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On the classification of tight contact structures I

Ko Honda

Mathematics Department, University of Georgia
Athens, GA 30602, USA

Email: honda@math.uga.edu

URL: http://www.math.uga.edu/~honda

Abstract

We develop new techniques in the theory of convex surfaces to prove complete classification results for tight contact structures on lens spaces, solid tori, and $T^2 \times I$.

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1 Introduction

It has been known for some time that, in dimension 3, contact structures fall into one of two classes: tight or overtwisted. A contact structure ξ is said to be *overtwisted* if there exists an embedded disk D which is tangent to ξ everywhere along ∂D, and a contact structure is *tight* if it is not overtwisted. This dichotomy was first discovered by Bennequin in his seminal paper [1], and further elucidated by Eliashberg [5]. In [2], Eliashberg classified overtwisted contact structures on closed 3–manifolds, effectively reducing the overtwisted classification to a homotopy classification of 2–plane fields on 3–manifolds. Eliashberg [5] then proceeded to classify tight contact structures on the 3–ball B^3, the 3–sphere S^3, $S^2 \times S^1$, and $\mathbb{R}^3$. In particular, he proved that there exists a unique tight contact structure on B^3, given a fixed boundary characteristic foliation — this theorem of Eliashberg comprises the foundational building block in the study of tight contact structures on 3–manifolds. Subsequent results on the classification of tight contact structures were: a complete classification on the 3–torus by Kanda [19] and Giroux (obtained independently), a complete classification on some lens spaces by Etnyre [6], and some partial results on solid tori $S^1 \times D^2$ by Makar–Limanov [22] and circle bundles over Riemann surfaces by Giroux. One remarkable discovery by Makar–Limanov [22] was that there exist tight contact structures which become overtwisted when pulled back to the universal cover $\widetilde{M}$ via the covering map $\pi: \widetilde{M} \to M$. This prompts us to define a *universally tight* contact structure to be one which remains tight when pulled back to $\widetilde{M}$ via π. We call a tight contact structure ξ *virtually overtwisted* if ξ becomes overtwisted when pulled back to a *finite* cover. It is not known whether every tight contact structure is either universally tight or virtually overtwisted, although this dichotomy holds when $\pi_1(M)$ is residually finite.

The goal of this paper is to give a complete classification of tight contact structures on lens spaces, as well as a complete classification of tight contact structures on solid tori $S^1 \times D^2$ and toric annuli $T^2 \times I$ with convex boundary. This completes the classification of tight contact structures on lens spaces, initiated by Etnyre in [6], as well as the classification of tight contact structures on solid tori (at least for convex boundary), initiated by Makar–Limanov [22]. We will also determine precisely which tight contact structures are universally tight and which are virtually overtwisted — all the manifolds we consider this paper will have residually finite $\pi_1(M)$, hence tight contact structures on these manifolds will either be universally tight or virtually overtwisted. Our method is a systematic application of the methods developed by Kanda [19], which in turn use Giroux's theory of convex surfaces [12]. In essence, we use Kanda's methods and apply them in Etnyre's setting: we decompose the 3–manifold M in a series of steps, along *closed convex surfaces* or *convex surfaces with Legendrian boundary*. The difference between Etnyre's approach and ours is that we require that the cutting surfaces have boundary consisting of *Legendrian curves*, whereas Etnyre used cutting surfaces which had *transverse curves* on the boundary. The Legendrian curve approach appears to be more efficient and yields fewer possible configurations than the transverse curve approach, although the author is not quite sure why this is the case.

The classification theorems will reveal a closer connection between contact structures and 3–dimensional topology than was previously expected. In particular, the geometry of $\pi_0(\mathrm{Diff}^+(T^2)) = SL(2;\mathbb{Z})$ (including the standard Farey tessellation) plays a significant role for the 3–manifolds studied in this paper — lens spaces have Heegaard decompositions into solid tori, and the toric annulus contains incompressible T^2. Unlike foliation theory (which is related to contact topology by the work of Eliashberg and Thurston [9]), contact topology has a built-in 'handedness', and we will see that the contact topology is determined in large part by *positive Dehn twists* in $\pi_0(\mathrm{Diff}^+(T^2)) = SL(2;\mathbb{Z})$. We believe the results in this paper represent a tiny fraction of a large and emerging theory of contact structures applied to three–manifold topology. The techniques developed in this paper are applied to other classes of 3–manifolds (circle bundles which fiber over closed oriented surfaces and torus bundles over S^1) in the sequel [17], and in [8] J. Etnyre and the author prove the non-existence of positive tight contact structures on the Poincare homology sphere for one of its orientations, thereby producing the first example of a closed 3–manifold which does not carry a tight contact structure.

Note E Giroux has independently obtained similar classification results. His approach and ours are surprisingly dissimilar, and the interested reader will certainly increase his understanding by reading his account [13] as well.

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2 Statements of results

In this paper all the 3–manifolds M are oriented and compact, and all the contact structures ξ are positive, ie, given by a global 1–form α with $\alpha \wedge d\alpha > 0$, and oriented. We will simply write 'contact structure', when we mean 'positive, oriented contact structure'.

2.1 Lens spaces

Consider the lens space $L(p,q)$, where $p > q > 0$ and $(p,q) = 1$. Assume $-\frac{p}{q}$ has the continued fraction expansion

$$-\frac{p}{q} = r_0 - \cfrac{1}{r_1 - \cfrac{1}{r_2 - \cdots \frac{1}{r_k}}},$$

with all $r_i < -1$. Then we have the following classification theorem for tight contact structures on lens spaces $L(p,q)$.

Theorem 2.1 *There exist exactly $|(r_0+1)(r_1+1)\cdots(r_k+1)|$ tight contact structures on the lens space $L(p,q)$ up to isotopy, where $r_0,\ldots,r_k$ are the coefficients of the continued fraction expansion of $-\frac{p}{q}$. Moreover, all the tight contact structures on $L(p,q)$ can be obtained from Legendrian surgery on links in S^3, and are therefore holomorphically fillable.*

Legendrian surgery is a contact surgery technique due to Eliashberg [3]. It produces contact structures which are holomorphically fillable, and are therefore tight, by a result of Eliashberg and Gromov [4, 15].

2.2 The thickened torus $T^2 \times I$

When we study contact structures on manifolds with boundary, we need to impose a boundary condition — a natural condition would be to ask that the boundary be *convex*. A closed, oriented, embedded surface Σ in a contact manifold (M,ξ) is said to be *convex* if there is a vector field v transverse to Σ whose flow preserves ξ. A generic surface Σ inside a contact 3–manifold is convex [12], so demanding that the boundary be convex presents no loss of generality.

A convex surface $\Sigma \subset (M,\xi)$ has a naturally associated family of disjoint embedded curves Γ_Σ, well-defined up to isotopy and called the *dividing curves* (for more details see Section 3.1.3). The dividing curves Γ_Σ separate the surface Σ into two subsurfaces R_+ and R_-. If ξ is tight and $\Sigma \neq S^2$, then the dividing curves Γ_Σ are homotopically essential, in the sense that none of them bounds an embedded disk in Σ. In particular, if Σ is a torus, Γ_Σ will consist of an even number of parallel essential curves.

Consider a tight contact structure ξ on $T^2 \times I = T^2 \times [0,1]$ with convex boundary. Fix an oriented identification between the torus T^2 and $\mathbb{R}^2/\mathbb{Z}^2$.

Given a convex torus T in $T^2 \times I$, its set of dividing curves is, up to isotopy, determined by the following data: (1) the *number* $\#\Gamma_T$ of these dividing curves and (2) their *slope* $s(T)$, defined by the property that each curve is isotopic to a linear curve of slope $s(T)$ in $T \simeq \mathbb{R}^2/\mathbb{Z}^2$.

2.2.1 Twisting

In order to state the classification theorem for $T^2 \times I$ it is necessary to define the notions of *twisting in the I–direction*, *minimal twisting in the I–direction*, and *nonrotativity in the I–direction*.

Given a slope s of a line in $\mathbb{R}^2$ (or $\mathbb{R}^2/\mathbb{Z}^2$), associate to it its standard angle $\bar\alpha(s) \in \mathbb{RP}^1 = \mathbb{R}/\pi\mathbb{Z}$. For $\bar\alpha_1$, $\bar\alpha_2 \in \mathbb{RP}^1$, let $[\bar\alpha_1, \bar\alpha_2]$ be the image of the interval $[\alpha_1, \alpha_2] \subset \mathbb{R}$, where $\alpha_i \in \mathbb{R}$ are representatives of $\bar\alpha_i$ and $\alpha_1 \leq \alpha_2 < \alpha_1 + \pi$. A slope s is said to be *between* s_1 and s_0 if $\bar\alpha(s) \in [\bar\alpha(s_1), \bar\alpha(s_0)]$.

Consider a tight contact structure ξ on $T^2 \times I$ with convex boundary and boundary slopes $s_i = s(T_i)$, $i = 0, 1$, where $T_i = T^2 \times \{i\}$. We say ξ is *minimally twisting* (in the I–direction) if every convex torus parallel to the boundary has slope s between s_1 and s_0. In particular, ξ is *nonrotative* (in the I–direction) if $s_1 = s_0$ and ξ is minimally twisting. Define the *I–twisting* of a tight ξ to be $\beta_I = \alpha(s_0) - \alpha(s_1) = \sum_{k=1}^{l} (\alpha(s_{\frac{k-1}{l}}) - \alpha(s_{\frac{k}{l}}))$, where (i) $s_{\frac{k}{l}} = s(T_{\frac{k}{l}})$, $k = 0, \ldots, l$, (ii) $T_0 = T^2 \times \{0\}$, $T_1 = T^2 \times \{1\}$, and $T_{\frac{k}{l}}$, $k = 1, \ldots, l-1$ are mutually disjoint convex tori parallel to the boundary, arranged in order from closest to T_0 to farthest from T_0, (iii) ξ is minimally twisting between $T_{\frac{k-1}{l}}$ and $T_{\frac{k}{l}}$, and (iv) $\alpha(s_{\frac{k}{l}}) \leq \alpha(s_{\frac{k-1}{l}}) < \alpha(s_{\frac{k}{l}}) + \pi$.

The following will be shown in Proposition 5.5:

(1) The I–twisting of ξ is well-defined, finite, and independent of the choices of l and the $T_{\frac{k}{l}}$.

(2) The I–twisting of ξ is always non-negative.

Notice that the I–twisting β_I is dependent on the particular identification $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. We therefore introduce $\pi_I(\xi) = \lfloor \frac{\beta_I}{\pi} \rfloor$, which is independent of the identification. Here $\lfloor\ \rfloor$ is the greatest integer function. Also, $\pi_I = 0$ is equivalent to minimal twisting.

2.2.2 Statement of theorem

After normalizing via $\pi_0(\mathrm{Diff}^+(T^2)) = SL(2, \mathbb{Z})$, we may assume that T_1 has dividing curves with slope $-\frac{p}{q}$, where $p \geq q > 0$, $(p, q) = 1$, and T_0 has slope

-1. Denote $T_a = T^2 \times \{a\}$. For this boundary data, we have the following:

Theorem 2.2 *Consider $T^2 \times I$ with convex boundary, and assume, after normalizing via $SL(2;\mathbb{Z})$, that Γ_{T_1} has slope $-\frac{p}{q}$, and Γ_{T_0} has slope -1. Assume we fix a characteristic foliation on T_0 and T_1 with these dividing curves. Then, up to an isotopy which fixes the boundary, we have the following classification:*

(1) *Assume either* (a) $-\frac{p}{q} < -1$ *or* (b) $-\frac{p}{q} = -1$ *and* $\beta_I > 0$. *Then there exists a unique factorization $T^2 \times I = (T^2 \times [0,\frac{1}{3}]) \cup (T^2 \times [\frac{1}{3},\frac{2}{3}]) \cup (T^2 \times [\frac{2}{3},1])$, where (i) $T_{\frac{i}{3}}$, $i = 0,1,2,3$, are convex, (2) $(T^2 \times [0,\frac{1}{3}])$ and $(T^2 \times [\frac{2}{3},1])$ are nonrotative, (3) $\#\Gamma_{T_{\frac{1}{3}}} = \#\Gamma_{T_{\frac{2}{3}}} = 2$, and (4) $T_{\frac{1}{3}}$ and $T_{\frac{2}{3}}$ have fixed characteristic foliations which are adapted to $\Gamma_{T_{\frac{1}{3}}}$ and $\Gamma_{T_{\frac{2}{3}}}$.*

(2) *Assume $-\frac{p}{q} < -1$ and $\#\Gamma_{T_0} = \#\Gamma_{T_1} = 2$.*

   (a) *There exist exactly $|(r_0+1)(r_1+1)\cdots(r_{k-1}+1)(r_k)|$ tight contact structures with $\beta_I = 0$. Here, $r_0,\dots,r_k$ are the coefficients of the continued fraction expansion of $-\frac{p}{q}$, and $-\frac{p}{q} < -1$.*

   (b) *There exist exactly 2 tight contact structures with $\beta_I = n$, for each $n \in \mathbb{Z}^+$.*

(3) *Assume $-\frac{p}{q} = -1$ and $\#\Gamma_{T_0} = \#\Gamma_{T_1} = 2$. Then there exist exactly 2 tight contact structures with $\beta_I = n$, for each $n \in \mathbb{Z}^+$.*

(4) *Assume $-\frac{p}{q} = -1$ and $\#\Gamma_{T_0} = 2n_0$, $\#\Gamma_{T_1} = 2n_1$. Then the nonrotative tight contact structures are in 1–1 correspondence with G, the set of all possible (isotopy classes of) configurations of arcs on an annulus $A = S^1 \times I$ with markings $\delta_i \subset S^1 \times \{i\}$, $i = 0,1$, which satisfy the following:*

   (a) *$|\delta_i| = 2n_i$, $i = 0,1$, where $|\ |$ denotes cardinality.*

   (b) *Every point of $\delta_0 \cup \delta_1$ is precisely one endpoint of one arc.*

   (c) *There exist at least two arcs which begin on δ_0 and end on δ_1.*

   (d) *There are no closed curves.*

2.3 Solid tori

Finally, we have the analogous theorem for solid tori. Fix an oriented identification of $T^2 = \partial(S^1 \times D^2)$ with $\mathbb{R}^2/\mathbb{Z}^2$, where $(1,0)^T$ corresponds to the meridian of the solid torus, and $(0,1)^T$ corresponds the longitudinal direction determined by a chosen framing. We consider tight contact structures ξ on $S^1 \times D^2$ with convex boundary T^2. Let the *slope* $s(T^2)$ of T^2 be the slope under the identification $T^2 \simeq \mathbb{R}^2/\mathbb{Z}^2$.

Theorem 2.3 *Consider the tight contact structures on $S^1 \times D^2$ with convex boundary T^2, for which $\#\Gamma_{T^2} = 2$ and $s(T^2) = -\frac{p}{q}$, $p \geq q > 0$, $(p,q) = 1$. Fix a characteristic foliation $\mathcal{F}$ which is adapted to Γ_{T^2}. There exist exactly $|(r_0+1)(r_1+1)\cdots(r_{k-1}+1)(r_k)|$ tight contact structures on $S^1 \times D^2$ with this boundary condition, up to isotopy fixing T^2. Here, $r_0, \ldots, r_k$ are the coefficients of the continued fraction expansion of $-\frac{p}{q}$.*

In other words, the number of tight contact structures for the solid torus with (a fixed) convex boundary with $\#\Gamma_{T^2} = 2$ and $s(T^2) = -\frac{p}{q}$ is the same as the number of tight contact structures on $T^2 \times I$ with (fixed) convex boundary, $\#\Gamma_{T_i} = 2$, $i = 0, 1$, slopes $s(T_1) = -\frac{p}{q}$ and $s(T_0) = -1$, and minimal twisting.

Via a multiplication by $\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \in SL(2,\mathbb{Z})$, $m \in \mathbb{Z}$, which is equivalent to a Dehn twist which induces a change of framing, all the boundaries of $S^1 \times D^2$ can be put in the form described in the theorem above. In addition, the choice of slope $-\frac{p}{q}$ with $p \geq q > 0$ is unique.

2.4 Strategy of proof

First consider $T^2 \times I$. We fix a boundary condition by prescribing dividing sets $\Gamma_i = \Gamma_{T_i}$, $i = 0, 1$. Also fix a boundary characteristic foliation which is compatible with Γ_i. Giroux's Flexibility Theorem, described in Section 3.1, roughly states that it is the *isotopy type* of the dividing set Γ which dictates the geometry of ξ, not the precise characteristic foliation which is compatible with Γ. This allows us to reduce the classification to one particular characteristic foliation compatible with Γ_i, and we choose a (rather non-generic) realization of a convex surface — one that is in *standard form* (see Section 3.2.1).

In Section 3.4 we introduce the notion of a *bypass*, which is the crucial new ingredient which allows us to successively peel off 'thin' $T^2 \times I$ layers which we call *basic slices*. We eventually obtain a factorization of a $(T^2 \times I, \xi)$ into basic $T^2 \times I$ slices, if ξ is tight and minimally twisting. This decomposition gives a possible upper bound for the number of tight contact structures on $T^2 \times I$ with given boundary conditions. These candidate tight contact structures are easily distinguished by the relative Euler class. We then successively embed $T^2 \times I \subset S^1 \times D^2 \subset L(p,q)$, and find that the upper bound is exact, since all of the candidate tight contact structures can be realized by Legendrian surgery. The remaining cases of Theorem 2.2 when the I-twisting is not minimal and when $\#\Gamma_i > 2$ are treated in Section 5.

3 Preliminaries

3.1 Convexity

In this section only (M, ξ) is a compact, oriented 3-manifold with a contact structure, tight or overtwisted.

An oriented properly embedded surface Σ in (M, ξ) is called *convex* if there is a vector field v transverse to Σ whose flow preserves ξ. This *contact vector field* v allow us to find an I-invariant neighborhood $\Sigma \times I \subset M$ of Σ, where $\Sigma = \Sigma \times \{0\}$. In most applications, our convex surface Σ will either be closed or compact with *Legendrian boundary*. The theory of closed convex surfaces appears in detail in Giroux's paper [12]. However, the same results for the Legendrian boundary case have not appeared in the literature, and we will rederive Giroux's results in this case.

3.1.1 Twisting number of a Legendrian curve

A curve γ which is everywhere tangent to ξ is called *Legendrian*. We define the *twisting number* $t(\gamma, Fr)$ of a closed Legendrian curve γ with respect to a given framing Fr to be the number of counterclockwise (right) 2π twists of ξ along γ, relative to Fr. In particular, if γ is a connected component of the boundary of a compact surface Σ, $T\Sigma$ gives a natural framing Fr_Σ, and if Σ is a Seifert surface of γ, then $t(\gamma, Fr_\Sigma)$ is the Thurston-Bennequin invariant $tb(\gamma)$. We will often suppress Fr when the framing is understood. Notice that it is easy to decrease $t(\gamma, Fr)$ by locally adding zigzags in a front projection, but not always possible to increase $t(\gamma, Fr)$.

3.1.2 Perturbation into a convex surface with Legendrian boundary

Giroux [12] proved that a closed oriented embedded surface Σ can be deformed by a C^∞-small isotopy so that the resulting embedded surface is convex. We will prove the following proposition:

Proposition 3.1 *Let $\Sigma \subset M$ be a compact, oriented, properly embedded surface with Legendrian boundary, and assume $t(\gamma, Fr_\Sigma) \leq 0$ for all components γ of $\partial\Sigma$. There exists a C^0-small perturbation near the boundary (fixing $\partial\Sigma$) which puts an annular neighborhood A of $\partial\Sigma$ into a standard form, and a* *subsequent C^1{small perturbation of the perturbed surface (xing the annular neighborhood of $@$), which makes convex. Moreover, if v is a contact vector eld de ned on a neighborhood of A and transverse to A , then v can be extended to a contact vector eld transverse to all of .*

Proof Assume that $t(\gamma; Fr\ ) < 0$, for all boundary components γ. After a C^0{small perturbation near the boundary (xing the boundary), we may assume that γ has a *standard annular collar* A. Here $A = S^1 \quad [0;1] = (\mathbb{R}=\mathbb{Z}) \quad [0;1]$ with coordinates $(x;y)$ and $\gamma = S^1 \quad f0g$. Its neighborhood $A \quad [-1;1]$ has coordinates $(x;y;t)$, and the contact 1{form on $A \quad [-1;1]$ is $= \sin(2 \ nx)dy + \cos(2 \ nx)dt$. The Legendrian curves $S^1 \quad f\mathrm{pt}g \quad A$ are called the *Legendrian rulings* and and $f\frac{k}{2n}g \quad [0;1] \quad A$, $k = 1;2; \quad ;2n$ are called the *Legendrian divides*.

Once we have standard annular neighborhoods of $@$, we use the following perturbation lemma, due to Fraser [10] | refer to Figure 1 for an illustration of *half-elliptic* and *half-hyperbolic* singular points.

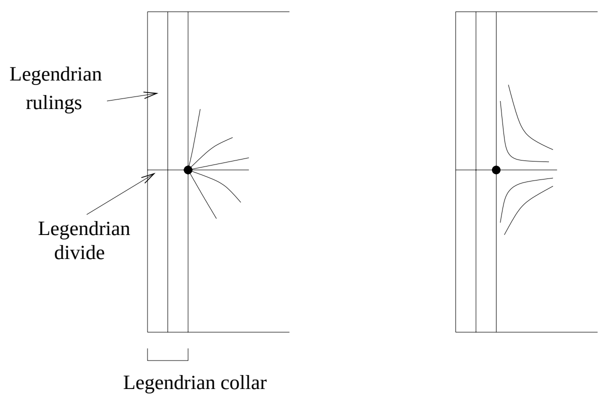


Figure 1: Half-elliptic point and half-hyperbolic point

Lemma 3.2 *It is possible to perturb , while xing the Legendrian collar, to make any tangency $(\frac{k}{2n};1)\ 2\ A = (\mathbb{R}=\mathbb{Z}) \quad [0;1]$ half-elliptic and any tangency half-hyperbolic.*

Proof It su ces, by a Darboux-type argument, to extend the contact structure on $S^1 \quad [0;1] \quad [-1;1]$ above to $S^1 \quad [0;2] \quad [-1;1]$, such that the characteristic foliation on $S^1 \quad [0;2] \quad f0g$ has a half-elliptic or a half-hyperbolic singularity. It therefore also su ces to treat the neighborhood of a Legendrian divide. Without loss of generality, let the Legendrian divide be $f0g \quad [0;1] \quad f0g$

$[-\varepsilon,\varepsilon]\times[0,1]-\{0\}$, with contact 1-form $\alpha' = dt + xdy$. Now extend to $\alpha' = dt - f(y)dx + xdy$ for a half-elliptic singularity and $\alpha' = dt + f(y)dx + xdy$ for a half-hyperbolic singularity, on $[-\varepsilon,\varepsilon]\times[0,2]-\{0\}$, where $f(y)=0$ on $[0,1]$ and $\frac{df}{dy}>0$ on $[1,2]$. □

Note M Fraser [10] obtained normal forms near the boundary, for Σ with Legendrian boundary, even when $t(\gamma)>0$ for some boundary component γ. In this case, Lemma 3.2 is no longer applicable. Instead, all the singularities must be half-hyperbolic, after appropriate cancellations. If $t(\gamma)>0$, Σ cannot be made convex.

When $t(\gamma)=0$, then perturb Σ, fixing γ, so that the contact structure is given by $\alpha = dt - ydx$ on $A\times[-1,1]$, where A is as before.

If Σ is compact with Legendrian boundary, and all the boundary components have $t\leq 0$, we use Lemma 3.2 if $t<0$, to make all the boundary tangencies of Σ half-elliptic (if $t=0$ use the paragraph above), and perturb to obtain Σ with characteristic foliation F which is Morse–Smale on the interior. This means that we have isolated singularities (which are 'hyperbolic', in the dynamical systems sense, not to be confused with elliptic vs. hyperbolic singular points, which will be written without quotes), no saddle–saddle connections, and all the sources or sinks are elliptic singularities or closed orbits which are Morse–Smale in the usual sense. This guarantees the convexity of Σ. The actual construction of the transverse contact vector field follows from Giroux's argument in [12] (Proposition II.2.6), where it is shown that Σ is convex if Σ is closed and the characteristic foliation is Morse–Smale.

The goal is to find some I–invariant contact structure ξ' (given by a 1–form α') which induces this characteristic foliation F on Σ. Orient the characteristic foliation so that the positive elliptic points are the sources and the negative elliptic points are the sinks. This will naturally identify which closed orbits are positive (sources) and which closed orbits are negative (sinks). Let X be a vector field which directs F and is nonzero away from the singularities of F. Consider the neighborhood $N(\Sigma)=\Sigma\times I$, where I has coordinate t. The 'hyperbolicity' of the singularities implies that if α is given by $\alpha = dt+\beta$ (here β has no dt–terms, but may be t–dependent), then $d\beta$ is nonzero near the singularity on $\Sigma\times\{0\}$. (This means X has positive *divergence* near the singularities.) Now let $U\subset\Sigma$ be the union of small neighborhoods of the half-elliptic or half-hyperbolic singularities, elliptic and hyperbolic singularities, the closed orbits, and neighborhoods of connecting orbits which connect between singularities of the same sign. Without loss of generality, restrict attention to

U_+, the components of U with positive singularities. Let β' be a 1–form on Σ given by $\beta' = i_X\omega$, where ω is an area form on Σ. The positive divergence ensures that $d\beta'$ is positive near the singular points. In a neighborhood $B = S^1 \times [-1,1]$ of a positive closed orbit $S^1 \times \{0\}$, with coordinates (x,y), let $X = \frac{\partial}{\partial x} + \lambda(x,y)\frac{\partial}{\partial y}$, and $\omega = dxdy$. Then $\beta' = i_X\omega$ satisfies $d\beta' > 0$ on B, since the Morse–Smale condition implies $\frac{\partial\lambda}{\partial y} > 0$. (However, away from the singularities and closed orbits, we do not know whether $d\beta'$ is positive.) We now take a positive function f for which f grows rapidly along X, ie, $df(X) >> 0$, and form $\beta'' = f\beta'$. Since $d\beta'' = df \wedge \beta' + f d\beta'$, we obtain $d\beta'' > 0$. Now let $\alpha' = dt + \beta''$.

Now, $\Sigma \setminus U$ consists of annuli $A' = (\mathbb{R}/\mathbb{Z}) \times I$, with coordinates (x,y) and $\mathcal{F}|_{A'}$ given by $x =$ const., and $A'' = I \times I$, with coordinates (x,y) and $\mathcal{F}|_{A''}$ also given by $x =$ const. Consider A'. The I–invariant contact structure ξ' is defined along $(\mathbb{R}/\mathbb{Z}) \times \{0\}$ by $f(x,y)dt - dx$ for some positive function $f(x,y)$ satisfying $\frac{\partial f}{\partial y} < 0$, and is defined along $(\mathbb{R}/\mathbb{Z}) \times \{1\}$ by $f(x,y)dt - dx$ for some negative function $f(y)$ satisfying $\frac{\partial f}{\partial y} < 0$. We simply interpolate f between $y = 0$ and $y = 1$, while keeping $\frac{\partial f}{\partial y} < 0$. A'' is similar, but we need to remember that f is already specified along $\{0,1\} \times I$.

We have therefore constructed an I–invariant contact structure ξ' such that $\xi'|_\Sigma = \mathcal{F}$ and $\xi = \xi'$ on a neighborhood of A. The proof of the proposition is complete once we have the following lemma.

Lemma 3.3 *Let Σ be closed or with collared Legendrian boundary. If ξ and ξ' are contact structures defined on a neighborhood of Σ, inducing the same characteristic foliation $\mathcal{F}$, then there exists a 1–parameter family of diffeomorphisms ϕ_s, $s \in [0,1]$, where $\phi_0 = id$, $\phi_1(\xi') = \xi$, and ϕ_s preserve $\mathcal{F}$. Moreover, if ξ and ξ' agree on the collared Legendrian boundary A, then ϕ_s can be made to have support away from A.*

The proof of this lemma uses Moser's method, and is proven exactly as in Proposition 1.2 of [12]. □

3.1.3 Dividing curves

A convex surface Σ which is closed or compact with Legendrian boundary has a *dividing set* Γ_Σ. We define a *dividing set* Γ_Σ for v to be the set of points x where $v(x) \in \xi(x)$. We will write Γ if there is no ambiguity of Σ. Γ is a union of smooth curves and arcs which are transverse to the *characteristic foliation*

$\xi|_\Sigma$. If Σ is closed, there will only be closed curves $\gamma \subset \Gamma$; if Σ has Legendrian boundary, $\gamma \subset \Sigma$ may be an arc with endpoints on the boundary. The isotopy type of Γ is independent of the choice of v — hence we will slightly abuse notation and call Γ *the dividing set* of Σ. Denote the number of connected components of Γ_Σ by $\#\Gamma_\Sigma$. $\Sigma \setminus \Gamma_\Sigma = R_+ - R_-$, where R_+ is the subsurface where the orientations of v (coming from the normal orientation of ξ) and the normal orientation of Σ coincide, and R_- is the subsurface where they are opposite.

3.1.4 Giroux's Flexibility Theorem

The following informal principle highlights the importance of the dividing set:

Key Principle It is the dividing set Γ_Σ (*not the exact characteristic foliation*) which encodes the essential contact topology information in a neighborhood of Σ.

To make this idea more precise, we will now present Giroux's Flexibility Theorem. If $\mathcal{F}$ is a singular foliation on Σ, then a disjoint union of properly embedded curves Γ is said to *divide* $\mathcal{F}$ if there exists some I–invariant contact structure ξ on $\Sigma \times I$ such that $\mathcal{F} = \xi|_{\Sigma \times \{0\}}$ and Γ is the dividing set for $\Sigma \times \{0\}$.

Theorem 3.4 (Giroux [12]) *Let Σ be a closed convex surface or a compact convex surface with Legendrian boundary, with characteristic foliation $\xi|_\Sigma$, contact vector field v, and dividing set Γ. If $\mathcal{F}$ is another singular foliation on Σ divided by Γ, then there is an isotopy ϕ_s, $s \in [0,1]$, of Σ such that $\phi_0(\Sigma) = \Sigma$, $\xi|_{\phi_1(\Sigma)} = \mathcal{F}$, the isotopy is fixed on Γ, and $\phi_s(\Sigma)$ is transverse to v for all s.*

An isotopy ϕ_s, $s \in [0,1]$, for which $\phi_s(\Sigma) \pitchfork v$ for all s is called *admissible*.

Proof Consider two I–invariant contact structures ξ_0 and ξ_1 on $\Sigma \times I$ which induce the same dividing set Γ on Σ. We may assume that $\xi_0 = \xi_1$ on $(N(\Gamma) \cup N(\partial\Sigma)) \times I$. Here $N(\Gamma)$ and $N(\partial\Sigma)$ are neighborhoods of Γ and $\partial\Sigma$ in Σ. Consider $\Sigma_0 \times I$, where Σ_0 is a connected component of $\Sigma \setminus N(\Gamma)$. Here ξ_s, $s = 0, 1$, will be given by $\alpha_s = dt + \beta_s$, $s = 0, 1$, where t is the variable in the I–direction, β_s is a 1–form on Σ_0 which is independent of t, and $d\beta_s > 0$. We interpolate ξ_0 and ξ_1 through $\alpha_s = (1 - s)\alpha_0 + s\alpha_1$, $s \in [0,1]$. Then

$\alpha_s = dt + \beta_s$, $s \in [0,1]$ are all contact and I-invariant. Also note that α_s is independent of s on $N(\partial\Sigma_0) \times I$. We use a Moser-type argument to obtain a 1-parameter family $\{\phi_s\}$ of diffeomorphisms satisfying

$$\phi_s^*(\alpha_s) = f_s \alpha_0, \tag{1}$$

where f_s is some function. Differentiating this equation, we obtain:

$$\phi_s^*\left(L_{X_s}\alpha_s + \frac{d\alpha_s}{ds}\right) = \frac{df_s}{ds}\alpha_0, \tag{2}$$

where X_s is the s-dependent vector field $\frac{d\phi_s}{ds}$, and L is the Lie derivative. Substituting Equation 1 into Equation 2 and removing ϕ_s^*, we obtain

$$L_{X_s}\alpha_s = -\frac{d\alpha_s}{ds} + g_s\alpha_s, \tag{3}$$

where g_s is some function. We may set $g_s = 0$, and solve the pair:

$$i_{X_s}(d\alpha_s) = -\frac{d\beta_s}{ds}, \tag{4}$$

$$i_{X_s}(dt + \beta_s) = 0. \tag{5}$$

It is important to note that, since β_s is constant along $N(\partial\Sigma_0) \cup N(\Gamma)$, $X_s = 0$ and ϕ_s leaves $(N(\partial\Sigma_0) \cup N(\Gamma)) \times I$ fixed. By construction, $\phi_s(\Sigma \times \{0\})$ is transverse to v. □

3.2 Convex surfaces in tight contact manifolds

From now on let (M, ξ) be a compact, oriented 3-manifold with a tight contact structure ξ. The following is Giroux's criterion for determining which convex surfaces have neighborhoods which are tight:

Theorem 3.5 (Giroux's criterion) *If $\Sigma \neq S^2$ is a convex surface (closed or compact with Legendrian boundary) in a contact manifold (M, ξ), then Σ has a tight neighborhood if and only if Γ_Σ has no homotopically trivial curves. If $\Sigma = S^2$, Σ has a tight neighborhood if and only if $\#\Gamma_\Sigma = 1$.*

We will prove the easy half of the theorem in Section 3.3.1.

Examples The following are some examples of convex surfaces that can exist inside tight contact manifolds.

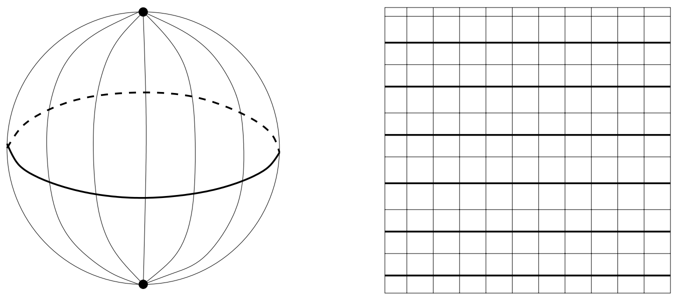


Figure 2: Dividing curves for S^2 and T^2

(1) $\Sigma = S^2$. Since $\#\Gamma_\Sigma = 1$, there is only one possibility. See Figure 2. Note that any time there is more than one dividing curve the contact structure is overtwisted. In Figure 2, the thicker lines are the dividing curves and the thin lines represent the characteristic foliation.

(2) $\Sigma = T^2$. Since there cannot be any homotopically trivial curves, Γ_Σ consists of an even number $2n > 0$ of parallel homotopically essential curves. Depending on the identification with $\mathbb{R}^2/\mathbb{Z}^2$ the dividing curves may look like as in Figure 2. Note that in our planar representation of T^2 the sides are identified and the top and bottom are identified.

3.2.1 Convex tori in standard form

One of the main ingredients in our study is the convex torus $\Sigma \subset M$ in *standard form*. Assume Σ is a convex torus in a tight contact manifold M. Then, after some identification of Σ with $\mathbb{R}^2/\mathbb{Z}^2$, we may assume Γ_Σ consists of $2n$ parallel homotopically essential curves of slope 0. The *torus division number* is given by $n = \frac{1}{2}(\#\Gamma_\Sigma)$. Using Giroux's Flexibility Theorem, we can deform Σ inside a neighborhood of $\Sigma \subset M$ into a torus which we still call Σ and has the same dividing set as the old Σ. The characteristic foliation on this new $\Sigma = \mathbb{R}^2/\mathbb{Z}^2$ with coordinates (x, y) is given by $y = rx + b$, where $r \neq 0$ is fixed, and b varies in a family, with tangencies $y = \frac{k}{2n}$, $k = 1, \ldots, 2n$. ($r = \infty$ will also be allowed, in which case we have the family $x = b$.) We say such a Σ is a *convex torus in standard form* (or simply *in standard form*). The horizontal Legendrian curves $y = \frac{k}{2n}$ are isolated and rather inflexible from the point of view of Σ (as well as nearby convex tori), and will be called *Legendrian divides*. The Legendrian curves that are in a family are much more flexible, and will be called *Legendrian rulings*. In particular, a consequence of Giroux's Flexibility Theorem is the following:

Corollary 3.6 (Flexibility of Legendrian rulings) *Let (T, ξ) be a torus in the above form, with coordinates $(x, y) \in \mathbb{R}^2/\mathbb{Z}^2$, Legendrian rulings $y = rx + b$ (or $x = b$), and Legendrian divides $y = \frac{k}{2n}$. Then, via a C^0–small perturbation near the Legendrian divides, we can modify the slopes of the rulings from $r \neq 0$ to any other number $r' \neq 0$ ($r = \infty$ included).*

We will also say that a convex annulus $A = S^1 \times I$ is in *standard form* if, after a diffeomorphism, $S^1 \times \{pt\}$ are Legendrian (ie, they are the Legendrian rulings), with tangencies $z = \frac{k}{2n}$ (Legendrian divides), where $S^1 = \mathbb{R}/\mathbb{Z}$ has coordinate z.

3.3 Convex decompositions

Let (M, ξ) be a compact, oriented, tight contact 3–manifold with nonempty convex boundary ∂M. Suppose Σ is a properly embedded oriented surface with $\partial \Sigma \subset \partial M$. In this section we describe how to perturb Σ into a convex surface with Legendrian boundary (after possible modification of the characteristic foliation on ∂M), and perform a *convex decomposition*.

3.3.1 Legendrian realization principle

In this section we present the *Legendrian realization principle* — a criterion for determining whether a given curve or a collection of curves and arcs can be made Legendrian after a perturbation of a convex surface Σ. The result is surprisingly strong — we can realize almost any curve as a Legendrian one. Our formulation of Legendrian realization is a generalization of Kanda's [20]. Call a union of disjoint properly embedded closed curves and arcs C on a convex surface Σ with Legendrian boundary *nonisolating* if (1) C is transverse to Γ_Σ, and every arc in C begins and ends on Γ_Σ, and (2) every component of $\Sigma \setminus (\Gamma_\Sigma \cup C)$ has a boundary component which intersects Γ_Σ. Here, $C \pitchfork \Gamma_\Sigma$, strictly speaking, makes sense only after we have fixed a contact vector field v. For the Legendrian realization principle and its corollary, the contact structure ξ does not need to be tight.

Theorem 3.7 (Legendrian realization) *Consider C, a nonisolating collection of disjoint properly embedded closed curves and arcs, on a convex surface Σ with Legendrian boundary. Then there exists an admissible isotopy ϕ_s, $s \in [0, 1]$ so that*

(1) $_0 = id$,

(2) $_s(\ )$ *are all convex,*

(3) $_1(\Gamma\ ) = \Gamma\ _{1(\ )}$,

(4) $_1(C)$ *is Legendrian.*

Therefore, in particular, a nonisolating collection C can be realized by a Legendrian collection C^0 with the same number of geometric intersections. A corollary of this theorem, observed by Kanda, is the following:

Corollary 3.8 (Kanda) *A closed curve C on which is transverse to Γ can be realized as a Legendrian curve (in the sense of Theorem 3.7), if $C \cap \Gamma \neq \emptyset$.*

Observe that if C is a Legendrian curve on a convex surface , then its twisting number $t(C; Fr\ ) = \frac{1}{2}\#(C \cap \Gamma\ )$, where $\#(C \cap \Gamma\ )$ is the geometric intersection number (signs ignored).

Proof By Giroux's Flexibility Theorem, it su ces to nd a characteristic foliation F on with (an isotopic copy of) C which is represented by Legendrian curves and arcs. We remark here that these Legendrian curves and arcs constructed will always pass through singular points of F. Consider a component $_0$ of $n(\Gamma\ \cup C)$ — let us assume $_0 \ R_+$, so all the elliptic singular points are sources. Denote $@\ _0 = \gamma^- - \gamma^+$, where γ^- consists of closed curves γ which intersect Γ , and γ^+ consists of closed curves $\gamma\ \ C$. This means that for $\gamma\ \ \gamma^-$, either $\gamma\ \ \Gamma$ or $\gamma = \ _1 \cup\ _2 \cup \ \cup\ _{2k}$, where $_{2i-1}$, $i = 1; \ ; k$, are subarcs of C, $_{2i}$, $i = 1; \ ; k$, are subarcs of Γ , and the endpoint of $_j$ is the initial point of $_{j+1}$. Since C is nonisolating, γ^- is nonempty. What the γ^- provide are 'escape routes' for the flows whose sources are γ^+ or the singular set of $_0$ — in other words, the flow would be exiting along Γ .

Construct F so that (1) the subarcs of γ^- coming from C are now Legendrian, with a single positive half-hyperbolic point in the interior of the arc, (2) the curves of $@\ _0$ contained in C are Legendrian curves, with one positive half-elliptic point and one positive half-hyperbolic point. If $\gamma\ \ \gamma^-$ intersects C, then we give a neighborhood $\gamma\ \ I$ a characteristic foliation as in Figure 3. After lling in this collar, we may assume that F is transverse to and flows out of γ^-. If γ^+ is empty, then we introduce a positive elliptic singular point on the interior of $_0$, and let γ^+ be a small closed loop around the singular point, transverse to the flow. At any rate, we may assume the flow enters through γ^+ and exits through γ^- — by lling in appropriate positive hyperbolic points we may extend F to all of $_0$. □

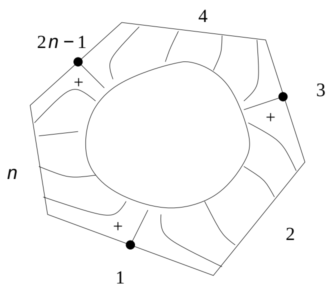


Figure 3: Characteristic foliation on $\gamma \quad I$

Actually, Kanda observes the following slightly stronger statement. The proof is identical | instead of single Legendrian curves, we insert a collar neighborhood.

Corollary 3.9 (Kanda) *Let $C \pitchfork \Gamma$ be a closed curve on which satis es $jC \setminus \Gamma \ j \quad 2$. Then C can be realized as a Legendrian curve, and, moreover, C can be made to have a standard annular collar neighborhood A consisting of a 1{parameter family of Legendrian ruling curves which are translates of C.*

We will now give a proof of one-half of Theorem 3.5, as a corollary of the Legendrian realization principle. The converse is more involved, and will be omitted (it will not be used in this paper).

Proof of Giroux's Criterion Assume Γ has a homotopically trivial curve γ which bounds a disk D. Then there exists a curve $\gamma^0 \quad nD$ parallel to γ, such that $\gamma^0 \setminus \Gamma \quad = ;$. Provided Γ does not consist solely of the homotopically trivial curve γ, γ^0 is nonisolating, and we may use Legendrian realization and assume, after modifying inside an I{invariant neighborhood, that γ^0 is Legendrian, and $t(\gamma^0) = 0$ with respect to . This implies that γ^0 bounds an overtwisted disk. The case $\#\Gamma \quad = 1$ requires a bit more work and one operation which is introduced later. We may assume is not a disk, since the boundary Legendrian curve would then bound an overtwisted disk. Take a closed curve which is homotopically essential, has no intersection with $\#\Gamma$, and does not separate (note that may be a boundary Legendrian curve). Use Legendrian realization to realize as a Legendrian curve with $t(\ ) = 0$. At this point, we will need to apply the 'folding' method for increasing the dividing curves described in Section 5.3.1. Each fold will introduce a pair of dividing curves parallel to . Now γ^0 is Legendrian-realizable. □

3.3.2 Cutting and rounding

Suppose M is a properly embedded oriented surface with $\partial \quad \partial M$, where ∂M is convex. Make $\partial \ \pitchfork \Gamma_{\partial M}$, and modify ∂ (by adding extraneous intersections) if necessary, so that $|\partial \ \cap \Gamma_{\partial M}| > 0$. Using the Legendrian realization principle, we may arrange C to be Legendrian on ∂M, with a standard annular collar, after perturbation.

C has a neighborhood $N(C)$ which is locally isomorphic to the neighborhood $\{x^2 + y^2 \quad \varepsilon\}$ of $M = \mathbb{R}^2 \quad (\mathbb{R}/\mathbb{Z})$ with coordinates (x, y, z) and contact 1–form $= \sin(2 \ nz)dx + \cos(2 \ nz)dy$, where $n = \frac{1}{2}|C \cap \Gamma_{\partial M}| \in \mathbb{Z}^+$. Here $C = \{x = y = 0\}$ and $\partial M \cap N(C) = \{x = 0\}$. Also let $\cap N(C) = \{y = 0\}$ and perturb the rest (xing $\cap N(C)$) so is convex with Legendrian boundary.

Lemma 3.10 *It is possible to arrange the transverse contact vector eld X for ∂M to be $\frac{\partial}{\partial x}$ and the transverse contact vector eld Y for to be $\frac{\partial}{\partial y}$.*

Proof Follows from Proposition 3.1. ☐

Now cut M along to obtain $M \setminus$ (which we really mean to be $M \setminus int(\quad I)$). Then round the edges using the following edge-rounding lemma:

Lemma 3.11 (Edge-rounding) *Let $_1$ and $_2$ be convex surfaces with collared Legendrian boundary which intersect transversely inside the ambient contact manifold along a common boundary Legendrian curve. Assume the neighborhood of the common boundary Legendrian is locally isomorphic to the neighborhood $N_\varepsilon = \{x^2 + y^2 \quad \varepsilon\}$ of $M = \mathbb{R}^2 \quad (\mathbb{R}/\mathbb{Z})$ with coordinates (x, y, z) and contact 1–form $= \sin(2 \ nz)dx + \cos(2 \ nz)dy$, for some $n \in \mathbb{Z}^+$, and that $_1 \cap N_\varepsilon = \{x = 0, 0 \quad y \quad \varepsilon\}$ and $_2 \cap N_\varepsilon = \{y = 0, 0 \quad x \quad \varepsilon\}$. If we join $_1$ and $_2$ along $x = y = 0$ and round the common edge (take $((_1 \cup \ _2) \setminus N \ ) \cup (\{(x - \ )^2 + (y - \ )^2 = \ ^2\} \cap N \ )$, where $< \varepsilon$), the resulting surface is convex, and the dividing curve $z = \frac{k}{2n}$ on $_1$ will connect to the dividing curve $z = \frac{k}{2n} - \frac{1}{4n}$ on $_2$, where $k = 0, \quad , 2n - 1$. Here we assume that the orientations of $_1$ and $_2$ are compatible and induce the same orientation after rounding.*

Refer to Figure 4.

Proof This follows from Lemma 3.10, and taking the transverse vector eld for $_1$ to be $\frac{\partial}{\partial x}$ and taking the transverse vector eld for $_2$ to be $\frac{\partial}{\partial y}$. The transverse vector eld for $\{(x - \ )^2 + (y - \ )^2 = \ ^2\} \cap N$ is the inward-pointing radial vector $-\frac{\partial}{\partial r}$ for the circle $\{(x - \ )^2 + (y - \ )^2 = \ ^2\}$. ☐

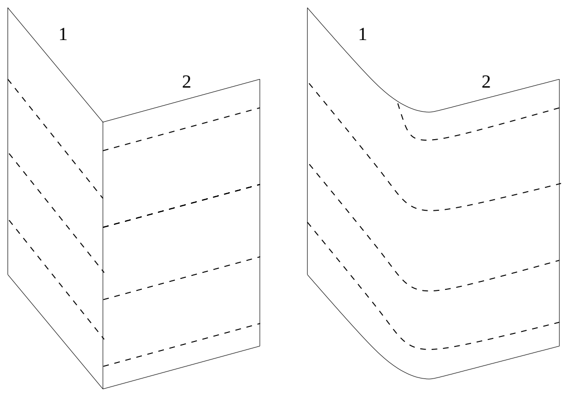


Figure 4: Edge rounding: Dotted lines are dividing curves.

3.4 Bypasses

Let M be convex surface (closed or compact with Legendrian boundary). A *bypass* for is an oriented embedded half-disk D with Legendrian boundary, satisfying the following:

(1) ∂D is the union of two arcs γ_1, γ_2 which intersect at their endpoints.

(2) D intersects transversely along γ_1.

(3) D (or D with opposite orientation) has the following tangencies along ∂D:

  (a) positive elliptic tangencies at the endpoints of γ_1 (= endpoints of γ_2),

  (b) one negative elliptic tangency on the interior of γ_1, and

  (c) only positive tangencies along γ_2, alternating between elliptic and hyperbolic.

(4) γ_1 intersects Γ exactly at three points, and these three points are the elliptic points of γ_1.

Refer to Figure 5 for an illustration. We will often also call the arc γ_2 a *bypass for* or a *bypass for* γ_1. We de ne the *sign* of a bypass to be the sign of the half-elliptic point at the center of the half-disk.

3.4.1 Bypass attachment lemma

Lemma 3.12 (Bypass Attachment) *Assume* D *is a bypass for a convex* . *Then there exists a neighborhood of* $[D$ M *di eomorphic to* $[0;1]$, *such*

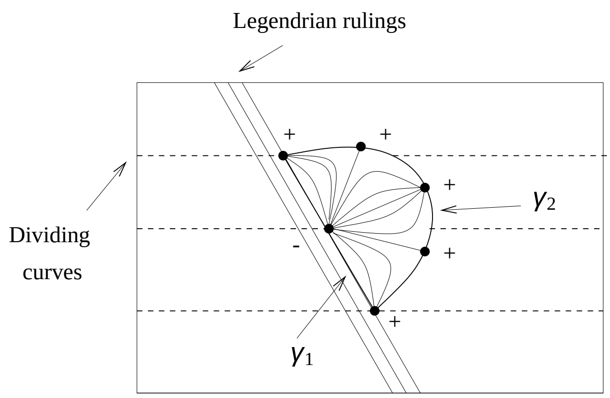


Figure 5: A bypass

that $_i = \{i\}$, $i = 0, 1$, *are convex,* $[0, \varepsilon]$ *is* I*–invariant,* $= \{\varepsilon\}$, *and* Γ_1 *is obtained from* Γ_0 *by performing the Bypass Attachment operation depicted in Figure 6 in a neighborhood of the attaching Legendrian arc* γ_1.

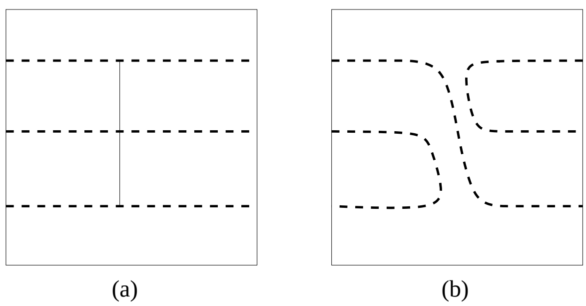


Figure 6: Bypass attachment: (a) Dividing curves on $_0$. (b) Dividing curves on $_1$. The dividing curves are dotted lines, and the Legendrian arc of attachment γ_1 is a solid line. We are only looking at the portion of $_i$ where the attachment is taking place.

Proof Extend γ_1 to a closed Legendrian curve γ on using the Legendrian Realization Principle. We may also assume that γ has an annular neighborhood of which is in standard form, and that D is a convex half-disk transverse to . Take an I–invariant one-sided neighborhood $[0, \varepsilon]$ of , where $= \{\varepsilon\}$. Now, $A' = \gamma \quad [0, \varepsilon] \quad [0, \varepsilon]$ is an annulus in standard form transverse to $\{0\}$. Form $A = A' \cup D$. A is convex, and we can take an I–invariant neighborhood $N(A)$ of A. If ∂A was smooth, then we take $(\{0\}) \cup N(A)$, and smooth out the four edges using the Edge-Rounding Lemma.

To smooth out ∂A, we use the Pivot Lemma, first observed by Fraser [10]. The proof is similar to the Flexibility Theorem.

Lemma 3.13 (Pivot) *Let S be an embedded disk in a contact manifold (M,ξ) with a characteristic foliation $\xi|_S$ which consists only of one positive elliptic singularity p and unstable orbits from p which exit transversely from ∂S. If γ_1,γ_2 are two unstable orbits meeting at p, and $\gamma_i \cap \partial S = p_i$, then, after a C^1–small perturbation of S fixing ∂S, we obtain S' whose characteristic foliation has exactly one positive elliptic singularity p' and unstable orbits from p' exiting transversely from ∂S, and for which the orbits passing through p_1, p_2 meet tangentially at p'.*

Now consider the half-elliptic singular points q_1, q_2 on D which are also the endpoints of γ_1. Modify D near q_i to replace q_i by a pair q_i^e, q_i^h, where q_i^e is a (full) elliptic point and q_i^h is a half-hyperbolic point as pictured in Figure 7. Use the Pivot Lemma to smooth the corners of A as in Figure 8. A is now

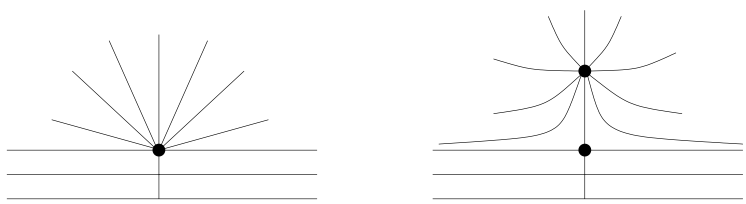


Figure 7: Replacing a half-elliptic point by a half-hyperbolic point and a full elliptic point

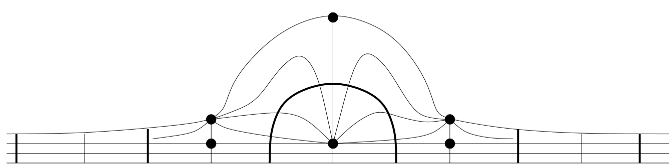


Figure 8: Smoothing the corners of A using the Pivot Lemma

convex with Legendrian boundary. The dividing curves on A are the thicker straight lines in Figure 8. Finally, we round the edges (see Figure 9) using the Edge-Rounding Lemma. □

We can also a define a *singular bypass* to be an immersion $D \to M$ which satisfies all the conditions of a bypass except one: D is an embedding away from $\gamma_1 \cap \gamma_2$, and these two points get mapped to one point on Σ. In this case, the Bypass Attachment Lemma would be as in Figure 10.

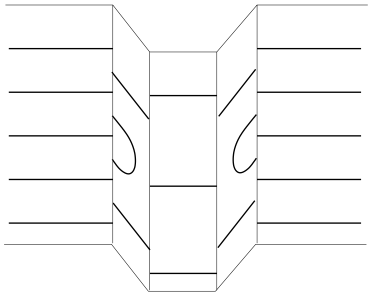


Figure 9: Rounding the edges will give the desired dividing set

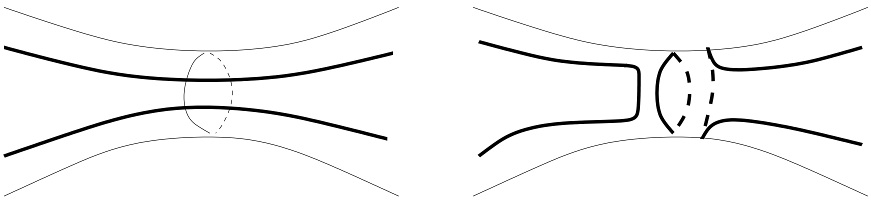


Figure 10: Edge-Rounding for a singular bypass

3.4.2 Tori

Let $\Sigma \subset \partial M$ be a convex torus in standard form, identified with $\mathbb{R}^2/\mathbb{Z}^2$. With this identification we will assume that the Legendrian divides and rulings are already linear, and will refer to *slopes* of Legendrian divides and Legendrian rulings. The slope of the Legendrian divides of Σ will be called the *boundary slope* s of Σ, and the slope of the Legendrian rulings will be the *ruling slope* r. Now assume, after acting via $SL(2;\mathbb{Z})$, that Σ has $s = 0$ and $r \neq 0$ rational. Note that we can normalize the Legendrian rulings via an element $\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \in SL(2;\mathbb{Z})$, $m \in \mathbb{Z}$, so that $-\infty < r \leq -1$.

In our later analysis on $T^2 \times I$ we will find an abundance of bypasses, and use them to stratify a given $T^2 \times I$ with a tight contact structure and convex boundary into thinner, more basic slices of $T^2 \times I$.

Lemma 3.14 (Layering) *Assume a bypass D is attached to $\Sigma = T^2$ with slope $s(T^2) = 0$, along a Legendrian ruling curve of slope r with $-\infty < r \leq -1$. Then there exists a neighborhood $T^2 \times I$ of $\Sigma \cup D \subset M$, with $\partial(T^2 \times I) = T_1 - T_0$, such that $\Gamma_{T_0} = \Gamma_\Sigma$, and Γ_{T_1} will be as follows, depending on whether $\#\Gamma_{T_0} = 2$ or $\#\Gamma_{T_0} > 2$:*

(1) *If* $\#\Gamma_{T_0} > 2$*, then* $s_1 = s_0 = 0$*, but* $\#\Gamma_{T_1} = \#\Gamma_{T_0} - 2$.

(2) *If* $\#\Gamma_{T_0} = 2$*, then* $s_1 = -1$*, and* $\#\Gamma_{T_1} = 2$.

Here s_i *is the boundary slope of* T_i.

Proof Follows from the Bypass Attachment Lemma. Refer to Figure 11 for the two possibilities. □

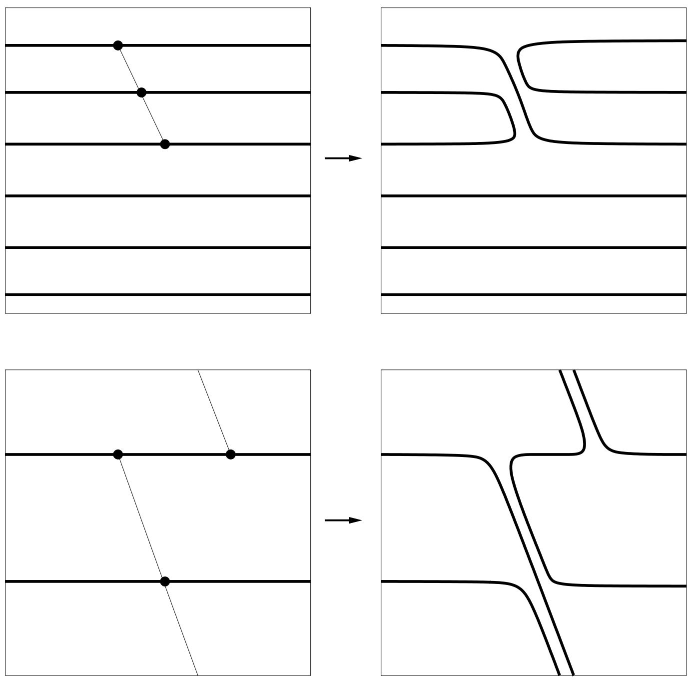


Figure 11: Bypass attachments along T^2

Notice that in the case $\#\Gamma_{T^2} = 2$, a bypass attachment effectively performs a *positive Dehn twist*.

3.4.3 Tessellation picture

In this section we interpret the Bypass Attachment Lemma in terms of the standard (Farey) tessellation of the hyperbolic unit disk $\mathbb{H}^2 = \{(x,y) \mid x^2 + y^2 \leq 1\}$. Recall we start by labeling $(1,0)$ as $0 = \frac{0}{1}$, and $(-1,0)$ as $\infty = \frac{1}{0}$. We inductively label points on $S^1 = \partial\mathbb{H}^2$ as follows (for $y > 0$): Suppose we have already labeled $\infty \geq \frac{p}{q} \geq 0$ (p, q relatively prime) and $\infty \geq \frac{p'}{q'} \geq 0$ (p', q' relatively prime) such that (p, q), (p', q') form a $\mathbb{Z}$–basis of $\mathbb{Z}^2$. Then, halfway between $\frac{p}{q}$ and $\frac{p'}{q'}$ along S^1 on the shorter arc (one for which $y > 0$ always), we label $\frac{p+p'}{q+q'}$. We then connect two points $\frac{p}{q}$ and $\frac{p'}{q'}$ on the boundary, if the corresponding shortest integral vectors form an integral basis of $\mathbb{Z}^2$. See Figure 12.

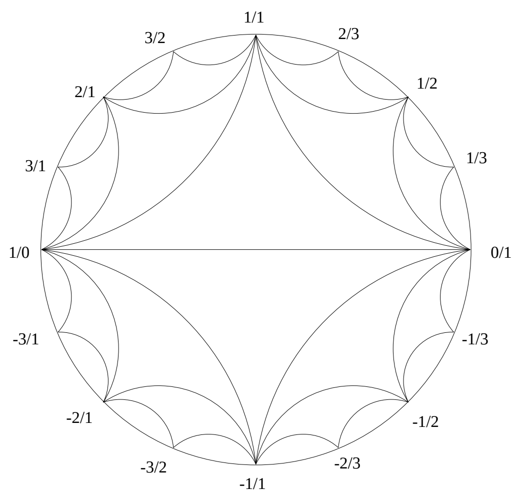


Figure 12: The standard tessellation of the hyperbolic unit disk

By transforming the situation in Lemma 3.14 via $SL(2, \mathbb{Z})$, we obtain the following rephrasing in more invariant language.

Lemma 3.15 *Let $\Sigma = T^2$ be a convex surface with $\#\Gamma_{T^2} = 2$ and slope $s = s(T^2)$. If a bypass D is attached to Σ along a Legendrian ruling curve of slope $r \neq s$, then the resulting convex surface Σ' will have $\#\Gamma_{T^2} = 2$ and slope s' which is obtained as follows: Take the arc $[r, s] \subset \partial\mathbb{H}^2$ obtained by starting from r and moving counterclockwise until we hit s. On this arc, let s' be the point which is closest to r and has an edge from s' to s.*

3.4.4 Abundance of bypasses

In this section we will demonstrate that bypasses are usually quite abundant. Suppose M is a 3–manifold with convex boundary, and we cut M along a convex surface with Legendrian boundary. The following are ways in which bypasses can occur.

Lemma 3.16 *Let $\Sigma = D^2$ be a convex surface with Legendrian boundary inside a tight contact manifold, and $t(\partial\Sigma, Fr\,\Sigma) = -n < 0$. Then every component of Γ_Σ is an arc which begins and ends on $\partial\Sigma$. There exists a bypass along $\partial\Sigma$ if $t(\partial\Sigma) < -1$.*

Proof If there is a closed dividing curve γ, then γ must bound a disk, contradicting Giroux's criterion. Therefore, every dividing curve must be an arc which begins and ends on the boundary. Now, if we have arranged Σ to have a collared Legendrian boundary and all half-elliptic points, then the endpoints of the dividing curves will lie between the half-elliptic points. There will be $2|t(\partial\Sigma)|$ endpoints for dividing curves, and hence $|t(\partial\Sigma)|$ curves. Now assume $t < -1$. Then there will exist an 'outermost' dividing curve γ — one that begins and ends on consecutive endpoints and cuts off a half-disk D_1 which does not contain any other dividing curve. Take an arc $\delta \subset \Sigma \setminus D_1$ which is parallel to γ and does not intersect Γ. Using the Legendrian realization principle (and the fact that $t < -1$, so that there are at least two half-elliptic points on $\Sigma \setminus D_1$), we can take δ to be a Legendrian arc after possible modification; δ cuts off a half-disk $D_2 \subset \Sigma$ (containing D_1) which is a bypass. ☐

Figure 13 illustrates a possible dividing set on $\Sigma = D^2$.

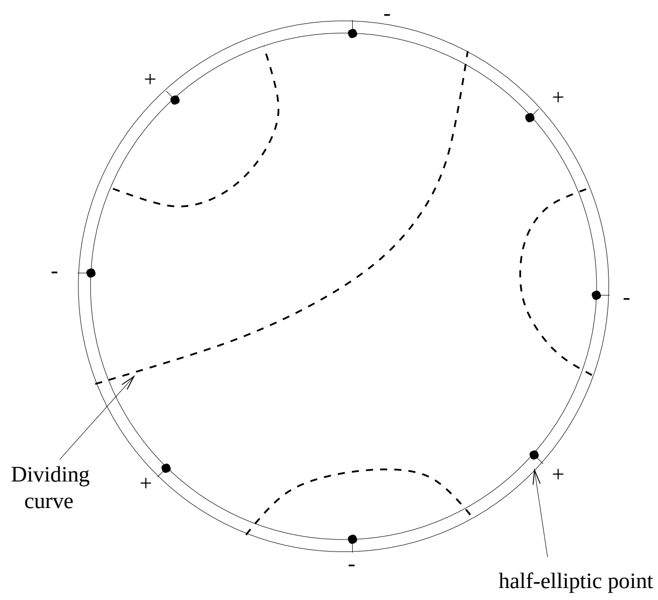


Figure 13: Standardized convex disk with Legendrian boundary

Proposition 3.17 (Imbalance Principle) *Let $\Sigma = S^1 \times [0,1]$ be convex with Legendrian boundary inside a tight contact manifold. If $t(S^1 \times \{0\}) < t(S^1 \times \{1\}) \leq 0$, then there exists a bypass along $S^1 \times \{0\}$.*

Proof Let $t_i = t(S^1 \times \{i\})$, $i = 0, 1$. There exist $2|t_0|$ endpoints of dividing curves on $S^1 \times \{0\}$ and $2|t_1|$ endpoints on $S^1 \times \{1\}$. If $t_0 < t_1$, then there exist two endpoints on $S^1 \times \{0\}$ which are connected by the same dividing arc γ. This γ must bound a half-disk D_1, and hence there is a Legendrian arc which bounds a bypass half-disk $D_2 \supset D_1$. □

Let Σ be a convex surface with (nonempty) Legendrian boundary, and γ be a dividing curve which cuts off a half-disk $D \subset \Sigma$ which has no other intersections with Γ_Σ. Such a dividing curve will be called a *boundary-parallel* dividing curve. We can generalize the above discussion and state the following (the proof is immediate):

Proposition 3.18 *Let Σ be a convex surface with Legendrian boundary, and γ a boundary-parallel dividing curve. If Σ is not a disk with $t(\partial\Sigma) = -1$, then there exists a bypass half-disk which contains the half-disk cut off by γ.*

4 Layering of $T^2 \times I$, $S^1 \times D^2$, and $L(p,q)$

4.1 Basic building blocks

In this section we will review the basic building blocks of tight contact manifolds.

4.1.1 3–ball

Recall the following fundamental theorem of Eliashberg [5]:

Theorem 4.1 *Assume there exists a contact structure ξ on a neighborhood of ∂B^3 which makes ∂B^3 convex with $\#\Gamma_{\partial B^3} = 1$. Then there exists a unique extension of ξ to a tight contact structure on B^3, up to an isotopy which fixes the boundary.*

The basic building blocks of tight contact manifolds are B^3, equipped with a unique tight contact structure if we prescribe the boundary.

4.1.2 Flexibility of characteristic foliation on boundary

Let M have nonempty boundary, and $\mathcal{F}$ be a characteristic foliation which is *adapted* to a dividing set $\Gamma_{\partial M}$. Denote by $\text{Tight}(M,\mathcal{F})$ the set of smooth contact 2–plane fields on M which induce a characteristic foliation $\mathcal{F}$ on ∂M. Then $\pi_0(\text{Tight}(M,\mathcal{F}))$ consists of the isotopy classes of tight contact structures on M with fixed boundary characteristic foliation $\mathcal{F}$. The Flexibility Theorem allows us to prove the following:

Proposition 4.2 *Let M be a compact, oriented 3–manifold with nonempty boundary. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two characteristic foliations on ∂M which are adapted to $\Gamma_{\partial M}$. There exists a bijection*

$$\phi_{12}: \pi_0(Tight(M,\mathcal{F}_1)) \rightarrow \pi_0(Tight(M,\mathcal{F}_2)).$$

Proof The map ϕ_{12} is defined as follows: Given any tight contact structure ξ in $\text{Tight}(M,\mathcal{F}_1)$, take an invariant neighborhood $\Sigma\times[0,\infty)\subset M$ for ξ, where $\Sigma\times\{0\}=\partial M$. Take a parallel copy $\Sigma_k=\Sigma\times\{k\}$, for some large k. Apply Giroux's Flexibility Lemma, with contact vector field $\frac{\partial}{\partial t}$, where t is the coordinate for $[0,\infty)$. Starting with Σ_k we obtain $\Sigma'\subset\Sigma\times(0,\infty)$ with characteristic foliation $\mathcal{F}_2$, after a $\frac{\partial}{\partial t}$–admissible isotopy (provided $k>>0$). Σ' divides $M=M_1\cup M_2$, where $M_1\subset\Sigma\times[0,\infty)$. We simply set $\phi_{12}(\xi)=\xi|_{M_2}$, where M_2 is identified with M and Σ' is identified with Σ via the flow of $\frac{\partial}{\partial t}$. ϕ_{12} does not depend on k, since we are considering contact 2–plane fields up to isotopy. We now prove that ϕ_{12} is independent of the choice of contact vector field X. Take a 1–parameter family of contact vector fields X_s, $s\in[0,1]$, which are transverse to Σ. Altering our perspective, this is equivalent to a 1–parameter family of $\frac{\partial}{\partial t}$–invariant contact 1–forms α_s, $s\in[0,1]$, on $\Sigma\times[0,\infty)\subset M$. The independence of the choice of vector field then follows from observing that the proof of the Flexibility Lemma also applies to a family of $\Sigma\times\mathbb{R}$'s. We now show that ϕ_{21} is the inverse of ϕ_{12}. Refer again to $\Sigma\times[0,\infty)$ for $\xi\in\text{Tight}(M,\mathcal{F}_1)$. Since $\frac{\partial}{\partial t}$ is also a transverse contact vector field for Σ', for sufficiently large $k'>>0$, $\Sigma_{k'}\subset M_2$. Finally observe that $\xi|_{M\setminus(\Sigma\times[k,k'])}$ is isotopic to ξ itself. □

In view of the proposition, we will often write $\text{Tight}(M,\Gamma)$ to stand for any of the $\text{Tight}(M,\mathcal{F})$, where $\mathcal{F}$ is adapted to Γ.

4.1.3 Standard neighborhoods of Legendrian curves

Let $\gamma\subset M$ be a Legendrian curve with a negative twisting number $t(\gamma)=n$ with respect to a fixed framing. The *standard tubular neighborhood* $N(\gamma)$ of a

Legendrian curve γ with $t(\gamma)$ negative is defined to be $S^1 \times D^2$ with coordinates $(z,(x,y))$ and contact 1–form $\alpha = \sin(2\pi nz)dx + \cos(2\pi nz)dy$. Here $\gamma = \{(z,(x,y)) | x = y = 0\}$. With respect to this fixed framing, we may identify $\partial(N(\gamma)) = \mathbb{R}^2/\mathbb{Z}^2$ by letting the meridian correspond to $\pm(1,0)^T$ and the longitude (from the framing) correspond to $\pm(0,1)^T$. With this identification, $s(\partial(N(\gamma))) = -\frac{1}{n}$. On the other hand, we have the following proposition, which is used by Kanda in [19], and is essentially proved in Makar–Limanov [22], although phrased a bit differently.

Proposition 4.3 *There exists a unique tight contact structure on* $S^1 \times D^2$ *with a fixed convex boundary with* $\#\Gamma_{\partial(S^1\times D^2)} = 2$ *and slope* $s(\partial(S^1 \times D^2)) = -\frac{1}{n}$*, where* n *is a negative integer. Modulo modifying the characteristic foliation on the boundary using the Flexibility Lemma, the tight contact structure is isotopic to the standard neighborhood of a Legendrian curve with twisting number* n*.*

Proof Using Proposition 4.2, we may assume that $T^2 = \partial(S^1 \times D^2)$ has Legendrian ruling curves of slope 0. Take a meridional disk D with one Legendrian ruling curve L on the boundary. There exists a collar annulus $A = L \times [0,1]$ with $L = L \times \{0\}$ transverse to T^2 along L. Using Proposition 3.1, we may perturb D to be convex with collared Legendrian boundary. Since $t(L, Fr_D) = -1$, there exists a unique dividing set Γ_D consisting of one arc from ∂D to ∂D. Using the Flexibility Lemma, we find that any D can be normalized to have a particular chosen characteristic foliation with this dividing set. Given any two tight contact structures ξ_1 and ξ_2 on $S^1 \times D^2$ with given boundary condition, we may match them up along $T^2 \cup D$, after an isotopy (not necessarily contact). The rest is a 3–ball $B^3 = (S^1 \times D^2)\setminus(T^2 \cup D)$ (after edge-rounding), and we find an isotopy which matches ξ_1 and ξ_2 on B^3 fixing ∂B^3, using Eliashberg's Theorem (Theorem 4.1). □

The following is a useful lemma:

Lemma 4.4 (Twist Number Lemma) *Let* (M,ξ) *be a tight manifold with a fixed framing* F*. Consider a Legendrian curve* γ *with* $t(\gamma, Fr) = n, n \in \mathbb{Z}$*, and a standard tubular neighborhood* V *of* γ *with boundary slope* $\frac{1}{n}$*. If there exists a bypass* D *which is attached along a Legendrian ruling curve of slope* r*, and* $\frac{1}{r} \geq n+1$*, then there exists a Legendrian curve with larger twisting number isotopic (but not Legendrian isotopic) to* γ*.*

Proof Follows immediately from Lemma 3.15. □

Notice that from this perspective the notion of *destabilization* due to Etnyre [7] is basically identical to our notion of a bypass.

4.2 Relative Euler class

Consider a tight contact structure ξ on a manifold M with convex boundary ∂M. Assume $\xi|_{\partial M}$ is trivializable, and choose a nowhere zero section s of ξ on ∂M. Then we may form the *relative Euler class* $e(\xi,s)\in H^2(M,\partial M;\mathbb{Z})$. Consider the following exact sequence:

$$\begin{array}{ccccccc} H^1(\partial M) & \to & H^2(M,\partial M) & \to & H^2(M) & \to & H^2(\partial M) \\ & & e(\xi,s) & \mapsto & e(\xi) & \mapsto & 0 \end{array}$$

This implies that a nonzero section s of ∂M allows for a lift of $e(\xi)$ to $e(\xi,s)$. Given two nonzero sections s_1 and s_2, $e(\xi,s_1)$ and $e(\xi,s_2)$ will differ by an element which is represented in $H^1(\partial M)=Map(\partial M,S^1)$. The relative Euler class can be evaluated as follows:

Proposition 4.5 *Let (M,ξ) be a contact manifold with convex boundary. Fix a nonzero section s of $\xi|_{\partial M}$.*

(1) *If $\Sigma\subset M$ is a closed convex surface with positive (resp. negative) region R_+ (resp. R_-) divided by Γ_Σ, then $\langle e(\xi),\Sigma\rangle=\langle e(\xi,s),\Sigma\rangle=\chi(R_+)-\chi(R_-)$.*

(2) *If $\Sigma\subset M$ is a compact convex surface with Legendrian boundary on ∂M and regions R_+ and R_-, and s is homotopic to s' which coincides with $\dot\gamma$ for every oriented connected component γ of $\partial\Sigma$, then $\langle e(\xi,s),\Sigma\rangle=\chi(R_+)-\chi(R_-)$.*

Proof (1) follows from perturbing Σ while fixing Γ so that Σ_ξ is singular Morse–Smale. Then use a standard computation which says that $\langle e(\xi,s),\Sigma\rangle=d_+-d_-$, where $d_\pm=e_\pm-h_\pm$, e_+ (resp. e_-) is the number of positive (resp. negative) elliptic points, and h_+ (resp. h_-) is the number of positive (resp. negative) hyperbolic points. (2) is almost identical. The only difference is that the half-elliptic and half-hyperbolic points must be counted properly. This is done in Kanda [20]. □

Let $T=\mathbb{R}^2/\mathbb{Z}^2$ be a component of a convex ∂M, where ξ is tight. Then, by the Flexibility Theorem, we may assume T is in standard form with slope $s(T)$

and Legendrian rulings with slope r. Take a nonzero section s of $\xi|_T$ given by the tangent field of the rulings. Let Σ be a compact surface with boundary along T. Starting with $T_0 = T$, there exists a 1–parameter family T_t, $t \in [0,1]$, of convex surfaces as in the Flexibility Theorem, so that T_1 is in standard form and has Legendrian rulings of slope r'. By excising and viewing T_t as the new boundary of M, we obtain a 1–parameter family of contact structures ξ_t with $\xi_0 = \xi$. If we take s' given by the tangent field of Legendrian rulings of slope r' on T_t, then

$$\langle e(\xi, s), \Sigma\rangle = \langle e(\xi_1, s'), \Sigma\rangle,$$

since the relative Euler class remains invariant under homotopy. This proves:

Lemma 4.6 *Let (M,ξ) be a tight contact manifold with convex boundary consisting of tori. Then the relative Euler class $\langle e(\xi, s), \Sigma\rangle$ is independent of the slope of the Legendrian rulings, if s is a nonzero section of ξ on a perturbation of ∂M, given by the tangent field of the rulings.*

We now explain how to compute relative Euler classes for spaces of interest. Assume ξ is tight. For $S^1 \times D^2$ with convex boundary, use the Flexibility Theorem to make the Legendrian rulings horizontal, take s to be tangent to $\partial(S^1 \times D^2)$, and compute the relative Euler class of a meridional convex disk Σ with Legendrian boundary by taking $\langle e(\xi, s), \Sigma\rangle = \chi(R_+) - \chi(R_-)$. $e(\xi, s) \in H^2(M, \partial M; \mathbb{Z}) = H_1(M; \mathbb{Z}) \simeq \mathbb{Z}$, so evaluation on a single meridional disk completely determines the relative Euler class.

Similarly, for $T^2 \times I$, we modify the boundary so the Legendrian rulings have the same slope r for both $T^2 \times \{0\}$ and $T^2 \times \{1\}$. Take a convex annulus $A = \gamma \times I$ with Legendrian boundary, where γ is a closed curve with slope r. If we compute $\langle e(\xi, s), A\rangle = \chi(R_+) - \chi(R_-)$ for two annuli of two different slopes, then this determines the element $e(\xi, s) \in H_1(T^2 \times I; \mathbb{Z}) \simeq H_1(T^2; \mathbb{Z}) \simeq \mathbb{Z}^2$.

4.2.1 Computation when $\partial(T^2 \times I)$ is nonsingular Morse–Smale

We explain how to relate the relative Euler class computations in the two settings: when $T_i = T^2 \times \{i\}$, $i = 0, 1$, have nonsingular Morse–Smale characteristic foliations versus when T_i are in standard form. We assume the dividing sets are unchanged when switching between cases. In the Morse–Smale case, take the nonzero section s_0' given by the nonsingular flow on T_i, or, equivalently, a nonzero section s' of ξ which is everywhere transverse to T_i. In the standard form situation, take the nonzero section s_0 given by the Legendrian rulings, or, equivalently, a nonzero section s of ξ which is transverse to the rulings and twists along the ruling curves. By comparing s and s', we see that '$s - s'$' is given by $\pm n\ PD(v_i) \in H^1(T_i; \mathbb{Z})$, where v_i is the shortest integral vector with slope $s(T_i)$ and n is the torus division number.

4.3 Basic slices

In what follows, we will fix an identification $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Let $T^2 \times I = \mathbb{R}^2/\mathbb{Z}^2 \times [0,1]$ with coordinates (x,y,z), and $T_s = T^2 \times \{s\}$, $s \in [0,1]$. Recall the *boundary slope* $s_i = s(T_i)$ is the slope of the dividing curves on T_i (defined only when T_i is convex). We will call $(T^2 \times I, \xi)$ a *basic slice* if

(1) ξ is tight.

(2) T_i are convex and $\#\Gamma_{T_i} = 2$, for $i = 0,1$.

(3) The minimal integral representatives of $\mathbb{Z}^2$ corresponding to s_i form a $\mathbb{Z}$–basis of $\mathbb{Z}^2$.

(4) ξ is *minimally twisting*, as defined in Section 2.2.1.

After a diffeomorphism of T^2, we may assume that a basic slice has boundary slopes $s_1 = -1$ and $s_0 = 0$. Denote the subset of minimally twisting tight contact structures in $\text{Tight}(T^2 \times I; F)$ by $\text{Tight}^{min}(T^2 \times I; F)$.

Proposition 4.7 *Let Γ_{T_i}, $i = 0,1$, satisfy $\#\Gamma_{T_i} = 2$ and $s_1 = -1$, $s_0 = 0$. Then $|\pi_0(Tight^{min}(T^2 \times I; \Gamma_{T_1} \cup \Gamma_{T_2}))| = 2$. (Here $|\ |$ denotes cardinality.) The two tight contact structures are universally tight, and the Poincare duals to the relative Euler classes are given by $\pm(0,1) \in H_1(T^2;\mathbb{Z})$.*

Proof We will prove this proposition in steps.

Step 1 We will show that $|\pi_0(\text{Tight}^{min}(T^2 \times I; \Gamma_{T_1} \cup \Gamma_{T_2}))| \leq 2$. Assume the contact structure ξ is tight. Take Γ_{T_i} to have $\#\Gamma_{T_i} = 2$ and $s_1 = -1$, $s_0 = 0$, and choose F_i adapted to Γ_{T_i}, $i = 0,1$, so that the Legendrian rulings for both T_i are vertical. Take a vertical annulus $A = \{0\} \times (\mathbb{R}/\mathbb{Z}) \times [0,1]$, whose boundary consists of two Legendrian ruling curves, each with twisting number -1 relative to T_i. After perturbation, A is convex with collared Legendrian boundary. Assume that the endpoints of Γ_A are $\{0\} \times \{0, \frac{1}{2}\} \times \{0,1\}$.

Claim *All the dividing curves on A must connect from T_0 to T_1, ie, there are no boundary-parallel dividing curves.*

Proof of Claim Otherwise, we obtain a singular bypass for T_0 attached along a vertical Legendrian ruling curve by using the Imbalance Principle. Using the Pivot Lemma, we smooth this bypass curve into a Legendrian curve γ which has slope ∞ when linearized. There exist $T_{\frac{1}{2}} \supset \gamma$ for which the twist number $t(\gamma)$ is zero with respect to $T_{\frac{1}{2}}$. Perturbing $T_{\frac{1}{2}}$ into a convex surface, we find that $s(T_{\frac{1}{2}}) = \infty$. Therefore, this contradicts the assumption that ξ is minimally twisting. □

Although the dividing curves connect from T_0 to T_1 and are parallel, there are still infinitely many possible configurations for Γ_A, distinguished by the *holonomy*. We can define the *holonomy* $k(A)$ as follows: pass to the cover $\{0\} \times \mathbb{R} \times I \subset S^1 \times \mathbb{R} \times I$ and let $k(A)$ be the integer such that there is a dividing curve which connects from $(0, 0, 0)$ to $(0, k(A), 1)$.

Claim *The holonomy function $k\colon \mathcal{A} \to \mathbb{Z}$ is surjective, where $\mathcal{A}$ is the set of convex annuli which have the same boundary as A and are isotopic to A.*

Proof of Claim We explain how to apply *sliding* to modify A to A' (with the same boundary) so that $k(A') = k(A) \pm 1$. This would then imply the surjectivity. Assume $A = \{0\} \times S^1 \times [0, 1]$ is convex with Legendrian boundary and holonomy $k(A)$. Let $N(A) = [-\varepsilon, \varepsilon] \times S^1 \times [0, 1]$ be an I–invariant neighborhood of A. Take $A' = (\{-\varepsilon\} \times S^1 \times [\varepsilon, 1 - \varepsilon]) \cup ((S^1 \setminus (-\varepsilon, 0)) \times S^1 \times \{\varepsilon, 1 - \varepsilon\}) \cup (\{0\} \times S^1 \times ([0, \varepsilon] \cup [1 - \varepsilon, 1]))$ and round the edges using the Edge-Rounding Lemma. Informally we are adjoining copies of T_0 and T_1 which are cut open along ∂A, and rounding. $k(A') = k(A) + 1.$ We can obtain A' with $k(A') = k(A) - 1$ similarly. □

Therefore, after an isotopy fixing its boundary, A can be put into standard form, with $k(A) = 0$ and vertical Legendrian rulings. Now cut along A to obtain $S^1 \times D^2$ with boundary slope $s(\partial(S^1 \times D^2)) = -2$ and vertical Legendrian rulings, after rounding the edges.

Next, using the Flexibility Lemma, we make the Legendrian rulings horizontal, and take a meridional disk D^2 of the solid torus, which we assume is convex with collared Legendrian boundary. There are two possible configurations of dividing curves, pictured in Figure 14. Now, given two tight contact structures ξ_1 and ξ_2 on $T^2 \times I$ with the given boundary conditions, ξ_1 and ξ_2 can be isotoped so that they agree on $T_0 \cup T_1 \cup A$. If the Γ_D are isotopic, then ξ_1 and ξ_2 can be matched up on D in addition, and Eliashberg's theorem (Theorem 4.1) implies that ξ_1 and ξ_2 are contact isotopic rel the boundary. Therefore we

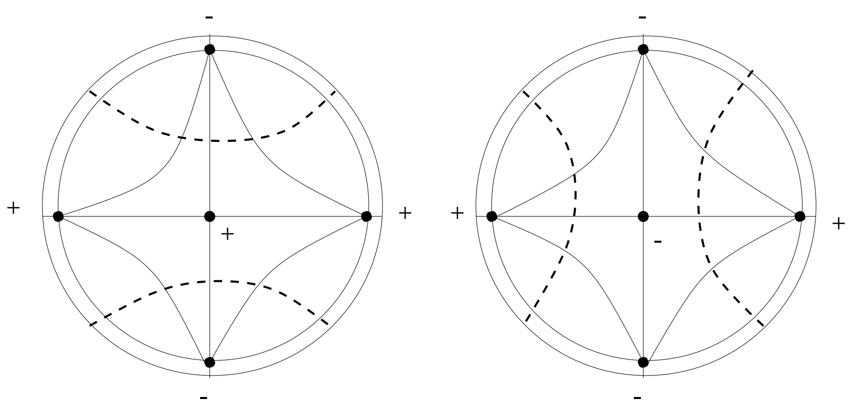


Figure 14: Two possibilities on D^2 with $t(\partial D) = -2$: The dotted lines are dividing curves.

have at most two tight structures on a basic slice up to an isotopy which fixes the boundary.

Step 2 Let us compute the relative Euler class. We already found that if the Legendrian rulings were made to have slope $r = \infty$, then the annulus $A = \gamma \times I$ with γ a closed curve of slope ∞ satisfies $\langle e(\xi, s), A\rangle = 0$. We now compute $\langle e(\xi, s), B\rangle$, for the annulus $B = \gamma \times [0,1]$, where γ is a closed curve with slope 1. Here the Legendrian rulings for T_0, T_1 have slope 1, and B is a convex surface with Legendrian boundary (we have fixed an orientation for B). Write $\gamma_i = \gamma \times \{i\}$, $i = 0, 1$. $t(\gamma_0) = -1$ and $t(\gamma_1) = -2$ with respect to B, so there exists a boundary-parallel dividing curve on B along γ_1 by the Imbalance Principle. We argue as in Step 1 to show that (1) two of the dividing curves on B must go across from γ_0 to γ_1; otherwise minimal twisting is violated, (2) we may normalize the holonomy $k(B)$ of the two dividing curves which go across, and (3) once B is normalized, the cut-open solid torus has boundary slope -1, hence is unique. Therefore, we find that $\langle e(\xi, s), B\rangle = \pm 1$, and $PD(e(\xi, s)) = \pm(0, 1)$.

Step 3 The two possible candidates for tight structures on the basic slice are tight (and even universally tight). We find an explicit model which can be embedded in (T^3, ξ_1), where $T^3 = \mathbb{R}^3/\mathbb{Z}^3$ has coordinates (x, y, z) and ξ_1 is given by the 1–form $\alpha_1 = \sin(2\pi z)dx + \cos(2\pi z)dy$. We can choose $T^2 \times [0, \frac{1}{8}] \subset T^3$, and perturb the boundary so that $\#\Gamma = 2$ and in standard form for both boundary components, with boundary slopes $s_{\frac{1}{8}} = -1$ and $s_0 = 0$. If we rotate this tight structure by π, then we obtain the other candidate.

Although not isotopic (distinguished by the relative Euler class), the two tight structures are diffeomorphic via a diffeomorphism isotopic to $-id$, where id is the identity map on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. The relative Euler class is computed by perturbing the boundary of $T^2 \times [0, \frac{1}{8}]$ so the characteristic foliation is Morse–Smale. The annulus $B = \gamma \times [0, \frac{1}{8}]$ with transverse boundary, where γ has slope 1, will give $\langle e(\xi, s'), B \rangle = 0$ if s' is tangent to the boundary. Converting this to s which is tangent to the Legendrian rulings for the characteristic foliation in standard form, we obtain $\langle e(\xi, s), B \rangle = \pm 1$.

Step 4 It remains to show that the tight structure on $N = T^2 \times [0, \frac{1}{8}] \subset T^3$ is minimally twisting. Assume the existence of a torus $T' \subset N$ parallel to $T_{\frac{1}{8}}$ and T_0, for which the boundary slope s' is not between -1 and 0. This is equivalent to the existence of a linear Legendrian curve $\gamma_0 \subset N$ with slope s' and $t(\gamma_0, Fr_{T^2}) = 0$. We will pass to the universal cover $(\tilde{N} = \mathbb{R}^2 \times [0, \frac{1}{8}], \tilde{\xi}_1)$ to find an overtwisted disk.

Assume $s' > 0$. Pick a point $p = (x_0, y_0, z_0)$ on γ_0 with the smallest z–coordinate, and view γ_0 as starting and ending at p. A lift $\tilde{\gamma}_0$ will have endpoints $\tilde{p}_1 = (x_1, y_1, z_0)$, $\tilde{p}_2 = (x_2, y_2, z_0)$ which are lifts of p. Let $\tilde{\gamma}_1$ be the linear Legendrian curve from $(x_1, y_1, 0)$ to (x_1, y_1, z_0), $\tilde{\gamma}_2$ be the linear Legendrian curve from (x_2, y_2, z_0) to $(x_2, y_2, 0)$, and $\tilde{\gamma}_3$ be the linear Legendrian curve from $(x_2, y_2, 0)$ to $(x_1, y_2, 0)$. Then the composite $\tilde{\gamma} = \tilde{\gamma}_1 + \tilde{\gamma}_0 + \tilde{\gamma}_2 + \tilde{\gamma}_3$ is a Legendrian curve which projects to a closed curve onto the xz–plane and has positive holonomy. It is easy to decrease its holonomy by adding a curve $\tilde{\gamma}'$ which projects to γ' in the xz–plane and satisfies $\gamma' = \partial\Omega$, where Ω is a region in the xz–plane. Therefore, we obtain an overtwisted disk bounded by $\tilde{\gamma} + \tilde{\gamma}'$. Notice that $t(\gamma_0) = 0$ translates to $tb(\tilde{\gamma} + \tilde{\gamma}') = 0$. We argue similarly for $s' < -1$, and find that N is minimally twisting. □

We also have the following corollary. A *pre-Lagrangian torus* is a torus with linear characteristic foliation.

Corollary 4.8 *Let $(T^2 \times I, \xi)$ be a basic slice, with boundary slopes s_0 and s_1. Then for any slope s between s_1 and s_0 (see Section 2.2.1 for the definition), there exists a convex torus T parallel to $T^2 \times \{pt\}$ with slope $s(T) = s$. For any slope s between s_1 and s_0 (but $\neq s_0, s_1$), there exists a pre-Lagrangian torus T parallel to $T^2 \times \{pt\}$ with slope s.*

Proof This follows from the explicit model in the proof of Proposition 4.7. A basic slice will have pre-Lagrangian tori of all slopes between s_1 and s_0, and any pre-Lagrangian torus can be perturbed into a convex torus with the same slope. □

4.4 Decomposition of $T^2 \times I$ into layers

Assume that ξ on $T^2 \times I$ is tight. In this section we will also assume the following: (1) $\#\Gamma_{T_i} = 2$, $i = 0, 1$, (2) ξ has minimal twisting. It is most convenient to arrange the boundary slopes, via an action of $SL(2, \mathbb{Z})$, as follows: $-\infty < s_1 \leq -1$ and $s_0 = -1$. Write $s_1 = -\frac{p}{q}$, where $p \geq q > 0$ are integers and $(p, q) = 1$.

4.4.1 Nonrotative case

Proposition 4.9 (Nonrotative case) *Let Γ_{T_i}, $i = 0, 1$, satisfy $\#\Gamma_{T_i} = 2$ and $s_0 = s_1 = -1$. Then there exists a holonomy map $k : \pi_0(Tight^{min}(T^2 \times I, \Gamma_{T_1} \cup \Gamma_{T_2})) \to \mathbb{Z}$ which is bijective.*

Proof Use the Flexibility Lemma to obtain rulings of slope $r_0 = r_1 = 0$, take a horizontal annulus $S^1 \times \{0\} \times I$ with Legendrian boundary, and perturb it into a convex surface with collared Legendrian boundary. If the dividing curves of the annulus A do not cross from T_0 to T_1, then, by Proposition 3.18, there exists a boundary-parallel dividing curve on A along T_1, and the corresponding singular bypass gives rise to a factoring $T^2 \times [0, 1] = T^2 \times [0, \frac{1}{2}] \cup T^2 \times [\frac{1}{2}, 1]$, where the intermediate layer $T_{\frac{1}{2}}$ is convex with slope $s_{\frac{1}{2}} = 0$. This contradicts our minimal twisting assumption. Therefore, both dividing curves on A cross from T_0 to T_1. Put A in standard form, cut along A, and perform Edge-Rounding to obtain a solid torus with boundary slope -1. There exists a unique tight contact structure on this solid torus by Proposition 4.3. This implies that, for every choice of Γ_A, there exists at most one tight contact structure.

Now define the *holonomy* $k(A)$ by passing to the cover $\mathbb{R} \times \{0\} \times I \subset \mathbb{R} \times S^1 \times I$ and letting $k(A)$ be the integer such that there exists a dividing curve connecting from $(0, 0, 0)$ to $(k(A), 0, 1)$ (assume that the endpoints of all the possible dividing curve configurations are fixed). To write down a tight contact structure ξ_0 with $k(A) = 0$, simply take the I–invariant neighborhood of a convex T^2 with $\#\Gamma = 2$, $s(T^2) = -1$, and horizontal Legendrian rulings. If we take ξ_0 and isotoped T_1 via $(x, y) \mapsto (x, y + k)$ $(k \in \mathbb{Z})$, while fixing T_0, then we obtain ξ_k with $k(A) = k$. The I–invariant contact structure is embeddable into a basic slice, hence it is universally tight. Moreover, since the basic slice is minimally twisting, so is the I–invariant tight structure.

We claim that $k(A)$ takes constant values in $\mathcal{A}$, the set of convex annuli which have the same boundary as A and are isotopic to A, provided ξ is fixed. Assume

$A' \in \mathcal{A}$ with $k(A') \neq 0$ (assume $k(A) = 0$). The proof follows a strategy due to Kanda [19]. The strategy is to pass to $\widetilde{M} = S^1 \times [-n, n] \times I$ (n large) and pick nonintersecting lifts $\widetilde{A'}$ and $\widetilde{A}$ of A' and A. If $k(A') > 0$, then take $\widetilde{A'}$ to lie above $\widetilde{A}$. Pick $N \subset \widetilde{M}$ bounded above by $\widetilde{A'}$ and below by $\widetilde{A}$, and round the edges. We find that the boundary slope of the rounded N is 0 or a positive integer. If the slope is zero, we have an overtwisted disk. Assume the slope is a positive integer. Make N have horizontal Legendrian rulings, and take a convex meridional disk D with a Legendrian collar boundary. There exists a bypass by Lemma 3.16, and, after bypass attachment, the slope is ∞. Therefore, N is the standard neighborhood of a Legendrian curve γ isotopic to $S^1 \times \{0\}$ with twist number 0, and $M \setminus N$ is the standard neighborhood of a Legendrian curve with twist number -1. This is a contradiction. □

Although the holonomy gives infinitely many tight contact structures up to isotopy (fixing the boundary), this turns out to be a special feature of the *nonrotative* case. In the *rotative* case, Proposition 4.7 allows us to reduce the infinitely many possible dividing sets to a finite collection.

4.4.2 Rotative case

Proposition 4.10 (Minimal twisting, rotative case) *Let Γ_{T_i}, $i = 0, 1$, satisfy $\#\Gamma_{T_i} = 2$ and $s_0 = -1$, $s_1 = -\frac{p}{q}$, where $p > q > 0$. Then*

$$|\pi_0(Tight^{min}(T^2 \times I; \Gamma_{T_1} \cup \Gamma_{T_2}))| \leq |(r_0 + 1)(r_1 + 1) \cdots (r_{k-1} + 1)(r_k)|, \quad (6)$$

where $-\frac{p}{q}$ has a continued fraction expansion

$$-\frac{p}{q} = r_0 - \cfrac{1}{r_1 - \cfrac{1}{r_2 - \cdots \cfrac{1}{r_k}}},$$

with all $r_i < -1$ integers.

The proof will consist of a factorization $T^2 \times I = \bigcup_{i=0}^{k} (T^2 \times [\frac{i}{k}, \frac{i+1}{k}])$ where $T_{\frac{i}{k}}$, $i = 0, \dots, k$, are convex with $\#\Gamma_{T_{i/k}} = 2$ and slopes $s_{\frac{i}{k}}$ arranged as $s_0 > s_{\frac{1}{k}} > s_{\frac{2}{k}} > \cdots > s_{\frac{k}{k}} = s_1$; this is followed by a shuffling argument which reorders the layers. This is sufficient to prove the upper bound in the proposition. The proof will occupy the next three sections. To prove that the upper bound is exact requires embeddings into lens spaces. This will be done in Section 4.6.2.

4.4.3 Factoring

Take $r_1 = r_0 = 0$ as before, and consider the horizontal annulus A. Since $t(S^1 \times \{0\} \times \{1\}) = -p < t(S^1 \times \{0\} \times \{0\}) = -1$, there must exist a bypass along T_1. Therefore, we can factor $T^2 \times I$ into $T^2 \times [0, \frac{1}{2}]$ and $T^2 \times [\frac{1}{2}, 1]$, where the latter is a basic slice. This follows from the following lemma:

Lemma 4.11 *$T_{\frac{1}{2}}$ will have boundary slope $-\frac{p'}{q'}$, where $pq' - qp' = 1$, $p > p' > 0$, and $q \geq q' > 0$.*

Proof In order to use Lemma 3.15, we need to reflect T_1 and transform via $SL(2, \mathbb{Z})$ so that the boundary slope is 0. Reflection gives us $-T_1$ with boundary slope $\frac{p}{q}$ and rulings of slope 0. Then $A_0 = \begin{pmatrix} p' & -q' \\ p & -q \end{pmatrix}$ sends $(q, p)^T \mapsto (-1, 0)^T$, $(1, 0)^T \mapsto (p', p)^T$. Since $p > p' > 0$, $\frac{p}{p'} > 1$, the boundary slope must be ∞ by Lemma 3.15. Now, $A_0^{-1} : (0, 1)^T \mapsto (q', p')^T$, and we have the lemma. □

Applying Lemma 4.11 inductively, we obtain basic slices whose boundary slopes increase from $-\frac{p}{q}$ to -1 in a finite number of steps.

Example Assume $s_1 = -\frac{10}{3}$ and $s_0 = -1$. Then the boundary slopes are $-\frac{10}{3}, -\frac{3}{1} = -3, -2, -1$, so we have a factorization into 3 layers.

4.4.4 Continued fractions

There exists a natural interpretation of the layering process in terms of continued fractions.

Let $-\frac{p}{q}$ have the following continued fraction expansion:

$$-\frac{p}{q} = r_0 - \cfrac{1}{r_1 - \cfrac{1}{r_2 - \cdots \cfrac{1}{r_k}}},$$

with all $r_i < -1$ integers. We identify $-\frac{p}{q}$ with $(r_0, r_1, \ldots, r_k)$. Then $-\frac{p'}{q'}$ as given in Lemma 4.11 will correspond to $(r_0, r_1, \ldots, r_k + 1)$, where we identify $(r_0, \ldots, r_{k-1} + 1) \simeq (r_0, \ldots, r_k + 1)$ if $r_k = -2$. This follows inductively from observing that if $\frac{a}{b}$, $\frac{a'}{b'}$ satisfy $ab' - ba' = 1$, then $r - \frac{1}{a/b} = \frac{ra-b}{a}$ and $r - \frac{1}{a'/b'} = \frac{ra'-b'}{a'}$ satisfy

$$(ra-b)a' - (ra'-b')a = 1.$$

Therefore, the boundary slopes of the factorization can be obtained in order by decreasing the last entry of the corresponding continued fraction expansion.

Notice that this layering process corresponds to taking a sequence $-\frac{p}{q} = -\frac{p_0}{q_0} < -\frac{p_1}{q_1} < \cdots < -1$ where the consecutive slopes correspond to pairs of vectors which form an integral basis of $\mathbb{Z}^2$. Moreover, the slopes on each basic slice represent a positive Dehn twist from the front face to the back face. Therefore, we have the following Factoring Lemma:

Lemma 4.12 *Let ξ be a minimally twisting tight contact structure on $T^2 \times I$. Then $T^2 \times I$ admits a decomposition $T^2 \times I = \bigcup_{i=0}^{k}(T^2 \times [\frac{i}{k}, \frac{i+1}{k}])$, where $T_{\frac{i}{k}}$, $i = 0, \dots, k$, are convex with $\#\Gamma_{T_{i/k}} = 2$ and slopes $s_{\frac{i}{k}}$. The sequence of slopes is obtained by taking the shortest sequence of positive Dehn twists from $-\frac{p}{q}$ to -1. Alternatively, in the tessellation picture, $s_1, s_{\frac{k-1}{k}}, \dots, s_0$, is the shortest sequence of hops along edges from s_1 to s_0, subject to the constraint that $s_{\frac{i}{k}}$ sit on the arc $[s_1, s_0] \subset \partial\mathbb{H}^2$ (counterclockwise starting from s_1).*

4.4.5 Sliding maneuver

There exists a natural grouping of the layers into blocks via continued fractions. The blocks are isomorphic to $T^2 \times I$ with minimal twisting, $\#\Gamma_{T_i} = 2$, $i = 0, 1$, and boundary slopes $s_1 = -m$, $s_0 = -1$, where $m \in \mathbb{Z}^+$, $m > 1$. Such blocks will be called *continued fraction blocks*, and are special because the basic layers that comprise a continued fraction block can be 'shuffled'.

Proposition 4.13 *Let $\Gamma_i = \Gamma_{T_i}$, $i = 0, 1$, be dividing sets satisfying $\#\Gamma_i = 2$, $s_0 = -1$, $s_1 = -m$, $m \in \mathbb{Z}^+$, $m > 1$. Then $|\pi_0(Tight^{min}(T^2 \times I; \Gamma_0 \cup \Gamma_1))| \leq m$.*

Proof Let $(T^2 \times I, \xi)$ have minimal twisting, $\#\Gamma_i = 2$, slopes $s_1 = -m$ and $s_0 = -1$, and coordinates $((x, y), z)$. Consider a convex annulus $A = S^1 \times \{0\} \times I$ with Legendrian boundary (after perturbation of the boundary) and oriented normal $\frac{\partial}{\partial y}$. The minimal twisting condition guarantees the existence of two dividing curves on A which go across from T_0 to T_1, and $m-1$ dividing curves from T_1 to itself. Since there must be at least one bypass, we can peel off a layer and obtain a basic slice $T^2\times[\frac{m-2}{m-1},1]$ with $s_1=-m$ and $s_{\frac{m-2}{m-1}}=-(m-1)$. The horizontal annulus from $T_{\frac{m-2}{m-1}}$ to T_1 will have $2(m-1)$ dividing curves which go across from $T_{\frac{m-2}{m-1}}$ to T_1, and 1 dividing curve from T_1 to itself, which is the boundary-parallel curve used to peel off $T^2\times[\frac{m-2}{m-1};1]$. The tight structure on the basic slice is determined by whether the half-disk separated by the boundary-parallel curve is positive or negative. (Recall that $PD(e(\xi,s))=\pm(0,1)\in H_1(T^2;\mathbb{Z})$ by Proposition 4.7.) In a similar manner, we successively peel off $T\times[\frac{i-1}{m-1},\frac{i}{m-1}]$, with boundary slopes $-i$ and $-(i+1)$. Let us say that the layer $T\times[\frac{i-1}{m-1},\frac{i}{m-1}]$ is positive (resp. negative) if the sign of half-disk separated by the boundary-parallel dividing curve is positive (resp. negative). The proof then follows from repeated applications of the following lemma. □

Lemma 4.14 (Shuffling) *Consider a minimally twisting tight $(T^2\times I,\xi)$ with $\#\Gamma_i=2$, $i=0,1$, boundary slopes $s_1=-k$ and $s_0=-k+2$, $m\geq k$, $k-2\geq 1$. Given a factorization $T^2\times I=N_1\cup N_2$, $N_1=T^2\times[0,\frac{1}{2}]$, $N_2=T^2\times[\frac{1}{2},1]$, into basic layers ($s_{\frac{1}{2}}=-k+1$), where N_1 is positive and N_2 is negative, there exists another factorization $T^2\times I=N_1'\cup N_2'$ so that N_1 is negative and N_2 is positive.*

Proof The lemma follows from applying the *sliding maneuver*, which we have already used once to prove Proposition 4.7. Assume the Legendrian rulings for $T_0,T_{\frac{1}{2}},T_1$ are all horizontal. Let $A_1=S^1\times\{0\}\times[0,\frac{1}{2}]$ and $A_2=S^1\times\{0\}\times[\frac{1}{2},1]$. A_1 (resp. A_2) has $2(k-2)$ (resp. $2(k-1)$) dividing curves which go across and 1 boundary parallel dividing curve along $T_{\frac{1}{2}}$ (resp. T_1). Let $L=A_1\cap T_{\frac{1}{2}}=S^1\times\{0\}\times\{\frac{1}{2}\}$, and $\Gamma_{A_1}\cap L=\{0,\frac{1}{2(k-1)},\frac{2}{2(k-1)},\dots,\frac{2(k-1)}{2(k-1)}\}\times\{0\}\times\{\frac{1}{2}\}$. We modify A_1 to A_1' by an isotopy which fixes ∂N_1 so that the boundary-parallel curve of $\Gamma_{A_1'}$ has endpoints which have been shifted by $\frac{2}{2(k-1)}$ along L.

Informally, we attach copies of T_0 and $T_{\frac{1}{2}}$ and round the edges. Let $N(A_1)=S^1\times[-\varepsilon,\varepsilon]\times[0,\frac{1}{2}]$ be an I-invariant neighborhood of A_1. Take $A_1'=(S^1\times\{\varepsilon\}\times[\varepsilon,\frac{1}{2}-\varepsilon])\cup(S^1\times(S^1\setminus(0,\varepsilon))\times\{\varepsilon,\frac{1}{2}-\varepsilon\})\cup(S^1\times\{0\}\times([0,\varepsilon]\cup[\frac{1}{2}-\varepsilon,\frac{1}{2}]))$, and round the edges using the Edge-Rounding Lemma. This moves the endpoints of the boundary-parallel curve by $-\frac{2}{2(k-1)}$ along L. See Figure 15 for an illustration. Note that the copy of T_0 is not attached in this picture, but we can still see that the bypass has been slid along L. Using the sliding maneuver, we may arrange $A_1\cup A_2$ so that the two dividing curves with endpoints on T_1 are not nested, ie, they are both boundary-compressible dividing curves for $A_1\cup A_2$. We then have the freedom to choose which bypass to peel off first. □

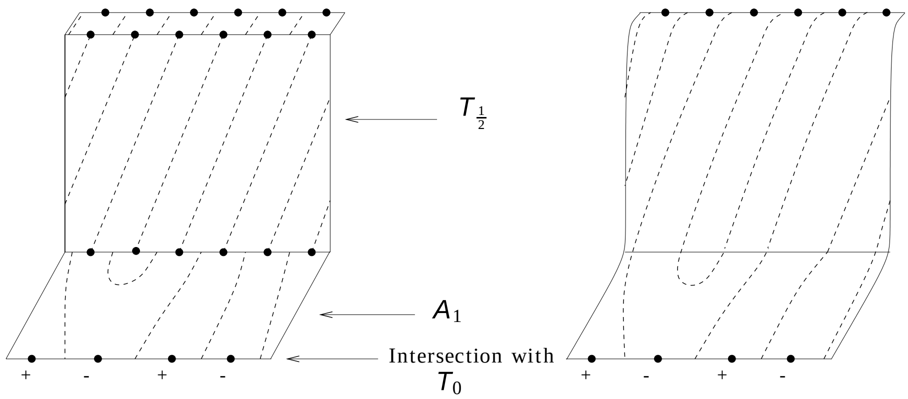


Figure 15: Sliding maneuver

Proof of Proposition 4.10 We now group the layers of $T^2 \times I$ with minimal boundary, minimal twisting, and boundary slopes $-\frac{p}{q}$ and -1 as follows: Act via $A_0 = \begin{pmatrix} -r_0 & 1 \\ -1 & 0 \end{pmatrix}$. Then $(1,-1)^T \mapsto (-r_0-1,-1)^T$ and $(q,-p)^T \mapsto (-r_0 q - p, -q)^T$. The boundary slopes are now $s_1 = \frac{q}{r_0 q + p} = r_1 - \frac{1}{r_2}$ and $s_0 = \frac{1}{r_0+1}$. Peel off a block with slopes $s_{\frac{1}{2}} = -1$ and $s_0 = \frac{1}{r_0+1}$, which is diffeomorphic to the form treated in Proposition 4.13, then continue. We will then obtain k blocks, each with minimal twisting, minimal boundary, and boundary slopes -1, $\frac{1}{r_i+1}$ (or, equivalently, $r_i + 1$ and -1), and one last block (at the very front) with boundary slopes r_k and -1. This completes the proof of Proposition 4.10. □

4.5 Factoring solid tori

Let $(S^1 \times D^2, \xi)$ be a solid torus with convex boundary T and $\#\Gamma_T = 2$. Fix a framing F so that the boundary slope $-\frac{p}{q}$ satisfies $-\infty < -\frac{p}{q} \leq -1$. This is possible by normalizing via a suitable element of $SL(2,\mathbb{Z})$. Here we view $T = \mathbb{R}^2/\mathbb{Z}^2$, where $(1,0)^T$ is the meridional circle and $(0,1)^T$ is the longitude with respect to F.

Proposition 4.15 *Let Γ_0, Γ_1 be dividing sets on $T^2 \simeq \mathbb{R}^2/\mathbb{Z}^2$ with $\#\Gamma_i = 2$, $i = 1,2$, and slopes $s_0 = -1$, $s_1 = -\frac{p}{q}$ ($-\infty < -\frac{p}{q} \leq -1$). Assume we have identified $\partial(S^1 \times D^2) \simeq \mathbb{R}^2/\mathbb{Z}^2$. Let ξ be a tight contact structure on $M \simeq S^1 \times D^2$ with convex boundary condition Γ_1. Then there exists a factorization $M = N \cup (M \setminus N)$, where N is the standard neighborhood of a core Legendrian* *curve with twist number* -1*,* $M \setminus N \simeq T^2 \times I$*, and* $\xi|_{T^2 \times I}$ *is minimally twisting with boundary dividing sets* Γ_0*,* Γ_1*. Hence we have*

$$|\pi_0(Tight(S^1 \times D^2; \Gamma_1))| \leq |\pi_0(Tight^{min}(T^2 \times I; \Gamma_0 \cup \Gamma_1))|$$

Proof Let γ be a Legendrian curve isotopic to the core S^1, satisfying $t(\gamma) = -m$, $m \in \mathbb{Z}^+$. Such a Legendrian curve exists because any closed curve C' is C^0–close approximated by a Legendrian curve C isotopic to C'. Take a standard neighborhood N' of γ so that $\partial N'$ is convex with $s(\partial N') = -\frac{1}{m}$ and $\#\Gamma_{\partial N'} = 2$. Now consider $M \setminus N'$ with boundary slopes $-\frac{p}{q}$ and $-\frac{1}{m}$.

We claim that the tight contact structure on $M \setminus N'$ is minimally twisting. Assume otherwise. Then there exists a factorization of $M \setminus N' = T^2 \times I$ as $(T^2 \times [0, \frac{1}{2}]) \cup (T^2 \times [\frac{1}{2}, 1])$, where $s_0 = -\frac{1}{m}$, $s_1 = -\frac{p}{q}$, and $s_{\frac{1}{2}}$ is not between s_1 and s_0. Proposition 4.16 below implies that there exists a convex torus with any slope between s_1 and $s_{\frac{1}{2}}$ and any slope between $s_{\frac{1}{2}}$ and s_0. In particular, $s_{\frac{1}{2}} = 0$ is realized. Now, a Legendrian divide on $T_{\frac{1}{2}}$ has twisting number zero with respect to a meridional disk it bounds. Hence $M \setminus N'$ is minimally twisting.

Since $-\frac{p}{q} < -1$, the layering procedure for $M \setminus N'$ will give us a convex torus T' with boundary slope -1, parallel to T. Factor $M = N \cup (M \setminus N)$ along T. By Proposition 4.3 N is a standard neighborhood of a Legendrian curve with twisting number -1. Hence, the number of potential tight structures on $S^1 \times D^2$ with $\#\Gamma_{\partial(S^1 \times D^2)} = 2$ and boundary slope $-\frac{p}{q}$ is bounded above by the number of minimally twisting tight contact structures on $T^2 \times I$ with $\#\Gamma_{T_i} = 2$ and boundary slopes $s_1 = -\frac{p}{q}$ and $s_0 = -1$. □

Proposition 4.16 *Let* $(T^2 \times I, \xi)$ *be tight with convex boundary, and let* s_0*,* s_1 *be the boundary slopes. Given any* s *between* s_1 *and* s_0*, there exists a convex torus parallel to* $T^2 \times \{pt\}$ *with slope* s*.*

Proof Let $s_0 = -1$, $s_1 = -\frac{p}{q}$, with $p > q$ positive integers. Let T_0, T_1 have Legendrian rulings of slope 0, and take a convex annulus $S^1 \times \{0\} \times I$ with Legendrian boundary which are ruling curves of T_i. There will exist a boundary-parallel dividing curve, and if we attach the corresponding bypass we obtain a slope $-\frac{p'}{q'}$ as in Lemma 4.11. After enough steps we arrive at a slope of -1. Now, by Corollary 4.8, any s between s_1 and s_0 is represented by a convex torus. □

4.6 Lens spaces

4.6.1 Decomposition of lens spaces

Consider the lens space $M = L(p,q)$, with $p > q > 0$. $L(p,q)$ is obtained by gluing two solid tori V_0 and V_1 via $A_0\colon \partial V_0 \to \partial V_1$ given by $\begin{pmatrix} -q & q' \\ p & -p' \end{pmatrix} \in -1\ SL(2,\mathbb{Z})$. Here, $(1,0)^T$ is the meridional direction of V_i, and $(0,1)^T$ is the direction of the core curve C_i of V_i. Note that A_0 is not unique — we can compose A_0 with Dehn twists to the left and the right. However, we will fix a framing for V_i, and assume $pq' - qp' = 1$, $p > p' > 0$ and $q \geq q' > 0$.

Proposition 4.17 *Let Γ_0, Γ_1 be dividing sets on $\partial(S^1 \times D^2) \simeq \mathbb{R}^2/\mathbb{Z}^2$ with $\#\Gamma_i = 2$, $i = 0, 1$, and slopes $s_0 = -1$, $s_1 = -\frac{p'}{q'}$ $(-\infty < -\frac{p'}{q'} \leq -1)$. Assume $-\frac{p}{q}$ has continued fraction representation $(r_0, \cdots, r_k)$ and $-\frac{p'}{q'}$ has continued fraction representation $(r_0, \cdots, r_k + 1)$. Then*

$$
\begin{aligned}
|\pi_0(Tight(L(p,q)))| &\leq |\pi_0(Tight(S^1 \times D^2, \Gamma_1))| && (7)\\
&\leq |\pi_0(Tight(T^2 \times I, \Gamma_0 \cup \Gamma_1))| && (8)\\
&\leq |(r_0 + 1)(r_1 + 1) \cdots (r_{k-1} + 1)(r_k + 1)|. && (9)
\end{aligned}
$$

Proof The proof is very similar to Proposition 4.15. The goal is to thicken the core Legendrian curve isotopic to C_0. Note that the meridional slope of V_0, when mapped to ∂V_1, will have slope $-\frac{p}{q}$ on ∂V_1. Let γ be a Legendrian curve in $M = L(p,q)$, isotopic to C_0, and with twisting number $n < 0$. Recall it is always possible to reduce the twisting number if necessary. Let V_0 to be the standard neighborhood of γ and $V_1 = M \setminus V_0$. Then A_0 maps $(n,1)^T \mapsto (-qn + q', pn - p')^T$, and the corresponding boundary slope on ∂V_1 is $\frac{pn-p'}{-qn+q'}$. Note that $-\frac{p'}{q'}$ is the point on $\partial\mathbb{H}^2$ with an edge in $\mathbb{H}^2$ to $-\frac{p}{q}$ which is closest to -1 on the arc $(-\frac{p}{q}, -1) \subset \partial\mathbb{H}^2$. There exists a convex torus $T \subset V_1$ with boundary slope $-\frac{p'}{q'}$, using the factorization in Proposition 4.15 and Corollary 4.8. Modify V_i so that M is split along T into V_0, V_1. Now $n = 0$ by Proposition 4.3 and the boundary slope of V_1 is $-\frac{p'}{q'}$. Now we count the number of (possible) tight structures on V_1 with $\#\Gamma_{\partial V_1} = 2$ and boundary slope $-\frac{p'}{q'}$. According to Proposition 4.10, an upper bound is given by $|(r_0 + 1)(r_1 + 1) \cdots (r_{k-1} + 1)(r_k + 1)|$, where $(r_0, \cdots, r_k)$ is the continued fraction representation of $-\frac{p}{q}$, and $(r_0, \cdots, r_{k-1}, r_k + 1)$ is the continued fraction representation of $-\frac{p'}{q'}$. □

Hence we have embedded a (candidate) minimally twisting tight contact structure on $T^2 \times I$ as follows:

$$T^2 \times I \hookrightarrow S^1 \times D^2 \hookrightarrow L(p,q).$$

It remains to prove:

Proposition 4.18 *$|\pi_0(Tight(L(p,q)))| \geq |(r_0+1)(r_1+1)\cdots(r_{k-1}+1)(r_k+1)|$, where $-\frac{p}{q}$ has continued fraction representation $(r_0, \cdots, r_k)$. All the tight contact structures in the lower bound are given by Legendrian surgery.*

The proof will be presented in the next section, after a discussion of Legendrian surgeries. Observe that Proposition 4.18 together with Proposition 4.17 prove Theorems 2.1 and 2.3 as well as Part 2(a) of Theorem 2.2.

4.6.2 Legendrian surgeries of S^3

In this section we will realize all of the possible tight structures from the previous sections inside Legendrian surgeries of links of unknots in S^3. Recall the following theorem due to Eliashberg [3].

Theorem 4.19 *Let $K_1, \cdots, K_n$ be mutually disjoint Legendrian knots in the standard tight contact structure ξ on S^3. Then M, obtained from B^3 by $(tb(K_i)-1)$–surgery (usually called* Legendrian surgery*) along all the K_i, $i = 1, \cdots, n$, is holomorphically fillable and therefore tight.*

Observe that for the lens space $L(p,q)$, $p > q > 0$, and the continued fraction expansion $(r_0, r_1, ..., r_k)$ for $-\frac{p}{q}$, we have a linked chain of unknots in S^3 with framings r_0, r_1, ..., r_k (in order along the chain), along which we can do Legendrian surgery to obtain $L(p,q)$. Denote the unknots by $\gamma_0, \cdots, \gamma_k$. See Figure 16. To perform Legendrian surgery, γ_i must have Thurston–Bennequin

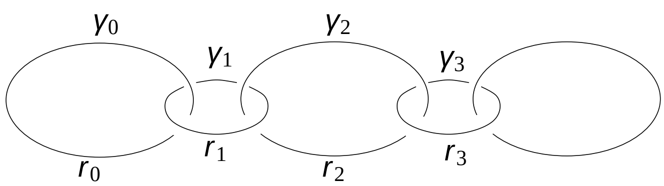


Figure 16: Surgery along link

invariant $tb(\gamma_i) = r_i + 1$. There however are $|r_i + 1|$ choices for the rotation number $r(\gamma_i)$: $r_i + 2, r_i + 4, \cdots, r_i + 2|r_i + 1|$.

Proof of Proposition 4.18. We will take an easy way out by using the following theorem, due to Lisca and Matic [21]:

Theorem 4.20 (Lisca–Matic) *Let X be a smooth 4–manifold with boundary. Suppose J_1, J_2 are two Stein structures with boundary on X. If the induced contact structures ξ_1, ξ_2 on ∂X are isotopic, then $c_1(J_1) = c_1(J_2)$.*

Let X be the Stein surface obtained from B^4 by attaching 2–handles $H_1, \dots, H_k$ corresponding to Legendrian surgeries with coefficients $r_1, \dots, r_k$ along the link in Figure 16. If $c_1(X)$ is the canonical class and h_i is a 2–dimensional class supported on H_i, then $\langle c_1(X), h_i\rangle = r(\gamma_i)$. For the various $r(\gamma_i)$, the $c_1(X)$ are distinct. □

Remark Theorems 2.2 and 2.3 can be thought of as a generalization of Eliashberg and Fraser's classification of Legendrian unknots [10].

4.7 Homotopy classification

In this section we will distinguish the minimally twisting tight structures on $T^2 \times I$ and tight structures on $S^1 \times D^2$ using the relative Euler class. Observe that the proof of Part 2(a) of Theorem 2.2 implies the following lemma:

Lemma 4.21 *Let $(T^2 \times I, \xi)$ be a contact manifold which admits a factorization $T^2 \times I = \cup_{i=0}^{k-1} N_i$, where each $N_i = T^2 \times [\frac{i}{k}, \frac{i+1}{k}]$ is a basic slice, and $s_0 = -1 > s_{\frac{1}{k}} > s_{\frac{2}{k}} > \cdots > s_1 = -\frac{p}{q}$, $p > q > 0$ integers, is obtained by taking the shortest counterclockwise sequence from s_1 to s_0 on $\partial\mathbb{H}^2$ as in Lemma 4.12. Then ξ is tight and minimally twisting. Moreover, such a factorization is unique up to a shuffling within a continued fraction block.*

Proof The fact that ξ is tight and minimally twisting follows from observing that Equation 6 is actually an equality. This means that every gluing of basic layers is tight, provided the slopes $s_{\frac{0}{k}}, s_{\frac{1}{k}}, \dots, s_{\frac{k}{k}}$ are obtained by taking the shortest counterclockwise sequence from s_1 to s_0 on $\partial\mathbb{H}^2$, since the number of contact structures obtained this way is at most the right-hand side of Equation 6. If the factorization was not unique up to a shuffling within a continued fraction block, the number of potential tight contact structures will be less than the actual number of tight contact structures, a contradiction. □

4.7.1 Minimally twisting $T^2 \times I$

Proposition 4.22 *The minimally twisting tight contact structures on $T^2 \times I$ with $\#\Gamma_{T_i} = 2$ and fixed s_0, s_1 can be distinguished by the relative Euler class.*

Proof For convenience, set $s_0 = -1$, $s_1 = -\frac{p}{q}$, $p > q > 0$ integers. Consider the factorization $T^2 \times I = \bigcup_{i=0}^{k-1} N_i$, where each $N_i = T^2 \times [\frac{i}{k}, \frac{i+1}{k}]$ is a *basic slice*, and $s_0 > s_{\frac{1}{k}} > s_{\frac{2}{k}} > \cdots > s_1$, is obtained by taking the shortest counterclockwise sequence from s_1 to s_0 on $\partial\mathbb{H}^2$. Let v_i the shortest integral vector with slope $s_{\frac{i}{k}}$ and negative x–coordinate. Then $PD(e(\xi_{N_i}, s)) = \pm(v_{i+1} - v_i)$, and

$$PD(e(\xi, s)) = \sum_{i=0}^{k-1} \pm(v_{i+1} - v_i), \tag{10}$$

for $(T^2 \times I, \xi)$.

Let A be a horizontal convex annulus with Legendrian boundary, after a perturbation of T_i. We claim that $\langle e(\xi, s), A\rangle$ are distinct for the different ξ. Let $(r_0, r_1, \cdots, r_k)$ be the continued fraction representation of $-\frac{p}{q}$. We will track the change in $\langle e(\xi|_{T^2 \times [0,i+1]}, s), A_i\rangle$, where A_i is the horizontal convex annulus for $N_0 \cup \cdots \cup N_i$, starting from the innermost layer with boundary slope -1, and moving out to $-\frac{p}{q}$. Consider the boundary slope $s_i = -\frac{p_i}{q_i} = \frac{-ar_j + b}{cr_j - d}$, corresponding to the continued fraction representation $(r_0, \cdots, r_j)$, where $(-c, a)$ and $(-d, b)$ form an oriented basis and $a > c \geq 0$, $b \geq d > 0$. Inductively we have $|\langle e(\xi|_{T^2 \times [0,i+1]}, s), A_i\rangle| < p_i$. Then $(r_0, \cdots, r_j - 1, -2)$ corresponds to $s_{i+1} = \frac{(-ar_j + b) + (-a(r_j - 1) + b)}{(cr_j - d) + (c(r_j - 1) - d)}$, and

$$
\begin{aligned}
|\langle e(\xi|_{T^2 \times [0,i+2]}, s), A_{i+1}\rangle - \langle e(\xi|_{T^2 \times [0,i+1]}, s), A_i\rangle| \\
&= (-a(r_j - 1) + b) && (11)\\
&\geq -ar_j + b && (12)\\
&> |\langle e(\xi|_{T^2 \times [0,i+1]}, s), A_i\rangle| && (13)
\end{aligned}
$$

We find that $\langle e(\xi, s), A\rangle$ determines the tight contact structure. □

4.7.2 Solid tori

Let us now give a homotopy classification of the potential tight structures on $S^1 \times D^2$ with $T = \partial(S^1 \times D^2)$, $\#\Gamma_T = 2$, and boundary slope $-\frac{p}{q}$.

Proposition 4.23 *The elements* $[\xi]$ *of* $\pi_0(Tight(S^1 \times D^2;\Gamma))$, $\#\Gamma = 2$, $s = -\frac{p}{q}$ *are distinguished by* $r(\partial D) = \langle e(\xi, s), D\rangle = \#(\text{Components of } R_+) - \#(\text{Components of } R_-)$, *where* D *is a convex meridional disk with Legendrian boundary. Here* r *denotes the rotation number.*

Proof Follows from Proposition 4.22 and noting that every connected component of $D \setminus \Gamma_D$ has Euler characterstic 1. □

4.7.3 Lens spaces

Proposition 4.24 *The homotopy classes of the tight contact structures on* $L(p,q)$ *are all distinct.*

Proof Let us use the same notation as before. In particular, V_0 is the standard neighborhood of the Legendrian core curve C_0 with the largest twisting number, and $V_1 = L(p,q) \setminus V_0$. Every tight contact structure is obtained by Legendrian surgery along γ_i, $i = 1,\dots,k$, in Figure 16. Let V_i' be small standard neighborhoods of $\gamma_i \subset S^3$, with boundary slopes $\frac{1}{r_i+1}$ (use the standard framing on S^3). Also let V_i'' be standard neighborhoods of Legendrian curves with twisting number -1. We remove V_i' from S^3, and glue in V_i'' by mapping $\partial V_i'' \to -\partial(S^3 \setminus V_i')$ via $\begin{pmatrix} -r_i & 1 \\ -1 & 0 \end{pmatrix}$. For $\partial V_i''$, $(1,0)^T$ is the meridian of V_i'' and $(0,1)^T$ the direction of the Legendrian core curve with twist number -1. For $-\partial(S^3 \setminus V_i') = \partial V_i'$, $(1,0)^T$ is the meridian of V_i' and $(0,1)^T$ the longitude for the framing for V_i'. We now identify $V_0'' \simeq V_0$ via a Dehn twist to match up the framings (the Legendrian core curve of minimized twisting number 0 for V_0 must go to the Legendrian core curve of twisting number -1 for V_0''). This then gives rise to a map $\partial V_0 \to \partial V_1 = \partial(S^3 \setminus V_0')$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -r_0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -r_0 & r_0+1 \end{pmatrix}.$$

Here, $(1,0)^T$ and $(0,1)^T$ are the same as before for ∂V_0, and for ∂V_1, $(0,1)^T$ is the meridional direction for $\partial V_0'$ and $(1,0)^T$ is the meridional direction of V_1. Now consider $T^2 \times I = S^3 \setminus (V_0' \cup V_1')$ with boundary slopes $r_0 + 1$ and $\frac{1}{r_1+1}$. There exist $|r_0+1|$ possibilities for $\langle e(\xi, s), A\rangle$ on a horizontal annulus $A = S^1 \times \{0\} \times I$ with Legendrian boundary, depending on the rotation number of γ_0. On the other hand, we have $|r_1+1|$ possibilities for a vertical annulus B, depending on the rotation number of γ_1. Therefore, we find that all $|(r_0+1)(r_1+1)|$ possible tight structures on $T^2 \times I$ with the given boundary slopes are realized.

Next, we transform $T^2 \times I$ via $\begin{pmatrix} 0 & -1 \\ 1 & -r_1 \end{pmatrix}$ to get boundary slopes $\frac{1-r_1(r_0+1)}{-(r_0+1)}$ and -1. Notice that γ_2 is now vertical, with boundary slope $\frac{1}{r_2+1}$. Consider a horizontal annulus A with Legendrian boundary for this (transformed) $T^2 \times I$. It will cut through V_2', and $\langle e(\xi, s), A\rangle$ will uniquely determine the homotopy class of the tight structure by Proposition 4.22. Now take $N = S^3 \setminus (V_0' \cup V_1' \cup V_2') \cup V_1''$, ie, we fill in V_1'' and remove V_2'. Consider the new horizontal annulus A', obtained by removing the meridional disk of V_2' and adding in the meridional disk of V_1''. Then $\langle e(\xi, s), A'\rangle = \langle e(\xi, s), A\rangle$, where the relative Euler class is taken in the respective manifolds. Now, $\langle e(\xi, s), B\rangle$ for the vertical annulus B with Legendrian boundary spanning from $\partial V_1''$ to $\partial V_2'$ corresponds to the rotation number of γ_2. Therefore we see that all $|(r_0+1)(r_1+1)(r_2+1)|$ possible tight structures are represented on N.

Take V_1 with convex boundary and horizontal Legendrian rulings, and perturb the characteristic foliation into a nonsingular Morse–Smale characteristic foliation; also take a meridional disk D for V_1 with Legendrian boundary and perturb into D' with transverse boundary. Etnyre in [6] relates the number of positive elliptic points on D (or the self-linking number $sl(\partial D')$) to the homotopy classes of 2–plane fields on $L(p, q)$, and shows, in particular, that the homotopy classes of tight structures on $L(p, q)$ are distinct if the self-linking numbers are distinct. Our proposition follows from observing that $r(\partial D)$ are distinct for the contact structures with distinct $r(\gamma_i)$ and using $sl(\partial D') = tb(\partial D) \pm r(\partial D)$. □

4.7.4 Gluing

As a consequence of the classification of minimally twisting tight contact structures on $T^2 \times I$ we have the following gluing theorem:

Theorem 4.25 (Gluing $T^2 \times I$) *Let ξ be a contact structure on $T^2 \times [0, n]$, where each $N_i = T^2 \times [i, i+1]$ is a basic slice. Assume all s_i lie on the counterclockwise arc $[s_n, s_0] \subset \partial\mathbb{H}^2$, and $s_n < s_{n-1} < s_{n-2} < \cdots < s_0$. Here we write $a < b$ if b is closer to s_0 than a is on the arc $[s_n, s_0]$. Then ξ is tight if and only if one of the following holds:*

(1) *$s_n, s_{n-1}, \dots, s_0$ is the shortest sequence from s_n to s_0.*

(2) *$s_n, \dots, s_0$ is not the shortest sequence and there is a triple s_{i+1}, s_i, s_{i-1} where s_i is removable from the sequence, ie, there exists an edge from s_{i+1} to s_{i-1} along $\partial\mathbb{H}^2$. $T^2 \times [i-1, i+1]$ is a basic slice and we shorten the sequence by omitting s_i. By repeating this procedure we get to Case (1).*

In Theorem 4.25, we may need to determine when $T^2 \times [i-1, i+1]$ is a basic slice, given that $N_{i-1} = T^2 \times [i-1, i]$ and $N_i = T^2 \times [i, i+1]$ are basic slices. The relative Euler class is useful for this. Let v_i be a shortest integral vector for s_i, chosen consistently so that there exists an element of $SL(2, \mathbb{Z})$ which maps v_{i+1}, v_i, v_{i-1} to $(1, 0), (1, 1), (0, 1)$. For the relative Euler classes of the two component basic slices to add up to a relative Euler class for basic slice we need $PD(e(\xi|_{N_{i-1}}, s)) = v_i - v_{i-1}$ and $PD(e(\xi|_{N_i}, s)) = v_{i+1} - v_i$, or both signs reversed.

5 Tight contact structures on $T^2 \times I$

5.1 Universal tightness

In this section we will precisely determine which minimally twisting tight structures on $T^2 \times I$, $S^1 \times D^2$, and $L(p, q)$ are universally tight. Let Σ be an annulus with a collared Legendrian boundary and negative twisting number on both boundary components. If Γ is the dividing set, then denote the connected components of $\Sigma \setminus \Gamma$ by Σ_i. We call Σ_i a *one-sided component* if it intersects only one boundary component of Σ. Σ_i is *boundary-parallel* if it is a half-disk which intersects a single dividing curve γ and γ is boundary-parallel.

Recall Σ_i is positive if the oriented flow exits from $\partial\Sigma$.

Proposition 5.1 (1) *There are exactly two tight constact structures on $M = T^2 \times I$ with minimal twisting, $\#\Gamma_i = 2$, and boundary slopes $s_1 = -\frac{p}{q}, s_0 = 0$ ($p > q > 0$ positive integers) which are universally tight. They satisfy $PD(e(\xi, s)) = \pm((-q, p) - (-1, 0))$.*

(2) *There are exactly two tight contact structures on $M = S^1 \times D^2$ with $\#\Gamma_{\partial M} = 2$, and boundary slope $s = -\frac{p}{q} < -1$ which are universally tight. (If $s = -1$ there is exactly one.)*

(3) *There are exactly two tight contact structures on $M = L(p, q)$ with $q \neq p-1$ which are universally tight. (If $p = p - 1$ there is exactly one.)*

The two universally tight structures on $T^2 \times I$ are diffeomorphic via $-id$, where id is the identity map on T^2.

Proof (1) Let $A = S^1 \times \{0\} \times I$. Consider its one-sided components A_i (they are all along $S^1 \times \{0\} \times \{1\}$). If $PD(e(\xi, s)) \neq \pm((-q, p) - (-1, 0))$, not all the one-sided components have the same sign. We have two possibilities: (A) there exists a positive one-sided A_1 and negative one-sided $A_2, \dots, A_k$ which lie further toward $S^1 \times \{0\} \times \{0\}$ as in the left-hand side of Figure 17 (or signs reversed), or (B) there is a positive boundary-parallel A_1 as well as a negative boundary-parallel A_2. Let γ_i be the dividing curve on A_i which is 'farthest' from $S^1 \times \{0\} \times \{1\}$ (ie, the half-disk cut off by γ_i contains the other dividing curves which bound A_i).

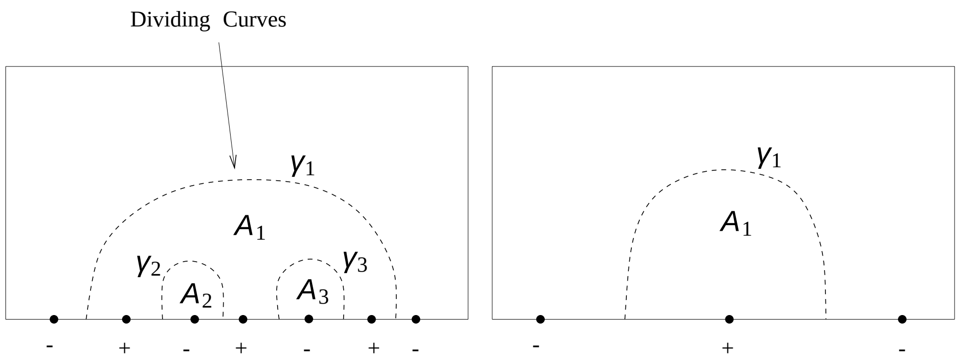


Figure 17: One-sided components

For both cases, pass to the cover $\widetilde{M} = S^1 \times \mathbb{R} \times I$. Let us first consider Case (B). There exist lifts $\widetilde{A} = S^1 \times \{0\} \times I$ and $\widetilde{A}' = S^1 \times \{m\} \times I$, $m \in \mathbb{Z}^+$, for which $N_1 = \widetilde{A} \cup \widetilde{A}' \cup (S^1 \times [0, m] \times \{1\})$, after rounding the edges, has a dividing curve γ which bounds a disk. In fact, a lift of γ_2 on $\widetilde{A}$ will connect up to a lift of γ_1 on $\widetilde{A}'$ for suitably chosen m. The existence of a null-homotopic dividing curve then implies that $\widetilde{M}$ is overtwisted.

For Case (A), take $\widetilde{A}$, $\widetilde{A}'$ as above, as well as lifts $\widetilde{\gamma}_i$ on $\widetilde{A}$ and $\widetilde{\gamma}_i'$ on $\widetilde{A}'$. Pick $m \in \mathbb{Z}^+$ so that $\widetilde{\gamma}_2$ connects the left endpoint of $\widetilde{\gamma}_1'$ to the left endpoint of $\widetilde{\gamma}_2'$, $\widetilde{\gamma}_3$ connects the right endpoint of $\widetilde{\gamma}_2'$ to the left endpoint of $\widetilde{\gamma}_3'$, and so on. What we still lack is a dividing curve connecting the right endpoint of $\widetilde{\gamma}_k'$ to the right endpoint of $\widetilde{\gamma}_1'$. Take $\widetilde{A}'' = S^1 \times \{m'\} \times I$, $m' \in \mathbb{Z}^-$, as well as $N_2 = \widetilde{A}' \cup \widetilde{A}'' \cup (S^1 \times [m', m] \times I)$, after rounding the edges. If we pick m' appropriately, we can make the desired connection along N_2. Now, the dividing curve sits on the branched surface $N_1 \cup N_2$, and there exists an overtwisted disk on this branched surface.

If the tight contact structure on M has a horizontal convex annulus A, all of whose one-side components are boundary-parallel with the same sign, then M can be embedded into, and is universally tight because (T^3, ξ_1) is.

(2) and (3) are left for the reader. □

Note Any tight contact structure ξ on $M = T^2 \times I$ with minimal twisting, $\#\Gamma_i = 2$, and boundary slopes $-\frac{p}{q}$ and -1 factors into continued fraction blocks of the form $N = (\mathbb{R}^2/\mathbb{Z}^2) \times I$ with boundary slopes $s_1 = -m$, $s_0 = -1$, $m \in \mathbb{Z}^+$. Consider the block N. According to Shuffling Lemma, we can arrange the dividing curves on a horizontal annulus A so we have the following: (1) two dividing curves γ_1, γ_2 which go across, (2) the rest are boundary-parallel curves. If there exist both positive and negative half-disk cut off by the boundary-parallel curves, then the double cover $\tilde{N} = (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/2\mathbb{Z}) \times I$ will be overtwisted, using the methods of Proposition 5.1 and the special form of the dividing curves on A. Hence, if any of the blocks of M have mixed signs, then a double cover of M is overtwisted.

5.2 Non-minimal twisting for $T^2 \times I$

We will now finish the proof of Parts 2(b) and 3 of Theorem 2.2. Consider a basic slice $(N_0 = T^2 \times I, \xi^-)$ with boundary slopes $s_1 = 0$, $s_0 = \infty$. If we fix a boundary characteristic foliation compatible with Γ_i, there are 2 possible tight structures on N_0. Let ξ^- be the tight structure on N_0 for which $PD(e(\xi, s)) = (1, -1) \in H_1(T^2; \mathbb{Z})$.

Let $N_{\frac{n}{2}}$ be N_0 rotated counterclockwise by $\frac{n\pi}{2}$, $n \in \mathbb{Z}$. Take $\xi_1^+ = N_0 \cup N_{\frac{1}{2}}$, $\xi_2^+ = N_0 \cup N_{\frac{1}{2}} \cup N_1 \cup N_{\frac{3}{2}}$, ..., $\xi_1^- = N_1 \cup N_{\frac{3}{2}}$, $\xi_2^- = N_1 \cup N_{\frac{3}{2}} \cup N_2 \cup N_{\frac{5}{2}}$, ..., where the T_0 of $N_{\frac{n}{2}}$ is identified with the T_1 of $N_{\frac{(n+1)}{2}}$. These ξ_n can be embedded inside some (T^3, ξ_m), $m \in \mathbb{Z}^+$, and are therefore universally tight.

Lemma 5.2 *A tight* $(M = T^2 \times I, \xi)$ *with* $\#\Gamma_{T_i} = 2$, $i = 0, 1$, *non-minimal twisting, and* $s_1 = s_0 = 0$ *is isotopic to one of the* ξ_n, $n \in \mathbb{Z}^+$.

Proof Let ξ be a tight structure on $T^2 \times I$ with minimal boundary and $s_1 = s_0 = 0$. Assume $r_1 = r_0 = \infty$. Let $B = \{0\} \times S^1 \times I$ be a vertical convex annulus with Legendrian boundary and oriented normal $\frac{\partial}{\partial x}$. Also assume that $\#\Gamma_B$ is minimal among all vertical convex annuli in its isotopy class rel boundary. See Figure 18 for possible configurations of dividing curves on B. If Γ_B does not have any boundary-parallel dividing curves, then $\#\Gamma_B = 2$ and the two dividing curves will go across from T_0 to T_1; rounding the edges, we find that we are in the minimally twisting, nonrotative case. Therefore Γ_B must have boundary-parallel dividing curves. We then cut along B and perform edge-rounding to

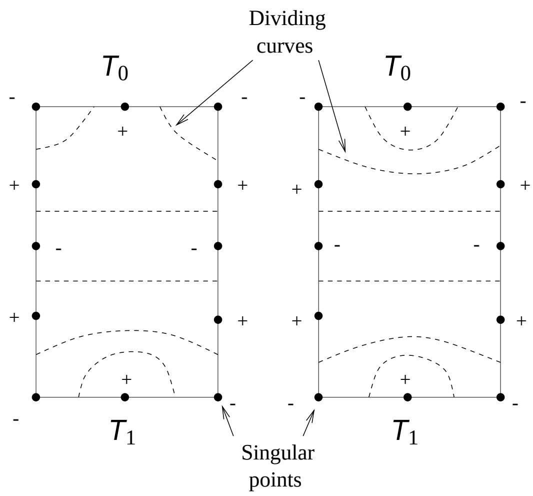


Figure 18: Configurations of dividing curves on B

obtain a solid torus $S^1 \times D^2$ with $2+2i$ vertical dividing curves, where i is the number of closed dividing curves (parallel to the boundary) on B.

Next cut $S^1 \times D^2$ along a meridional disk D after modifying the boundary to be standard with horizontal rulings. The configuration of dividing curves on D is completely determined by the condition that the number of dividing curves on B be minimal. Let γ_0 and γ_1 be the dividing curves on $\partial(S^1 \times D^2)$ which intersect T_0 and T_1 (ie, $\Gamma_{T_0}, \Gamma_{T_1}$ become part of γ_0, γ_1 after edge-rounding). Then all $\gamma \in \Gamma_D$ must separate $D \cap \gamma_1$ from $D \cap \gamma_0$ (hence the dividing curves of D are parallel segments, with only two boundary-parallel components, each containing one $D \cap \gamma_i$ as the half-elliptic point on the interior); otherwise there would exist a bypass which allows for a reduction in the number of dividing curves on B.

Therefore, the tight structure ξ on M depends only on Γ_B, which in turn is determined by the sign of the boundary-parallel component of B along T_1, together with $i+2 = \#\Gamma_B$. If the sign is $+$ ($-$), then $\xi = \xi^+_{i+1}$ (ξ^-_{i+1}). □

Lemma 5.3 *The ξ_n, $n \in \mathbb{Z}^+$ are distinct.*

Proof We distinguish among the four classes ξ^+_{2m-1}, ξ^-_{2m-1}, ξ^+_{2m}, ξ^-_{2m}, $m \in \mathbb{Z}$, according to whether attaching $N_{-\frac{1}{2}}$ to the front preserves tightness (they do for ξ^+_n) and whether attaching N_0 to the back preserves tightness (they do for ξ^+_{2m} and ξ^-_{2m-1}). In the cases when tightness is not preserved, we can find horizontal annuli with a dividing curve bounding a disk.

In each case, m determines the twisting. For example, consider ξ^+_{2m}. If we glue the front and back via the identity map, we obtain the tight contact structure (T^3, ξ_m) described previously, and the ξ_m are distinguished by the minimal twisting number for closed curves isotopic to $S^1 = I/\sim$. (This is due to Kanda [19].) □

Proposition 5.4 *A tight $(M = T^2 \times I, \xi)$ with $\#\Gamma_{T_i} = 2$, $s_0 = 0$, $s_1 = -\frac{p}{q}$, $p > q > 0$, and non-minimal twisting is universally tight. Moreover, there exists a splitting $T^2 \times I = (T^2 \times [0, \frac{2}{3}]) \cup (T^2 \times [\frac{2}{3}, 1])$ where $T_{\frac{2}{3}}$ is convex with $\#\Gamma_{\frac{2}{3}} = 2$, $T^2 \times [\frac{2}{3}, 1]$ is minimally twisting, and $T^2 \times [0, \frac{2}{3}]$ is isotopic to some ξ_n.*

Proof Given such $(T^2 \times I, \xi)$, there exist enough bypasses to factor M into $M_1 = T^2 \times [0, \frac{1}{3}]$, $M_2 = T^2 \times [\frac{1}{3}, \frac{2}{3}]$, and $M_3 = T^3 \times [\frac{2}{3}, 1]$, where $s_0 = 0$, $s_{\frac{1}{3}} = -\frac{p}{q}$, $s_{\frac{2}{3}} = 0$, $s_1 = -\frac{p}{q}$, and M_1, M_3 are minimally twisting. Notice that the tight structure on $M_2 \cup M_3$ is one of the ξ_n as in Lemma 5.3, and is universally tight. By Proposition 5.1, this reduces the possibilities on M_3 to two. A consideration of the signs will reveal that ξ on M is universally tight, and can be split into M_3 with minimal twisting, and $M_1 \cup M_2$ with some ξ_n. There will be two such, according to whether the horizontal bypasses on M_1 are all positive or all negative. This n is unique — this is proved in the same way as Lemma 5.3. □

Proposition 5.5 *The I–twisting β_I of a tight contact structure ξ on $T^2 \times I$ is well-defined and finite. In particular, β_I is independent of the factorization $T^2 \times I = \cup_{k=0}^{l-1}(T^2 \times [\frac{k}{l}, \frac{k+1}{l}])$ into minimally twisting slices.*

Proof The finiteness follows (provided β_I is well-defined) from the factorization in Proposition 5.4 into a minimally twisting $T^2 \times I$ and a slice with some ξ_n, which can be factored into basic slices as above. It remains to prove that β_I remains invariant under subdivisions. But this follows from observing that β_I is well-defined on a minimally twisting $T^2 \times I$, since any factorization will satisfy $s_1 < s_{\frac{l-1}{l}} < s_{\frac{l-2}{l}} < \cdots < s_0$ where $a < b$ if there exists a counterclockwise subarc of $[s_1, s_0] \subset \partial\mathbb{H}^2$ from a to b. □

5.3 Non-minimal boundary

5.3.1 Model for increasing the torus division number

Let T^2 be a convex torus in standard form with $s = \infty$, $r = 0$, and $\#\Gamma = 2n$. Since T^2 is convex, there is a universally tight, I–invariant neighborhood $T^2 \times [-\varepsilon, \varepsilon]$ of $T^2 = T_0$. The horizontal annulus $A = S^1 \times \{0\} \times [-\varepsilon, \varepsilon]$ has parallel dividing curves from $T_{-\varepsilon}$ to T_ε. We will find T' C^0–close to T_0 so that the division number of T' is $n + 1$. Modify T_0 near one of its Legendrian divides to increase $\#\Gamma$ by 2, as in Figure 19. Here, T_0, T' are invariant in the y–direction, and their projections to A are as shown. One of the modifications

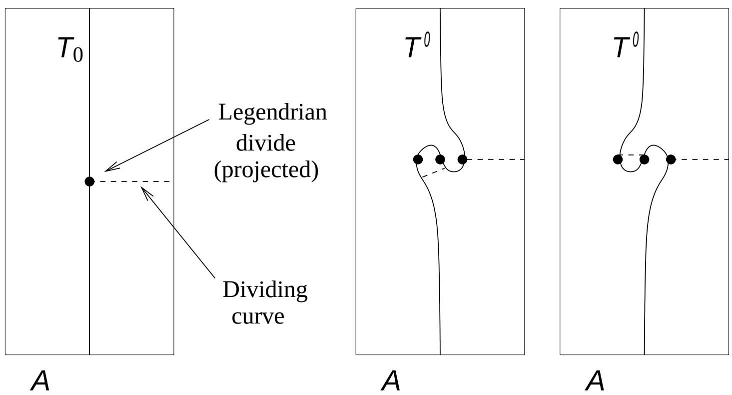


Figure 19: Perturbation to increase the division number

will increase $\#R_+$ by 1, and the other will increase $\#R_-$ by 1. Now perturb T' so it is standard. The region bounded by T' and $T_{-\varepsilon}$ will be universally tight. Note that we can insert a bypass to create any possible configuration for $T^2 \times I$ with no twisting, $n_1 = n + 1$, $n_0 = n$, $s_1 = s_0 = \infty$. Here n_i is the torus division number for T_i.

By iterating this procedure, we find that any $(N = T^2 \times I, \xi)$ with $n_1 \geq n_0$, $s_1 = s_0$, and bypasses on a horizontal annulus only along T_1, can be obtained as a universally tight structure inside a translation invariant one on $T^2 \times I$. Moreover, for any (M, ξ) tight and ∂M a union of tori in standard form, attaching layers of the same type as N is an operation which preserves tightness, since the resulting manifold and contact structure can be found inside (M, ξ) due to convexity.

5.3.2 Template matching

In the next section we shall reduce the problem of classifying nonrotative contact structures to a 2–dimensional problem which we treat first. Given an oriented compact surface Σ with boundary, and a finite subset $\Delta \subset \partial\Sigma$ which we call the *markings*, define $C(\Sigma)$ to be the set of configurations Γ, where each configuration is a set of arcs with endpoints on Δ, and every point of Δ is used as an endpoint exactly once. (In particular, $|\Delta|$ must be even.) If Σ and Σ' are glued along a boundary component C, then there exists a natural map:

$$G\colon C(\Sigma)\times C(\Sigma') \to C(\Sigma\cup_C \Sigma'),$$
$$(\Gamma,\Gamma') \mapsto (\Gamma\cup\Gamma')_0,$$

where $()_0$ means throw away any closed curves. If Σ is an annulus, and if the number of markings on the two boundary components are $m \leq n$, then we define $C_0(\Sigma) \subset C(\Sigma)$ to be the set of configurations where m of the markings on on one boundary component are connected to m markings on the other boundary component via an arc.

Consider annuli $A = S^1 \times [0,1]$ and $B = S^1 \times [1,2]$. Fix markings $p_1,\dots,p_{2m}$ (in cyclical order) each on $S^1\times\{0\}$ and $S^1\times\{2\}$, and markings $q_1,\dots,q_{2n}$ (in cyclical order) on $S^1\times\{1\}$, where $m<n$. If $\mathcal{C} \subset C(A)$, then we define the *dual of* $\mathcal{C}$ to be $\mathcal{C}^* = \{\Gamma \in C(B) | \Gamma' \cup \Gamma = id, \forall \Gamma' \in \mathcal{C}\}$, where $id \in C(A\cup B)$ is the unique element (up to changes in holonomy).

Lemma 5.6 (Reflexive Property) *Consider* $\mathcal{A} = \{\Gamma_0\} \subset C_0(A)$*, for any element* Γ_0*. Then* $(\mathcal{A}^*)^* = \mathcal{A}$.

What this lemma says is that Γ_0 can be detected externally by considering the space of *templates* on B which give $2m$ parallel curves (and no closed homotopically trivial curves) when glued to Γ_0.

Proof By induction on $n-m$. Assume first $n-m=1$. Then Γ_0 will consist of $2m$ curves which cross from $S^1\times\{0\}$ to $S^1\times\{1\}$, and one boundary-parallel curve from q_k to q_{k+1}. $\mathcal{A}^*$ will have two configurations, $\Gamma_0^{\pm}$, both with $2m$ curves crossing from $S^1\times\{1\}$ to $S^1\times\{2\}$. Γ_0^+ (Γ_0^-) will have a boundary-parallel curve from q_{k+1} to q_{k+2} (resp. q_{k-1} to q_k). $(\mathcal{A}^*)^* = \{\Gamma_0^+\}^* \cap \{\Gamma_0^-\}^* = \mathcal{A}$.

Suppose the lemma is true for all Γ_0 with $n-m=l$. Now assume Γ_0 has $n-m=l+1$. We claim any $\Gamma \in (\mathcal{A}^*)^*$ will have a factorization $A = (S^1$

$[0,\frac{1}{2}]) \cup (S^1 \times [\frac{1}{2},1])$, $\Gamma = \Gamma_{l-1} \cup \Gamma_l$, where Γ_l consists of $2(n-1)$ curves which go across and 1 boundary-parallel curve, and Γ_{l-1} consists of $2m$ curves which go across. Moreover, the boundary-parallel curve on Γ_l will coincide with a boundary-parallel curve on Γ_0. This can be seen as follows: Let q_i be the point on $S^1 \times \{1\}$ which is connected to p_i by an arc of Γ_0. Look at two consecutive q_i, q_{i+1}. If $|q_{i+1} - q_i| > 1$, then there exists a boundary-parallel arc of Γ_0 inbetween. Every boundary-parallel arc with endpoints on the interval $[q_i, q_{i+1}]$ can be incorporated into an element of A, except when the endpoints are exactly the endpoints of . If $|q_{i+1} - q_i| = 1$, and $m > 1$, then we take parallel curves from q_i and q_{i+1} to $S^1 \times \{2\}$, and extend to an element of A. Now consider $\Gamma \in (A)$. The discussion above restricts the possible positions of the boundary-parallel arcs of Γ. If there is a boundary parallel arc which is not a boundary-parallel arc for Γ_0 as well, then take q_i, q_{i+1} so that $\partial \quad [q_i, q_{i+1}]$. If $|q_{i+1} - q_i| > 1$, then there is only one position where a closed homotopically trivial curve is not created by summing with some $\Gamma' \in A$. If $|q_{i+1} - q_i| = 1$, and $m > 1$, then there exists $\Gamma' \in A$ such that summing creates a boundary-parallel arc along $S^1 \times \{2\}$. If $|q_{i+1} - q_i| = 1$ and $m = 1$, then the only Γ which is not immediately factorable is one where Γ has endpoints q_i, q_{i+1}, and there are no other boundary-parallel arcs (hence all the other arcs with endpoints on $S^1 \times \{1\}$ are concentric arcs). This can be eliminated by taking Γ' which extends the union of two arcs $_1$ (with endpoints q_{i-1}, q_i) and $_2$ (with endpoints q_{i+1}, q_{i+2}). Therefore, Γ can be factored as claimed above and we are done by induction. □

5.3.3 Factorization

For $(T^2 \times I, \ )$ with convex boundary, we set $T_i = T^2 \times \{i\}$, $\Gamma_i = \Gamma_{T_i}$, $s_i = s(T_i)$ (slopes of the dividing sets), $r_i = r(T_i)$ (slopes of the Legendrian rulings), and $n_i = \frac{1}{2}(\#\Gamma_{T_i})$ (torus division number).

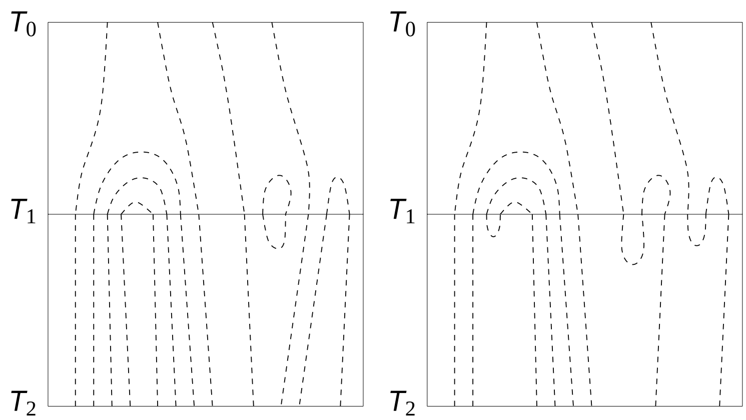
Lemma 5.7 *Let $Tight^0(T^2 \times I;\Gamma)$ be the space of nonrotative tight contact structures with fixed boundary condition $\Gamma = \Gamma_0 \cup \Gamma_1$, $n_0 \ \ n_1$, $s_0 = s_1 = \infty$, and $r_0 = r_1 = 0$, and G is the set of dividing sets Γ_A on an annulus A with a fixed number of endpoints on each component of ∂A, subject to the condition that such that at least two dividing curves go across from T_0 to T_1. There exists a bijection*

$$\Psi: \pi_0(Tight^0(T^2 \times I;\Gamma)) \to G.$$

Proof Let $\in$ Tight$^0(T^2 \times I;\Gamma)$. Let $A_{[0,1]} = S^1 \times \{0\} \times [0,1]$ be a horizontal convex annulus with Legendrian boundary on $T^2 \times [0,1]$. Since is nonrotative,

$\Gamma_{A_{[0,1]}}$ must have at least two dividing curves which go across. Then $\Gamma_{A_{[0,1]}}$ completely determines the isotopy type of ξ, since $(T^2 \times I)\backslash A_{[0,1]}$ is a solid torus which has boundary slope $-\frac{1}{k}$ after rounding (here $2k$ is the number of dividing curves which go across). In particular, a tight contact structure $\xi(\Gamma_{A_{[0,1]}})$ which has dividing set $\Gamma_{A_{[0,1]}}$ is isotopic to an S^1–invariant tight contact structure on $S^1 \times A_{[0,1]}$, all of whose cross sections $\{pt\} \times A$ have the same dividing set $\Gamma_{A_{[0,1]}}$.

It remains to show that $\Gamma_{A_{[0,1]}}$ is uniquely determined by $\xi \in \text{Tight}^0(T^2 \times I;\Gamma)$. Assume first that there exist no boundary-parallel components on $\Gamma_{A_{[0,1]}}$ along T_0. We prove that there cannot exist $A'_{[0,1]}$ with a different $\Gamma_{A'_{[0,1]}}$. The idea is to take advantage of the fact that ξ is S^1–invariant and apply dimensional reduction. We attach various $T^2 \times [1,2]$ with $n_2 < n_1$, $s_2 = s_1 = \infty$, and no twisting, onto $T^2 \times [0,1]$. Equivalently, set $\mathcal{A} = \Gamma_{A_{[0,1]}}$ and consider all possible gluings to $A_{[1,2]}$ with dividing set $\Gamma' \in \mathcal{A}$. The elements Γ' correspond to all the gluings which (1) do not produce an overtwisted disk after gluing and (2) do not produce a bypass along T_0 after gluing. See Figure 20(A) for an illustration. Any gluing which does not produce a dividing curve on $A_{[0,1]} \cup A_{[1,2]}$ bounding a



(A) (B)

Figure 20: Gluing

disk will yield a universally tight structure — this follows from observing that both contact structures are invariant in the S^1-direction and using Giroux's criterion for tightness of I–invariant neighborhoods of ξ. Now apply Lemma 5.6 and obtain that $\Gamma_{A_{[0,1]}}$ is completely determined by the isotopy type of ξ (modulo holonomy). To take show that the holonomy is the same, we use the same technique as in Proposition 4.9.

In the general case we first factor $T^2\times[0,1]=(T^2\times[0,\frac{1}{2}])\cup(T^2\times[\frac{1}{2},1])$ and $A_{[0,1]}=A_{[0,\frac{1}{2}]}\cup A_{[\frac{1}{2},1]}$ so that $\Gamma_{A_{[0,\frac{1}{2}]}}$, $\Gamma_{A_{[\frac{1}{2},1]}}$ have no boundary-parallel components along $S^1\times\{0\}\times\{\frac{1}{2}\}$. Suppose there exists another $A'_{[0,1]}$ with the same boundary. By passing to a large enough cover $S^1\times[-N,N]\times I$ as in Proposition 4.9, we take disjoint copies $\widetilde{A_{[0,1]}}$ and $\widetilde{A'_{[0,1]}}$. If the minimal geometric intersection number $2m=\#(\Gamma_{A_{[0,1]}}\cap(S^1\times\{\frac{1}{2}\})$ is not equal to $2m'=\#(\Gamma_{A'_{[0,1]}}\cap(S^1\times\{\frac{1}{2}\})$ (say $m>m'>0$), then we can cut along $\widetilde{A_{[0,1]}}$, round the edges, and obtain a Legendrian curve of twisting number $-m'$ inside a standard neighborhood of a Legendrian curve of twisting number $-m'$. Therefore, $m=m'$. Next, use Legendrian realization to take curves γ, γ' on $\widetilde{A_{[0,1]}}$, $\widetilde{A'_{[0,1]}}$ with twist number m, take a convex annulus interpolating from γ to γ'. Since there cannot exist any boundary-parallel dividing curves (this would imply a bypass), the dividing curves must connect between γ and γ'. This implies that there exists a splitting $T^2\times[0,1]=(T^2\times[0,\frac{1}{2}])\cup(T^2\times[\frac{1}{2},1])$ which simultaneously splits $A'_{[0,1]}=A_{[0,\frac{1}{2}]}\cup_{\gamma}A_{[\frac{1}{2},1]}$ and $A'_{[0,1]}=A'_{[0,\frac{1}{2}]}\cup_{\gamma'}A'_{[\frac{1}{2},1]}$ where there are no boundary-parallel components along γ, γ', and γ, γ' can be identified without loss of generality. Finally, apply template matching (Lemma 5.6) to both components of the splitting. □

This proves Theorem 2.3, Part (4).

Proposition 5.8 *There exists a unique factorization of a tight contact manifold (M,ξ) with convex $\partial M=\cup_{i=1}^{n}T_i^2$ so that $M=(\cup_i N_i)\cup M_0$, where (1) $N_i\simeq T_i^2\times I$ with identical boundary slopes on both boundary components and no twisting, (2) $T_i^2\times\{1\}=T_i^2$, (3) $T_i^2\times\{0\}$ has the minimum possible torus division number, where the minimum is taken over all $T_i^2\times I\subset(M\setminus\cup_{k=1}^{i-1}N_k)$ satisfying (1) and (2), and (4) $M_0=M\setminus(\cup_i N_i)$.*

Proof Let us show that the first factorization is unique. Suppose $M=N_1\cup M_0=N_1'\cup M_0'$. Then the cross-sectional annuli for N_1 and N_1' must have identical dividing sets, by using the template technique. Therefore, N_1 and N_1' can be matched up using an isotopy. It then remains to show that M_0 and M_0' are isotopic. This follows from attaching a template $T_1^2\times[1,2]$ such that $T_1^2\times[0,2]$ is now an I–invariant neighborhood of T_i^2. N_1 and N_1' are therefore isotopic. □

Proof of Theorem 2.2(1) Consider $M=T^2\times I$ with convex boundary and boundary slopes $-\frac{p}{q}$ and -1. If $-\frac{p}{q}<-1$ or $-\frac{p}{q}=-1$ and $\beta_I>0$, then there exist nonrotative outer layers $T^2 \times [0, \frac{1}{3}]$ and $T^2 \times [\frac{2}{3}, 1]$, where $T_{\frac{i}{3}}$, $i = 0, 1, 2, 3$, are convex and $\#\Gamma_{\frac{1}{3}} = \#\Gamma_{\frac{2}{3}} = 2$. Moreover, Proposition 5.8 indicates that the factorization is unique up to isotopy rel boundary.

If $p = q = 1$ with no twisting, then M will have an inner layer $T^2 \times [\frac{1}{3}, \frac{2}{3}]$ with boundary slopes -1 and torus division number n, together with a horizontal convex annulus, all of whose dividing curves go across. This is the only time for $T^2 \times I$ that the minimal possible torus division number is not necessarily 1. In both cases, Proposition 5.8 allows us to factor M into an essential inner layer, together with universally tight outer layers which can be thought of as decoration. □

6 Remarks and questions

The results in this paper are best thought of as *building blocks* for a more topological (cut-and-paste) theory of tight contact structures on 3–manifolds. Using the techniques presented here, we completely classify tight contact structures on the following classes of 3–manifolds in subsequent papers:

- Torus bundles which fiber over the circle [17].
- Circle bundles which fiber over closed Riemann surfaces [17].
- Some Seifert fibered spaces over S^2, such as the Poincare homology sphere [8].

We also list some classes of 3–manifolds which are more stubborn, for which only (very weak) partial results are known.

- Genus g handlebodies where $g > 1$.
- Circle bundles which fiber over surfaces with boundary (even for the 3–holed sphere).
- Seifert fibered spaces.
- $\Sigma \times I$, where Σ is a closed surface of genus $g > 1$.
- Surface bundles over the circle with pseudo-Anosov monodromy.

Here are a few facts and questions.

Proposition 6.1 *Let M be a genus $g > 1$ handlebody and Γ be a dividing set for ∂M. Then $|\pi_0(Tight(M, \Gamma))|$ is finite.*

This follows from the fact that there exist g compressing disks $D_1, \dots, D_g$ so that $M \setminus (D_1 \cup \cdots \cup D_g)$ is a 3–ball, and that the number of possible dividing sets on each D_i is finite.

Question 1 *Can every tight (M, ξ), where M is a genus g handlebody, be embedded inside a symplectically semi-fillable (M', ξ')?*

By Theorem 2.3, when $g = 1$ every tight (M, ξ) can be embedded inside a lens space $L(p, q)$ with a tight contact structure which is holomorphically fillable. [1]

Question 2 *Can every tight (M, ξ), where $M = S^1 \times \Sigma$ and Σ is a 3–holed sphere, be embedded inside a symplectically semi-fillable (M', ξ')?*

The author believes the answer is no.

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