\bigcap_{i=1}^{\infty} K_i = \bigcap_{i=1}^{\infty} h(K_i) \neq \emptyset \quad (3.4)$$

for any possible option of the hyperconvex hulls $h(K_i)$.

Proof. In order to prove that $\bigcap_{i=1}^{\infty} K_i$ is nonempty and hyperconvex, it is enough, by Baillon’s intersecting theorem [3], to prove that it coincides with the intersection of a decreasing family of nonempty bounded hyperconvex sets. However, it is not hard to see that the sequence of hyperconvex hulls $(h(K_i))$ may be chosen, so it is also a descending sequence of sets (see [11, Chapter 13] for details). Hence, to complete this part of the proof, it suffices to prove that

$$\bigcap_{i=1}^{\infty} K_i = \bigcap_{i=1}^{\infty} h(K_i), \tag{3.5}$$

where the sequence of hyperconvex hulls has been fixed as above.

The inclusion “ \supseteq ” is trivial. In order to prove the other inclusion, let $\varepsilon > 0$ and $x \in \bigcap_{i=1}^{\infty} h(K_i)$. Then $B(x, \varepsilon) \cap K_i \neq \emptyset$ for every i such that $\mu(K_i) < \varepsilon$, where $B(x, \varepsilon)$ stands for the closed ball of center x and radius ε . Moreover, since $K_{i+1} \subset K_i$, $B(x, \varepsilon)$ contains points of each set K_i . Thus, since K_i is closed, $x \in K_i$ for every $i \in \mathbb{N}$. Hence, $x \in \bigcap_{i=1}^{\infty} K_i$ and the claim is proved.

To finish the proof, it suffices to prove the equality between the intersections for the case when the sequence of hyperconvex hulls is not a decreasing one. But, since $\bigcap_{i=1}^{\infty} K_i \neq \emptyset$, we also have that $\bigcap_{i=1}^{\infty} h(K_i) \neq \emptyset$. The proof follows just as above. □

The following theorem improves [Theorem 3.4](#) as it is not required for the metric space to be compact. Notice also that any mapping, under the hypothesis of [Theorem 3.4](#), must also satisfy the hypothesis of the next theorem.

THEOREM 3.7. *Let A be a nonempty bounded and complete metric space, and let $f : A \rightarrow A$ be a continuous and both α - and μ -contractive. Then f has a fixed-point.*

Proof. Since f is μ -contractive, we may apply [Lemma 3.6](#) to deduce that the set $L = \bigcap_{n=0}^{\infty} \overline{f^n(A)}$ is a nonempty hyperconvex set. Moreover, from the α -contractiveness of f and the very well-known property that the Kuratowski measure of noncompactness is monotone (see [11, Chapter 8] for details), L is compact. Hence, L is an f -invariant compact hyperconvex set and so f has a fixed-point in L . □

In what follows, we state different variants of [Theorem 3.7](#). The next is related to the theory of *limit compact* mappings. These mappings were introduced by Sadovskii in [12], in order to obtain new versions of Schauder’s theorem. The hyperconvex version of those mappings was first studied in [8], see also [11, Chapter 13].

Definition 3.8. Let X be a metric space and $f : X \rightarrow X$ a mapping. The *γ -transfinite iterates* of X through f are defined as

$$f(f^{\gamma-1}(X)) \quad \text{if } \gamma \text{ is an ordinal with antecedent} \tag{3.6}$$

or

$$\bigcap_{\beta < \gamma} f^\beta(X) \quad \text{if } \gamma \text{ is a limit ordinal.} \tag{3.7}$$

Definition 3.9. Let X be a metric space. Then a mapping $f : X \rightarrow X$ is said to be an *extended μ -contractive mapping* if

$$\liminf_{\gamma} \mu(f^\gamma(X)) = 0. \tag{3.8}$$

(We understand here that $\mu(\emptyset) = +\infty$.)

The mapping f is said to be an *extended α -contractive mapping* if

$$\inf \{ \alpha(f^\gamma(X)) : \gamma \text{ is ordinal number with } f^\gamma(X) \neq \emptyset \} = 0. \tag{3.9}$$

Note that the transfinite sequence of the γ -iterates of X through f is eventually constant (either as the empty set or as a nonempty set) from a certain ordinal number on. The following lemma is analogous to [Lemma 3.6](#) in this transfinite setting. We omit the proof since the argument to follow is not very different from that of [Lemma 3.6](#).

LEMMA 3.10. *Let $\{K_\gamma\}$ be a descending transfinite sequence of nonempty bounded closed subsets of a metric space X such that $\liminf_{\gamma} \mu(K_\gamma) = 0$. Then $\bigcap K_\gamma$ is hyperconvex and*

$$\bigcap K_\gamma = \bigcap h(K_\gamma) \neq \emptyset. \tag{3.10}$$

This lemma allows us to give an analogous theorem to [Theorem 3.7](#) for extended contractive mappings.

THEOREM 3.11. *Let A be a nonempty bounded and complete metric space, and let $f : A \rightarrow A$ be a continuous and both extended α -contractive and extended μ -contractive mapping. Then f has a fixed-point.*

Proof. The proof of this theorem follows introducing small changes in that of [Theorem 3.7](#). □

The next theorem takes place in the linear context since it is an extension of those theorems treated by Cano in [\[5\]](#), where the author worked with the concept of measure of nonconvexity. If A is a subset of a linear metric space, then the measure of nonconvexity $\beta(A)$ of A is given by

$$\beta(A) = H(A, \text{conv} A), \tag{3.11}$$

where H is again the Hausdorff distance and $\text{conv} A$ stands for the convex hull of A .

Definition 3.12. Let A be a subset of a linear metric space. Then a mapping $f : A \rightarrow A$ is said to be β -contractive if

$$\liminf_{n \rightarrow \infty} \beta(f^n(A)) = 0. \tag{3.12}$$

THEOREM 3.13. *Let A be a nonempty and closed subset of a Banach space, and let $f : A \rightarrow A$ be a continuous and β -contractive mapping. If there exists a point x_0 in A such that the implication*

$$(V = \text{conv } f(V)) \quad \text{or} \quad (V = f(V) \cup \{x_0\}) \implies V \text{ is relatively compact} \tag{3.13}$$

holds for every subset V of A , then f has a fixed-point.

Proof. Arguing similarly as in [15], we infer that there exists a set $Z \subset A$ such that $f(Z) = Z$. Let $D = \bigcap_{n=0}^{\infty} f^n(A)$. Obviously, $f(D) \subset D$ and $Z \subset D$. By Cano’s lemma from [5], D is convex. Let $R(X) = \text{conv } f(X)$ for $X \subset A$ and let Ω denote the family of all subsets X of A such that $Z \subset X$ and $R(X) \subset X$. Since $Z \subset D$ and $\text{conv } f(D) \subset D$, so Ω is nonempty. Denote by V the intersection of all subsets of the family Ω . As $Z \subset V$, V is nonempty and $Z = f(Z) \subset R(Z) \subset R(V)$. Since $R(V) \subset R(X) \subset X$ for all $X \in \Omega$, $R(V) \subset V$, and therefore $V \in \Omega$. Moreover, $R(R(V)) \subset R(V)$, and therefore $R(V) \in \Omega$. Consequently, $V = R(V)$, that is, $V = \text{conv } f(V)$. In view of (3.13), this implies that \bar{V} is a compact subset of D . Now the Schauder fixed-point theorem implies that f has a fixed-point in \bar{V} . \square

Theorem 3.13 extends [15, Theorem 1]. Note that a mapping f satisfying (3.13) does not have to be α -contractive as the following example shows.

Example 3.14. Let $X = \{x = (x_n) \in l_\infty : 0 \leq x_n \leq 1 \text{ for } n \in \mathbb{N}\}$. Define

$$f(x) = (0, \sqrt{x_1}, \sqrt{x_2}, \dots), \quad x \in X. \tag{3.14}$$

It can be shown that f satisfies (3.13) but still $\alpha(f^n(X)) = 1$ for each $n \in \mathbb{N}$ (see [4, Example 4]).

We finish this paper with the following theorem which is an extension of [4, Theorem 3].

THEOREM 3.15. *Let A be a nonempty complete metric space, and let $f : A \rightarrow A$ be a continuous and μ -contractive mapping. Suppose that there exists a point x_0 in A such that for $V \subset A$, the equality $f(V) \cup \{x_0\} = V$ implies the relative compactness of V , and that only relatively compact sets V can be equal to $h(f(V))$. Then f has a fixed-point.*

Proof. Again let Z be a subset of A such that $f(Z) = Z$ and let $D = \bigcap_{n=0}^{\infty} \overline{f^n(A)}$. Obviously, $f(D) \subset D$ and $Z \subset D$. By Lemma 3.6, it is hyperconvex. Denote by Ω the family of all sets $H \subset A$ such that $Z \subset H$, H is hyperconvex and $f(H) \subset H$. Obviously, the family Ω is nonempty because $D \in \Omega$. Further, we argue similarly as in [4, Theorem 3]. \square

Remark 3.16. Although Cano's paper [5] is the actual motivation for this paper, the definitions of β -contractive and μ -contractive mappings given here do not quite fit that one of β -contractive mapping originally given by Cano. In fact, as Cano defined it, a mapping $f : X \rightarrow X$ is said to be β -contractive if $\inf\{\mu(f^n(X)) : n \in \mathbb{N}\} = 0$. Note that since the measure of nonconvexity, as well as the measure of nonhyperconvexity, is nonmonotone, both definitions of β -contractive mappings are different. Gaps in the proof of Theorem 2 in Cano's paper made us redefine the notion of β -contractive mapping. Notice also that [5, Theorem 2] becomes doubtless true under this new definition.

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