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Basicity of System of Exponents with Complex Coefficients in Generalized Lebesgue Spaces

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Abstract

We consider a perturbed system of exponents $\left\{e^{i\lambda_n t}\right\}_{n\in Z}$*, when the sequence* $\{\lambda_n\}_{n\in Z}$ *has a definite asymptotics. We study basis properties (completeness, minimality, basicity) of this system in Lebesgue space of functions* $L_{p(\cdot)}(-\pi,\pi)$ *with variable summability exponent* $p(\cdot)$*, subject to parameters contained in the asymptotics* $\{\lambda_n\}_{n\in Z}$*.*

Keywords: *System of exponents, Perturbation, Variable exponent, Basis properties.*

1 Introduction

We consider the perturbed system of exponents

$$\left\{e^{i\lambda_n t}\right\}_{n\in Z}, \tag{1}$$

where Z is the set of integrals and $\{\lambda_n\} \subset R$ is some sequence of real numbers for which, it holds the asymptotics

$$\lambda_n = n - \alpha\, sign\, n + O\left(|n|^{-\beta}\right),\ \ n \to \infty, \tag{2}$$

$\lambda, \beta \in R$ are real parameters.

Starting with Paley-Wiener's basic result [1], a great number of papers have been devoted to studying basis properties (completeness, minimality, basicity) in $L_p = L_p(-\pi;\pi)$ of the system of exponents of the form (1). Some results in this direction may be found in R. Young's monograph [2]. The basicity criterion of the system (1) in L_p, $1 < p < +\infty$, in the case when $\lambda_n = n - \alpha sign\, n$, was obtained in the papers [3;4]. The same problem in Lebesgue space $L_{p(\cdot)}$ with variable summability exponent $p(\cdot)$ at $\lambda_n = n - \alpha sign\, n$, $n \in Z$, was studied in the papers [5-7].

The present paper is devoted to studying the basicity of the system (1) in the space $L_{p(\cdot)} \equiv L_{p(\cdot)}(-\pi,\pi)$, when the sequence $\{\lambda_n\}_{n\in Z}$ has asymptotics (2). It should be noted that similar problems were studied in L_p in the papers [8-12]. It also should be noted that partially, these results previously published in [16].

2 Necessary Information

Let $p : [-\pi,\pi] \to [1,+\infty)$ be some Lebesgue measurable function. We denote the class of all measurable on $[-\pi,\pi]$ (with respect to Lebesgue measure) functions by $\mathscr{L}_0$. Accept the denotation

$$I_p(f) \overset{def}{\equiv} \int_{-\pi}^{\pi} |f(t)|^{p(t)}\, dt.$$

Assume

$$\mathscr{L} \equiv \{f \in \mathscr{L}_0 : I_p(f) < +\infty\}.$$

Accept

$$p^- = \inf_{[-\pi,\pi]} vrai\, p(t) \ \ , p^+ = \sup_{[-\pi,\pi]} vrai\, p(t) \ \ .$$

For $p^+ < +\infty$, with respect to ordinary linear operations of addition of functions and multiplication by a scalar, $\mathscr{L}$ turns into a linear space. By the norm

$$\|f\|_{p(\cdot)} \overset{def}{\equiv} \inf\left\{\lambda > 0 : I_p\left(\frac{f}{\lambda}\right) \le 1\right\},$$

$\mathscr{L}$ is a Banach and we denote it by $L_{p(\cdot)}$. Denote

$$WL \overset{def}{\equiv} \{p : p(-\pi) = p(\pi); \exists C > 0, \quad \forall t_1, t_2 \in [-\pi,\pi] : |t_1 - t_2| \le \tfrac{1}{2} \Rightarrow$$
$$\Rightarrow |p(t_1) - p(t_2)| \le \tfrac{C}{-\ln|t_1 - t_2|}\right\}.$$

Everywhere, $q(t)$ denotes a function conjugated to $p(t)$: $\frac{1}{p(t)} + \frac{1}{q(t)} \equiv 1$. It holds the Holder generalized inequality

$$\int_{-\pi}^{\pi} |f(t)\, g(t)|\, dt \le c\left(p^-; p^+\right) \|f\|_{p(\cdot)} \|g\|_{q(\cdot)},$$

where $c(p^-; p^+) = 1 + \frac{1}{p^-} - \frac{1}{p^+}$.

We need some notion and facts from the theory of close bases .

The system $\{x_n\}_{n\in N} \subset X$ *is said to be* ω*-linear independent in Banach space* X *if* $\sum_{n=1}^{\infty} a_n x_n = 0$ *holds only for* $a_n = 0, \forall n \in N$.

While proving the main theorem we use the following fact.

Let X be B-space with the norm $\|\cdot\|$.

The systems $\{x_n\}_{n\in N}$; $\{y_n\}_{n\in N} \subset X$ *are said to be* p*-close if*

$$\sum_{n=1}^{\infty} \|x_n - y_n\|^p < +\infty.$$

The basis $\{x_n\}_{n\in N} \subset X$ *in* X *with the biorthogonal system* $\{x_n^*\}_{n\in N} \subset X^*$ *is called* p*- basis if it holds*

$$\left\|\{x_n^*(x)\}_{n\in N}\right\|_{l_p} \le c\,\|x\|, \quad \forall x \in X,$$

where c *is some constant.*

The following statement is valid

Statement 2.1 *Let* X *be* B*-space with* p*-basis* $\{x_n\}_{n\in N} \subset X$ *and* $\{y_n\}_{n\in N} \subset X$ *be a system* q*-close to* $\{x_n\}_{n\in N}$*. Then the following properties of the system* $\{y_n\}_{n\in N}$ *in* X *are equivalent. i)* $\{y_n\}_{n\in N}$ *is complete; ii)* $\{y_n\}_{n\in N}$ *is minimal; iii)* $\{y_n\}_{n\in N}$ *is* ω*- linear independent; iv)* $\{y_n\}_{n\in N}$ *forms a basis isomorphic to* $\{x_n\}_{n\in N}$*.*

We also need the easily provable

Lemma 2.2 *Let* X *be a Banach space with the basis* $\{x_n\}_{n\in N} \subset X$ *and* $F : X \to X$ *be a Fredholm operator. Then the following properties of the system* $\{y_n = Fx_n\}_{n\in N}$ *in* X *are equivalent:*

1) $\{y_n\}_{n\in N}$ *is complete;*

2) $\{y_n\}_{n\in N}$ *is minimal;*

3) $\{y_n\}_{n\in N}$ *is* ω*-linear independent;*

4) $\{y_n\}_{n\in N}$ *is a basis isomorphic to* $\{x_n\}_{n\in N}$*.*

The validity of the following lemma is easily established.

Lemma 2.3 *Let* X *be* B*-space with the basis* $\{x_n\}_{n\in N}$ *and* $\{y_n\}_{n\in N} \subset X$: $card\{K = \{n : y_n \ne x_n\}\} < +\infty$*. Then the expression* $Fx = \sum_{n=1}^{\infty} x_n^*(x)\, y_n$ *generates Fredholm operator* $F : X \to X$*, where* $\{x_n^*\}_{n\in N} \subset X^*$*.*

Indeed

$$Fx = \sum_{n=1}^{\infty} x_n^* (x) (x_n + y_n - x_n) =$$

$$= \sum_{n=1}^{\infty} x_n^* (x) + \sum_{n \in K} x_n^* (x) (y_n - x_n) = x + Tx = (I + T) x,$$

where

$$Tx = \sum_{n \in K} x_n^* (x) (y_n - x_n) .$$

It is clear that T is a finite-dimensional operator and as a result of that F is Fredholm.

For more detailed information on these results see the papers [2;11-15].

3 Basic Results

Before proving the main result we prove the following theorem that we will use.

Theorem 3.1 *Let $p \in WL$, $p^- > 1$ and $|A|^{\pm 1}$; $|B|^{\pm 1} \in L_\infty (-\pi, \pi)$. Then if the system (1) forms a basis in $L_{p(t)} \equiv L_{p(t)} (-\pi, \pi)$, then it is isomorphic in $L_{p(t)}$ to classic system of exponents $\{e^{int}\}_{n \in Z}$ and the isomorphism is prescribed by the operator*

$$Sf = A (t) \sum_{0}^{\infty} \left(f, e^{inx}\right) e^{int} + B (t) \sum_{1}^{\infty} \left(f, e^{-inx}\right) e^{int}, \tag{3}$$

where

$$(f, g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f (t) \overline{g (t)} dt.$$

Proof. Let the system

$$\left\{A (t) e^{int}; B (t) e^{-i(n+1)t}\right\}_0^{\infty} \tag{4}$$

form a basis in $L_{p(\cdot)} (-\pi, \pi)$. Denote by S an operator determined by formula (3). It is clear that S is a linear operator acting from $L_{p(\cdot)}$ to $L_{p(\cdot)}$. From basicity of the system of exponents $\{e^{int}\}_{-\infty}^{+\infty}$ and from condition (2) it follows that S is a bounded operator in $L_{p(\cdot)}$. Furthermore, from (3) it immediately follows

$$S \left[e^{int}\right] = A (t) e^{int}, S \left[e^{-i(n+1)t}\right] = B (t) e^{-i(n+1)t}, n = \overline{0, \infty}.$$

Now prove that for $\forall g \in L_{p(\cdot)}$ the equation $Sf = g$ has a solution. As the system (4) forms a basis in $L_{p(\cdot)}$, then

$$A(t)\sum_{0}^{\infty} a_n e^{int} + B(t)\sum_{1}^{\infty} b_n e^{-int} = g(t), \tag{5}$$

where a_n, b_n are biorthogonal coefficients. Denote

$$F(\xi) = \sum_{0}^{\infty} a_n \xi^n, G(\xi) = \sum_{0}^{\infty} \bar{b}_{n+1}\xi^n, |\xi| = 1.$$

As the series $\sum_0^\infty a_n e^{int}$ and $\sum_1^\infty b_n e^{-int}$ converge in $L_{p(\cdot)}(-\pi,\pi)$, we get

$$\int_{|\xi|=1} F(\xi)\,\xi^k d\xi = \sum_{0}^{\infty} a_n \int_{|\xi|=1} \xi^{n+k} d\xi = 0, \forall k \geq 0; \tag{6}$$

$$\int_{|\xi|=1} G(\xi)\,\xi^k d\xi = \sum_{0}^{\infty} \bar{b}_{n+1} \int_{|\xi|=1} \xi^{n+k} d\xi = 0, \forall k \geq 0.$$

From condition (6) it follows [6] that there exist the functions $F(z)$ and $G(z)$ from the Hardy class $L_{p(\cdot)}$ such that

$$F^+(\xi) = F(\xi), G^+(\xi) = G(\xi),$$

where $F^+(\xi)$, $G^+(\xi)$ are the boundary values from within a unit circle of the functions $F(z)$ and $G(z)$, respectively. Thus, we get that the conjugate problem

$$A(t)\,F^+\left(e^{it}\right) + B(t)\,e^{-it}\overline{G^+\left(e^{it}\right)} = g(t),$$

has a solution in $L_{p(\cdot)}$ for $\forall g \in L_{p(\cdot)}$.

As is known [6], for $\forall F(z) \in H_{p(\cdot)}$ it holds

$$\lim_{r\to 1-0} \int_{-\pi}^{\pi} \left|F\left(re^{it}\right) - F^+\left(e^{it}\right)\right|^{p(\cdot)} dt = 0.$$

Hence we get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F^+\left(e^{it}\right)\, e^{-int} dt = \begin{cases} a_n\,,\ n \geq 0\,, \\ 0\,,\ n < 0\,, \end{cases} \tag{7}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{G^+\left(e^{it}\right)}\, e^{int} dt = \begin{cases} b_{n+1}\,,\ n \geq 0\,, \\ 0\,,\ n < 0\,, \end{cases}$$

and so

$$F^+\left(e^{it}\right) = \sum_{0}^{\infty} a_n e^{int},$$

$$e^{-it}\overline{G^{+}\left(e^{it}\right)}=\sum_{0}^{\infty}b_{n+1}e^{-i(n+1)t}.$$

Denote

$$\Phi\left(z\right)=z\,G\left(z\right).$$

It is clear that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{\Phi^{+}\left(e^{it}\right)}\,e^{int}dt=\left\{\begin{array}{l} b_n\,,\ n\geq 1\,,\\ 0\,,\ n<1\,,\end{array}\right. \tag{8}$$

and $\Phi\left(z\right)\in H_{p(\cdot)}$.

Introduce the following function

$$f\left(t\right)=F^{+}\left(e^{it}\right)+\overline{\Phi^{+}\left(e^{it}\right)}.$$

From expressions (7), (8) it immediately follows that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}f\left(t\right)e^{-int}dt=\left\{\begin{array}{l} a_n\,,\ n\geq 0\,,\\ b_{-n}\,,\ n<0\,.\end{array}\right.$$

Then from (5) it follows that $Sf=g$. As a result, from the Banach theorem we get that there exists a bounded inverse operator S^{-1}.

The theorem is proved.

By using the above statements we prove the following basic theorem.

Theorem 3.2 ***I.*** *Let it hold asymptotics* (2) *and the inequalities*

$$-\frac{1}{2p\left(\pi\right)}<\alpha<\frac{1}{2q\left(\pi\right)}\ ,\ \beta>\frac{1}{\tilde{p}} \tag{9}$$

be fulfilled, where $\tilde{p}=\min\left\{p^{-};2\right\}$.

Then the following properties of system (1) *in* $L_{p(t)}$ *are equivalent:*
i) system (1) *is complete;*
ii) system (1) *is minimal;*
iii) system (1) *is* ω*-linear independent;*
iv) system (1) *is a basis isomorphic to* $\{e^{int}\}_{n\in N}$*;*
v) $\lambda_i\neq\lambda_j$ *for* $i\neq j$.

II. *Let* $\beta>1$. *Then for* $-\frac{1}{2p(\pi)}\leq\alpha<\frac{1}{2q(\pi)}$ *the following properties are equivalent :*

1. *system* (1) *is complete;*
2. *system* (1) *is minimal;*
3. $\lambda_i\neq\lambda_j$ *for* $i\neq j$.

For $\alpha=-\frac{1}{2p(\pi)}$ *system* (1) *doesn't form a basis in* $L_{p(t)}$.

III. *Let* $\beta>1$ *and* $\lambda_i\neq\lambda_j$ *for* $i\neq j$. *Then system* (1) *is complete and minimal in* $L_{p(t)}$ *only for* $-\frac{1}{2p(\pi)}\leq\alpha<\frac{1}{2q(\pi)}$, *and for* $\alpha<-\frac{1}{2p(\pi)}$ *it is not complete, but minimal; while for* $\alpha\geq\frac{1}{2q(\pi)}$ *is complete but not minimal in* $L_{p(t)}$.

Proof. Consider case I . Assume
$\mu_n=n-\alpha signn, n\in Z$. It is clear that it holds

$$|\lambda_n-\mu_n|\leq cn^{-\beta}, n\in Z\backslash\{0\},$$

where c (and later on also may be different at various places) is a constant independent of n. We have

$$\left|e^{i\lambda_n t}-e^{i\mu_n t}\right|=\left|e^{i(\lambda_n-\mu_n)t}-1\right|\leq\pi\left|\lambda_n-\mu_n\right|+\frac{\pi\left|\lambda_n-\mu_n\right|^2}{2!}+.....=$$

$$=\pi\left|\lambda_n-\mu_n\right|\left(1+\frac{\pi\left|\lambda_n-\mu_n\right|}{2!}+.....\right)\leq mn^{-\beta},$$

where m is some constant. Consequently

$$\sum_n\left\|e^{i\lambda_n t}-e^{i\mu_n t}\right\|_{p(\cdot)}^p\leq c\sum_n n^{-\beta p}<+\infty, \tag{10}$$

if $\beta>\frac{1}{p}$, let $p=p(\pi)$ and$q=q(\pi)$. From condition (9) it follows that the system $\left\{e^{i\mu_n t}\right\}_{n\in Z}$ forms a basis in $L_p\equiv L_p(-\pi,\pi)$ (see e.i. [4]). Then by the results of the paper [11], the system $\left\{e^{i\mu_n t}\right\}_{n\in Z}$ is isomorphic in $L_p(-\pi,\pi)$ to the classic system of exponents $\left\{e^{int}\right\}_{n\in Z}$, and as a result it is clear that the Hausdorff-Young type statement is valid for it, i.e. if $1<p\leq 2$ and $f\in L_p$, the sequence of biorthogonal coefficients $\{f_n\}_{n\in Z}$ of the function f in the system $\left\{e^{i\mu_n t}\right\}_{n\in Z}$ belongs to l_q and it holds

$$\left\|\{f_n\}_{n\in Z}\right\|_{l_q}\leq c\left\|f\right\|_p\ ,\ \forall f\in L_p.$$

Thus, if $p^-\geq p$, there hold the continuous imbeddings

$$\left\|f\right\|_p\leq c_1\left\|f\right\|_{p^-}\leq c_2\left\|f\right\|_{p(\cdot)}\ ,\ \forall f\in L_{p(\cdot)}.$$

Consequently

$$\left\|\{f_n\}_{n\in Z}\right\|_{l_q}\leq c\left\|f\right\|_{p(\cdot)}\ ,\ \forall f\in L_{p(\cdot)}.$$

Having paid attention to relation (10), we get that the systems $\left\{e^{i\lambda_n t}\right\}_{n\in Z}$ and $\left\{e^{i\mu_n t}\right\}_{n\in Z}$ are p-close in $L_{p(\cdot)}$, and $\left\{e^{i\mu_n t}\right\}_{n\in Z}$ forms a q-basis in $L_{p(\cdot)}$. The basicity of the system $\left\{e^{i\mu_n t}\right\}_{n\in Z}$ in $L_{p(\cdot)}$ follows from the result of the papers

[12;13]. Then the validity of part I immediately follows from Statement 2.1. Case II differs from case I by the value $\alpha = -\frac{1}{2p(\pi)}$. In this case, by the results of the papers [5;12;13], the system $\{e^{i\mu_n t}\}_{n\in Z}$ is complete and minimal in $L_{p(\cdot)}$, but doesn't form a basis in it. Then it will ascertain similar to the previous case by using Theorem 3.1. Case III is derived from case II, using periodicity of the system of exponents.

The theorem is proved.

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