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On a Subclass of Meromorphic Function with Fixed Second Coefficient Involving Fox-Wright's Generalized Hypergeometic Function

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Abstract

In this paper, we have introduced and studied new subclass of meromorphic function with fixed second coefficient involving Fox-Wright's generalized hypergeometric function. We have obtained coefficient estimates, extreme points, growth and distortion theorems, radii of meromorphically starlikeness and convexity for this new subclass and other interesting properties.

Keywords: *Meromorphic functions, Hadamard product, Fixed second coefficient, coefficient inequalities, radii of meromorphically starlikeness and convexity.*

1 Introduction

Let Σ denote the class of normalized meromorphic functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \tag{1}$$

defined on the punctured unit disk $\Delta^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$. Afunction $f \in \Sigma$ is *meromorphic starlike of order* $\alpha, (0 \leq \alpha < 1)$ if $-\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha (z \in \Delta^* = \Delta\backslash\{0\})$. The class of all such functions is denoted by $\Sigma^*(\alpha)$. A function $f \in \Sigma$ is$meromorphic convix of order \alpha, (0 \leq \alpha < 1)$

If$-\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, (z \in \Delta^* = \Delta\backslash\{0\})$. Let Σ_p be the class of functions $f \in \Sigma\ with\ a_n \geq 0$. The subclass of Σ_p consisting of starlike functions of order α is denoted by $\Sigma_p^*(\alpha)$ and convex functions of order $\alpha$by $\Sigma_p^k(\alpha)$.Various subclasses of Σ have been defined and studied by various authors (see [1, 2, 3, 4,5, 6, 7, 8, 9, 10, 11, 12, 13]).

For functions$f(z)$ given by (1) and $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n$we define the Hadamard product or convolution of f and g by

$(f * g) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n$.

For positive real parameters $\alpha_1, A_1, \ldots, \alpha_l, A_l, \beta_1, B_1, \ldots, \beta_m, B_m (l, m \in \mathbb{N} = \{1,2,3, \ldots\})$ such that

$1 + \sum_{k=1}^{m} B_k - \sum_{k=1}^{l} A_k \geq 0, z \in \{z \in \mathbb{C}: 0 < |z| < 1\}$ the Wright's generalized hypergeometric function

${}_l\Psi_m[(\alpha_1, A_1), \ldots, (\alpha_l, A_l); (\beta_1, B_1), \ldots, (\beta_m, B_m); z] =_l \Psi_m\left[(\alpha_t, A_t)_{1,l}, (\beta_t, B_t)_{1,m}; z\right]$

is defined by

${}_l\Psi_m\left[(\alpha_t, A_t)_{1,l}, (\beta_t, B_t)_{1,m}; z\right] = \sum_{k=0}^{\infty}\left\{\prod_{t=0}^{l} \Gamma(\alpha_t + kA_t)\right\}\left\{\prod_{t=0}^{m} \Gamma(\beta_t + kB_t)\right\}^{-1} \frac{z^k}{k!}$.

If $A_t = 1\ (t = 1,2, \ldots, l) and B_t = 1 (t = 1,2, \ldots, m)$ we have the relationship

$$\Omega_l\Psi_m\left[(\alpha_t, A_t)_{1,l}, (\beta_t, B_t)_{1,m}; z\right] \equiv_l F_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$$

$$= \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \ldots (\alpha_l)_k}{(\beta_1)_k \ldots (\beta_m)_k} \frac{z^k}{k!}.$$

$$(l \leq m+1; l, m \in \mathbb{N}_0 = \mathbb{N} = \{0,1,2,\dots\}; z \in \Delta).$$

This is the generalized hypergeometric function (see [6]). Here (α_n) is thePochammer symbol and

$$\Omega = \left(\prod_{t=0}^{l} \Gamma(\alpha_t)\right)^{-1} \left(\prod_{t=0}^{m} \Gamma(\beta_t)\right).$$

Using the generalized hypergeometric function, we define a linear operator

$$\mathcal{V}\left[(\alpha_t, A_t)_{1,l}, (\beta_t, B_t)_{1,m}\right]: \Sigma_P \to \Sigma_P.$$

By

$$\mathcal{V}\left[(\alpha_t, A_t)_{1,l}, (\beta_t, B_t)_{1,m}\right] f(z) = z^{-1} \Omega_l \Psi_m \left[(\alpha_t, A_t)_{1,l}, (\beta_t, B_t)_{1,m}; z\right] * f(z) \quad (2)$$

For convenience, we denote$\mathcal{V}\left[(\alpha_t, A_t)_{1,l}, (\beta_t, B_t)_{1,m}\right]$by $\mathcal{V}[\alpha_1]$. If f has the form (1) then,

$$\mathcal{V}[\alpha_1] f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \sigma_n(\alpha_1) a_n z^n\ , \quad (3)$$

Where

$$\sigma_n(\alpha_1) = \frac{\Omega \Gamma(\alpha_1 + A_1(n+1)) \dots \Gamma(\alpha_l + A_l(n+1))}{(k+1)! \Gamma(\beta_1 + B_1(n+1)) \dots \Gamma(\beta_l + B_l(n+1))}. \quad (4)$$

Now, we define a new subclass ofΣ_P by using the linear operator $\mathcal{V}[\alpha_1]$as follows.

For $0 \leq \eta < 1$ and $0 \leq \lambda < 1$ we let $\mathcal{N}(\lambda, \eta)$ denote a subclass of Σ_p consisting functions of the form (1) satisfying the condition that

$$\Re\left(\frac{z(\mathcal{V}[\alpha_1] f(z))'}{(\lambda-1)(\mathcal{V}[\alpha_1] f(z)) + \lambda z(\mathcal{V}[\alpha_1] f(z))'}\right) > \eta \quad (5)$$

.
Where $A_t = 1\ (t = 1,2,\dots,l) and B_t = 1 (t = 1,2,\dots,m)$.

Now we prove the coefficient inequality for $f \in \mathcal{N}(\lambda, \eta)$.

2 Coefficients Inequalities

Our first theorem gives a necessary and sufficient condition for a function f to be in the class $\mathcal{N}(\lambda, \eta)$.

Theorem 1: *Let* $f \in \Sigma_P$ *be given by* (1). *Then* $f \in \mathcal{N}(\lambda, \eta)$ *ifand only if*

$$\sum_{n=1}^{\infty}\{n+\eta-\eta\lambda(1+n)\}\sigma_n(\alpha_1)a_n \leq (1-\eta) \tag{6}$$

Proof: At first suppose that $f \in \Sigma_{\mathrm{P}}$ given by (1) is in the class $\mathcal{N}(\lambda,\eta)$, Then by (5) we have

$$\Re\left(\frac{z\big(\mathcal{V}[\alpha_1]f(z)\big)'}{(\lambda-1)\big(\mathcal{V}[\alpha_1]f(z)\big)+\lambda z\big(\mathcal{V}[\alpha_1]f(z)\big)'}\right) > \eta$$

$$\Re\left(\frac{-1+\sum_{n=1}^{\infty} n\sigma_n(\alpha_1)a_n z^{n+1}}{-1+\sum_{n=1}^{\infty}(\lambda-1+\lambda n)\sigma_n(\alpha_1)a_n z^{n+1}}\right) > \eta$$

.

If $z \to 1^-$, we have

$$\Re\left(\frac{-1+\sum_{n=1}^{\infty} n\sigma_n(\alpha_1)a_n}{-1+\sum_{n=1}^{\infty}(\lambda-1+\lambda n)\sigma_n(\alpha_1)a_n}\right) > \eta.$$

This means that (6) holds, conversely suppose that the inequality (6) holds. Let

$$\omega = \frac{z\big(\mathcal{V}[\alpha_1]f(z)\big)'}{(\lambda-1)\big(\mathcal{V}[\alpha_1]f(z)\big)+\lambda z\big(\mathcal{V}[\alpha_1]f(z)\big)'}$$

We have to prove that $\Re\omega > \eta$ It is enough to prove that

$$|\omega-1| < |\omega+1-2\eta|$$

$$\left|\frac{\omega-1}{\omega+1-2\eta}\right| = \left|\frac{z\big(\mathcal{V}[\alpha_1]f(z)\big)'-(\lambda-1)\big(\mathcal{V}[\alpha_1]f(z)\big)+\lambda z\big(\mathcal{V}[\alpha_1]f(z)\big)'}{z\big(\mathcal{V}[\alpha_1]f(z)\big)'+(1-2\eta)(\lambda-1)\big(\mathcal{V}[\alpha_1]f(z)\big)+\lambda z\big(\mathcal{V}[\alpha_1]f(z)\big)'}\right|$$

$$= \left|\frac{\sum_{n=1}^{\infty}(1-\lambda)(n+1)\,\sigma_n(\alpha_1)a_n z^{n+1}}{-2(1-\eta)+\sum_{n=1}^{\infty}[n(1+(1-2\eta)\lambda)+(1-2\eta)(\lambda-1)]\sigma_n(\alpha_1)a_n z^{n+1}}\right|$$

$$\leq \left|\frac{\sum_{n=1}^{\infty}(1-\lambda)(n+1)\,\sigma_n(\alpha_1)a_n}{2(1-\eta)-\sum_{n=1}^{\infty}[n(1+(1-2\eta)\lambda)+(1-2\eta)(\lambda-1)]\sigma_n(\alpha_1)a_n}\right| \leq 1$$

.

Thus we have $f \in \mathcal{N}(\lambda,\eta)$. ■

From (6) we have

$$\sigma_1 a_1 \leq \frac{(1-\eta)}{1+\eta-2\eta\lambda} \tag{7}$$

$$\sigma_1 a_1 \leq \frac{(1-\eta)c}{1+\eta-2\eta\lambda}, 0 < c < 1. \tag{8}$$

Definition 1: *The subclass $\mathcal{N}(\lambda,\eta,c)$of $\mathcal{N}(\lambda,\eta)$ consists of all functions of the form*

$$f(z) = \frac{1}{z} + \frac{(1-\eta)c}{1+\eta-2\eta\lambda} + \sum_{n=2}^{\infty} \sigma_n(\alpha_1) a_n z^n, 0 < c < 1 \tag{9}$$

We now obtain the coefficient estimates, growth and distortion bounds, extreme points, radii of mero-morphically starlikenss and convexity for the class $\mathcal{N}(\lambda,\eta)$ by fixing the second coefficient**.**

We now prove the coefficient inequality.

Theorem 2: *Let f be defined by (9). Then f∈ $\mathcal{N}(\lambda,\eta,c)$ if and only if*

$$\sum_{n=2}^{\infty} \{n+\eta-\eta\lambda(1+n)\}\sigma_n(\alpha_1) a_n \leq (1-\eta)(1-c). \tag{10}$$

The result is sharp.

Proof: *f*∈ $\mathcal{N}(\lambda,\eta,c)$ implies *f*∈ $\mathcal{N}(\lambda,\eta)$. Therefore by (6)

$$(1+\eta-2\eta\lambda)\sigma_1(\alpha_1)a_1 + \sum_{n=2}^{\infty}\{n+\eta-\eta\lambda(1+n)\}\sigma_n(\alpha_1)a_n \leq (1-\eta)$$

Using (8)

$$(1-\eta)c + \sum_{n=2}^{\infty}\{n+\eta-\eta\lambda(1+n)\}\sigma_n(\alpha_1)a_n \leq (1-\eta).$$

From which we get (10). The result is sharp for the function

$$f(z) = \frac{1}{z} + \frac{(1-\eta)c}{1+\eta-2\eta\lambda} z + \frac{(1-\eta)(1-c)}{[n+\eta-\eta\lambda(1+n)]\sigma_n(\alpha_1)} z^n, \ n \geq 2. \blacksquare \tag{11}$$

Corollary 3: *If f defined by (9) is in the class $\mathcal{N}(\lambda,\eta,c)$, then*

$$a_n \leq \frac{(1-\eta)(1-c)}{[n+\eta-\eta\lambda(1+n)]\sigma_n(\alpha_1)}, \ n \geq 2 \tag{12}$$

The result is sharp for the function given by (11).

3 Growth and Distortion Theorems

A growth and distortion property for the function $f \in \mathcal{N}(\lambda,\eta,c)$ is given as follows:

Theorem 4: *If f given by (9) is in the class* $\mathcal{N}(\lambda,\eta,c)$ *then for* $0<|z|=r<1$

$$|f(z)|\geq\frac{1}{r}-\frac{(1-\eta)c}{1+\eta-2\eta\lambda}r-\frac{(1-\eta)(1-c)}{2+\eta-3\eta\lambda}r^2 \tag{13}$$

and

$$|f(z)|\leq\frac{1}{r}+\frac{(1-\eta)c}{1+\eta-2\eta\lambda}r+\frac{(1-\eta)(1-c)}{2+\eta-3\eta\lambda}r^2. \tag{14}$$

The result is sharp for $f(z)=\frac{1}{z}+\frac{(1-\eta)c}{1+\eta-2\eta\lambda}z+\frac{(1-\eta)(1-c)}{2+\eta-3\eta\lambda}z^2$.

Proof: Since $f\in\mathcal{N}(\lambda,\eta,c)$ by Theorem 2

$$\sigma_n(\alpha_1)a_n=\frac{(1-\eta)(1-c)}{[n+\eta-\eta\lambda(1+n)]}. \tag{15}$$

For $0<|z|=r<1$,

$$|f(z)|\leq\frac{1}{|z|}+\frac{(1-\eta)c}{1+\eta-2\eta\lambda}|z|+\sum_{n=2}^{\infty}\sigma_n(\alpha_1)a_n|z|^n$$

$$\leq\frac{1}{r}+\frac{(1-\eta)c}{1+\eta-2\eta\lambda}r+r^2\sum_{n=2}^{\infty}\sigma_n(\alpha_1)a_n$$

$$\leq\frac{1}{r}+\frac{(1-\eta)c}{1+\eta-2\eta\lambda}r+\frac{(1-\eta)(1-c)}{2+\eta-3\eta\lambda}r^2.$$

Similarly,

$$|f(z)|\geq\frac{1}{|z|}-\frac{(1-\eta)c}{1+\eta-2\eta\lambda}|z|-\sum_{n=2}^{\infty}\sigma_n(\alpha_1)a_n|z|^n$$

$$\geq\frac{1}{r}-\frac{(1-\eta)c}{1+\eta-2\eta\lambda}r-r^2\sum_{n=2}^{\infty}\sigma_n(\alpha_1)a_n$$

$$\geq\frac{1}{r}-\frac{(1-\eta)c}{1+\eta-2\eta\lambda}r-\frac{(1-\eta)(1-c)}{2+\eta-3\eta\lambda}r^2. \blacksquare$$

A distortion theorem for the function *f to* be in the class $\mathcal{N}(\lambda,\eta,c)$ is given as follow:

Theorem 5: *If f given by (9) is in the class $\mathcal{N}(\lambda,\eta,c)$ then for $0<|z|=r<1$*

$$|f'(z)| \geq \frac{1}{r^2} - \frac{(1-\eta)c}{1+\eta-2\eta\lambda} - \frac{(1-\eta)(1-c)}{2+\eta-3\eta\lambda} r \tag{16}$$

and

$$|f'(z)| \leq \frac{1}{r^2} + \frac{(1-\eta)c}{1+\eta-2\eta\lambda} + \frac{(1-\eta)(1-c)}{2+\eta-3\eta\lambda} r. \tag{17}$$

The result is sharp for $f(z) = \frac{1}{z} + \frac{(1-\eta)c}{1+\eta-2\eta\lambda} z + \frac{(1-\eta)(1-c)}{2+\eta-3\eta\lambda} z^2.$

4 Extreme Points

In this section, we determine the extreme points for functions in the class $\mathcal{N}(\lambda,\eta,c)$.

Theorem 6: *Let* $f_1(z) = \frac{1}{z} + \frac{(1-\eta)c}{1+\eta-2\eta\lambda} z$, *and*

$$f_n(z) = \frac{1}{z} + \frac{(1-\eta)c}{1+\eta-2\eta\lambda} z + \sum_{n=2}^{\infty} \frac{(1-\eta)(1-c)}{\big(n+\eta-\eta\lambda(1+n)\big)\sigma_n(\alpha_1)} z^n for n \geq 2.$$

Then $f \in \mathcal{N}(\lambda,\eta,c)$ if and only if it can be expressed as

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z), \mu_n \geq 0, \sum_{n=1}^{\infty} \mu_n = 1.$$

Proof: Suppose $f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z), \mu_n \geq 0, \sum_{n=1}^{\infty} \mu_n = 1$. Then

$$f_n(z) = \frac{1}{z} + \frac{(1-\eta)c}{1+\eta-2\eta\lambda} z + \sum_{n=2}^{\infty} \frac{(1-\eta)(1-c)}{\big(n+\eta-\eta\lambda(1+n)\big)\sigma_n(\alpha_1)} \mu_n z^n.$$

Now

$$\sum_{n=2}^{\infty} \frac{(1-\eta)(1-c)\mu_n}{\big(n+\eta-\eta\lambda(1+n)\big)\sigma_n(\alpha_1)} \frac{\big(n+\eta-\eta\lambda(1+n)\big)\sigma_n(\alpha_1)}{(1-\eta)(1-c)} = \sum_{n=2}^{\infty} \mu_n = 1-\mu_1 \leq 1.$$

This implies $f \in \mathcal{N}(\lambda,\eta,c)$. Conversely, let $f \in \mathcal{N}(\lambda,\eta,c)$. Then

$$a_n \leq \frac{(1-\eta)(1-c)}{[n+\eta-\eta\lambda(1+n)]\sigma_n(\alpha_1)a_n}, \qquad n \geq 2.$$

Set $\mu_n = \frac{(n+\eta-\eta\lambda(1+n))\sigma_n(\alpha_1)}{(1-\eta)(1-c)} a_n,\ n \geq 2$ and $\mu_1 = 1 - \sum_{n=2}^{\infty} \mu_n$. Then

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z). \blacksquare$$

Theorem 7: *The class $\mathcal{N}(\lambda, \eta, c)$ is closed under convex combination.*

Proof: Let $f, g \in \mathcal{N}(\lambda, \eta, c)$ such that

$$f(z) = \frac{1}{z} + \frac{(1-\eta)c}{1+\eta-2\eta\lambda} + \sum_{n=2}^{\infty} a_n z^n$$

and

$$g(z) = \frac{1}{z} + \frac{(1-\eta)c}{1+\eta-2\eta\lambda} + \sum_{n=2}^{\infty} b_n z^n.$$

For $0 \leq \mu \leq 1$,let

$$h(z) = \mu f(z) + (1-\mu)g(z).$$

Then

$$h(z) = \frac{1}{z} + \frac{(1-\eta)c}{1+\eta-2\eta\lambda} + \sum_{n=2}^{\infty} [a_n\mu + (1-\mu)b_n] z^n.$$

Therefore

$$\sum_{n=2}^{\infty} \{n + \eta - \eta\lambda(1+n)\}\sigma_n(\alpha_1)[a_n\mu + (1-\mu)b_n] \leq (1-\eta)(1-c).$$

This implies $h(z) = \mu f(z) + (1-\mu)g(z) \in \mathcal{N}(\lambda, \eta, c)$. Hence $\mathcal{N}(\lambda, \eta, c)$ is closed under convex combination. $\blacksquare$

5 Radii of Meromorphically Starlikeness and Convexity

The radii of starlikeness and convexity for the class $\mathcal{N}(\lambda, \eta, c)$ is given by the following theorem:

Theorem 8: *Let $f \in \mathcal{N}(\lambda, \eta, c)$.Then f is meromorphically starlike of order $\delta(0 \leq \delta < 1)$ in the disk $|z| < r_1(\lambda, \eta, c, \delta)$,where $r_1(\lambda, \eta, c, \delta)$ is the largest value for which*

$$\left(\frac{(3-\delta)(1-\eta)c}{1+\eta-2\eta\lambda}\right) r^2 + \left(\frac{(n+2-\delta)(1-\eta)(1-c)}{(n+\eta-\eta\lambda(1+n))}\right) r^{n+1} \leq 1-\delta, n \geq 2. \qquad (18)$$

Proof: It is enough to show that

$$\left|\frac{zf'(z)}{f(z)}+1\right| \le (1-\delta) \tag{19}$$

$$\left|\frac{zf'(z)}{f(z)}+1\right| = \left|\frac{zf'(z)+f(z)}{f(z)}\right| = \left|\frac{\frac{2(1-\eta)cz^2}{1+\eta-2\eta\lambda}+\sum_{n=2}^{\infty}(n+1)\sigma_n(\alpha_1)a_n z^{n+1}}{1+\frac{(1-\eta)cz}{1+\eta-2\eta\lambda}+\sum_{n=2}^{\infty}\sigma_n(\alpha_1)a_n z^{n+1}}\right|$$

Then we write (19) as

$$\left|\frac{2(1-\eta)cz^2}{1+\eta-2\eta\lambda}+\sum_{n=2}^{\infty}(n+1)\sigma_n(\alpha_1)a_n z^{n+1}\right| \le (1-\delta)\left|1+\frac{(1-\eta)cz}{1+\eta-2\eta\lambda}+\sum_{n=2}^{\infty}\sigma_n(\alpha_1)a_n z^{n+1}\right|.$$

That is

$$\frac{(3-\delta)(1-\eta)c}{1+\eta-2\eta\lambda}r^2+\sum_{n=2}^{\infty}(n+2-\delta)a_n r^{n+1} \le 1-\delta.$$

From Theorem 1, we may take

$$a_n = \frac{(1-\eta)(1-c)}{[n+\eta-\eta\lambda(1+n)]\sigma_n(\alpha_1)a_n}\mu_n, \qquad n \ge 2, \mu_n \ge 0, \qquad \sum_{n=2}^{\infty}\mu_n = 1.$$

For each fixed r, we choose the positive integer $n_0 = n_0(r)$for which $\frac{(n+2-\delta)\sigma_n(\alpha_1)}{(n+\eta-\eta\lambda(1+n))}r^{n+1}$ is maximal. This implies

$$\sum_{n=2}^{\infty}(n+2-\delta)a_n r^{n+1} \le \frac{(n_0+2-\delta)(1-\eta)(1-c)}{(n_0+\eta-\eta\lambda(1+n))}r^{n_0+1}.$$

Then f is starlike of order δ in$0 < |z| < r_1(\lambda,\eta,c,\delta)$ If

$$\frac{(3-\delta)(1-\eta)c}{1+\eta-2\eta\lambda}r^2+\frac{(n_0+2-\delta)(1-\eta)(1-c)}{(n_0+\eta-\eta\lambda(1+n))}r^{n_0+1} \le 1-\delta.$$

We have to find the value of $r_0 = r_0(\lambda,\eta,c,\delta)$ and the corresponding integer $n_0(r_0)$ so that

$$\frac{(3-\delta)(1-\eta)c}{1+\eta-2\eta\lambda}r^2+\frac{(n_0+2-\delta)(1-\eta)(1-c)}{(n_0+\eta-\eta\lambda(1+n))}r^{n_0+1} = 1-\delta. \tag{20}$$

It is the value for which $f(z)$ is starlike of order δ in $0 < |z| < r_0$.■

We now state a result for radius of meromorphic convexity for the class $\mathcal{N}(\lambda,\eta,c)$for which the proof is similar to above.

Theorem 9: Let $f\in \mathcal{N}(\lambda,\eta,c)$.*Then f is meromorphically convex of order* $\delta(0\le\delta<1)$*in the disk* $|z|<r_2(\lambda,\eta,c,\delta)$*where*$r_2(\lambda,\eta,c,\delta)$*is the largestvalue for* $n\ge 2$

$$\left(\frac{(3-\delta)(1-\eta)c}{1+\eta-2\eta\lambda}\right)r^2+\left(\frac{n(n+2-\delta)(1-\eta)(1-c)}{\left(n+\eta-\eta\lambda(1+n)\right)}\right)r^{n+1}\le 1-\delta. \qquad (21)$$

6 Integral Operators

In this section, we consider integral operators of functions in the class $\mathcal{N}(\lambda,\eta,c)$.

Theorem 10: *Let* $f\in \mathcal{N}(\lambda,\eta,c)$.*Thenthe integral operator*

$$h(z)=x\int_0^1 u^x f(uz)du\,(0<u\le 1, 0<x<\infty)$$

is in $\mathcal{N}(\lambda,\eta,c)$, *where*

$$\varphi\le\frac{(x+n+1)\left(n+\eta-\eta\lambda(1+n)\right)-xn(1-\eta)(1-c)}{x(1-\eta)(1-c)[1-\lambda(1+n)]+(x+n+1)\left(n+\eta-\eta\lambda(1+n)\right)}.$$

The result is sharp for$f(z)=\frac{1}{z}+\frac{(1-\eta)c}{1+\eta-2\eta\lambda}z+\frac{(1-\eta)(1-c)}{2+\eta-3\eta\lambda}z^2$.

Proof: Let $f\in\mathcal{N}(\lambda,\eta,c)$. Then

$$h(z)=x\int_0^1 u^x f(uz)du=\frac{1}{z}+\frac{(1-\eta)c}{1+\eta-2\eta\lambda}z+\sum_{n=2}^{\infty}\frac{x}{x+n+1}\sigma_n(\alpha_1)a_n z^n.$$

It is sufficient to show that

$$\sum_{n=2}^{\infty}\frac{x[n+\varphi-\varphi\lambda(1+n)]\sigma_n(\alpha_1)}{(x+n+1)(1-\varphi)(1-c)}a_n\le 1. \qquad (22)$$

Since $f\in\mathcal{N}(\lambda,\eta,c)$, we have

$$\sum_{n=2}^{\infty}\frac{\left(n+\eta-\eta\lambda(1+n)\right)\sigma_n(\alpha_1)}{(1-\eta)(1-c)}a_n\le 1.$$

Therefore (22) is true if

$$\frac{x[n+\varphi-\varphi\lambda(1+n)]\sigma_n(\alpha_1)}{(x+n+1)(1-\varphi)(1-c)} \leq \frac{\big(n+\eta-\eta\lambda(1+n)\big)\sigma_n(\alpha_1)}{(1-\eta)(1-c)}.$$

Solving for φ, we have

$$\varphi \leq \frac{(x+n+1)\big(n+\eta-\eta\lambda(1+n)\big)-xn(1-\eta)(1-c)}{x(1-\eta)(1-c)[1-\lambda(1+n)]+(x+n+1)\big(n+\eta-\eta\lambda(1+n)\big)} = \Psi(n).$$

A simple computation will show that $\Psi(n)$ is increasing and $\Psi(n) \geq \Psi(1)$. ■

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