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Two New Fixed Point Theorems

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Abstract

The purpose of this paper is to prove two theorems which generalize the corresponding results of Khojesteh et al [1].

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1 Introduction

Let T be a selfmap of a complete metric space. Of the thousands of papers containing fixed point theorems for such a map, the authors of [1] have categorized such theorems into four broad classes: (1) those for which T has a unique fixed point, and for which $\{T^n x\}$converges to the fixed point beginning with any $x \in X$; (2) T has a unique fixed point, but $\{T^n x\}$ need not converge for every $x \in X$; (3) T has more than one fixed point, but $\{T^n x\}$ converges for every $x \in X$; and (4) T may have more than one fixed point and $\{T^n x\}$ does not necessarily converge to a fixed point.

The authors of [1] have proved a new fixed point theorem for a single-valued map in category (3). Specifically, Theorem 1 of [1] reads as follows.

Theorem 1.1 *Let (X, d) be a complete metric space and let T be a selfmap of X satisfying*

$$d(Tx, Ty) \leq \Big(\frac{d(x, Ty) + d(y, Tx)}{d(x, Tx) + d(y, Ty) + 1}\Big) d(x, y) \tag{1}$$

for all $x, y \in X$. Then

(a) T has at least one fixed point $p \in X$;
(b) $\{T^n x\}$ converges to a fixed point for each $x \in X$;
(c) if p and q are two distinct fixed points of T, then $d(p, q) \geq 1/2$.

The second theorem of [1] deals with a multivalued map in category (3), and it will be quoted in the next section.

2 Main Results

The first theorem of this paper extends Theorem 1 to two maps and to a much wider class of maps, while using essentially the same proof technique.

For any map T, the symbol $F(T)$ denotes the set of fixed points of T.

Theorem 2.1 *Let (X, d) be a complete metric space, S, T selfmaps of X satisfying*

$$d(Sx, Ty) \leq N(x, y)m(x, y) \quad \text{for all} \quad x, y \in X, \tag{2}$$

where

$$N(x, y) := [max\{d(x, y), d(x, Sx) + d(y, Ty), d(x, Ty) + d(y, Tx)\}] \div [d(x, Sx) + d(y, Ty) + 1] \tag{3}$$

and

$$m(x, y) := \max\{d(x, y), d(x, Sx), d(y, Ty), [d(x, Ty) + d(y, Sx)]/2\}. \tag{4}$$

Then

(a) S and T have at least one common fixed point $p \in X$.
(b) For n even, $\{(ST)^{n/2}x\}$ and $T(ST)^{n/2}x\}$ converge to a common fixed point for each $x \in X$.
(c) If p and q are distinct common fixed points of S and T, then $d(p, q) \geq 1/2$.

The following Lemma will shorten the proof of Theorem 2.

Lemma 2.2 *Suppose that S and T satisfy the hypotheses of Theorem 2. Then each fixed point of S is a fixed point of T, and conversely.*

Proof of Lemma 1: Let $u \in F(S)$ and suppose that $u \notin F(T)$. From (3),

$$N(u, u) = \frac{max\{0, 0 + d(u, Tu), d(u, Tu) + 0\}}{0 + d(u, Tu) + 1} < 1,$$

and, from (4),

$$m(u,u) = \max\{0, 0, d(u,Tu), [d(u,Tu)+0]/2\} = d(u,Tu).$$

Substituting into (2) gives

$$d(u,Tu) < d(u,Tu),$$

a contradiction. Therefore $u \in F(T)$. Similarly, it can be shown that, if $v \in F(T)$, then $v \in F(S)$.

Proof of Theorem 2: Let $x_0 \in X$ and define $\{x_n\}$ by

$$x_{2n+1} = Sx_{2n}, x_{2n+2} = Tx_{2n+1} \quad \text{for all} \quad n \geq 0. \tag{5}$$

Suppose that there exists a value of n for which $x_{2n+1} = x_{2n+2}$. Then, from (5), $x_{2n+1} = Tx_{2n+1}$ and $x_{2n+1} \in F(T)$. By Lemma 1, $x_{2n+1} \in F(S)$, and (a) is satisfied.

Similarly, if there exists a value of n for which $x_{2n} = x_{2n+1}$, then $x_{2n} \in F(S) \cap F(T)$, and again (a) is satisfied.

Therefore we shall assume that

$$x_n \neq x_{n+1} \quad \text{for all} \quad n \geq 0. \tag{6}$$

From (2),

$$d(x_{2n+1}, x_{2n+2}) = d(Sx_{2n}, Tx_{2n+1}) \leq N(x_{2n}, x_{2n+1})m(x_{2n}, x_{2n+1}). \tag{7}$$

Defining $d_n := d(x_n, x_{n+1})$, from (3),

$$\begin{aligned} N(x_{2n}, x_{2n+1}) &= \frac{\max\{d_{2n}, d_{2n} + d_{2n+1}, d(x_{2n}, x_{2n+2}) + 0\}}{d_{2n} + d_{2n+1} + 1} \\ &= \frac{d_{2n} + d_{2n+1}}{d_{2n} + d_{2n+1} + 1} := \beta_{2n}. \end{aligned} \tag{8}$$

From (4),

$$m(x_{2n}, x_{2n+1}) = \max\{d_{2n}, d_{2n}, d_{2n+1}, [d(x_{2n}, x_{2n+2}) + 0]/2\} = \max\{d_{2n}, d_{2n+1})\}. \tag{9}$$

Substituting (8) and (9) into (7) gives

$$d_{2n+1} \leq \beta_{2n} \max\{d_{2n}, d_{2n+1}\} = \beta_{2n} d_{2n}, \tag{10}$$

since $0 < \beta_{2n} < 1$ and, from (6), $d_{2n+1} \neq 0$.

Similarly, it can be shown that

$$d_{2n} \leq \beta_{2n-1} \max\{d_{2n-1}, d_{2n}\} = \beta_{2n-1} d_{2n-1}. \tag{11}$$

Therefore, from (10) and (11) it follows that

$$d_n \leq \beta_{n-1} \max\{d_{n-1}, d_n\} < d_{n-1} \quad \text{for all} \quad n > 0. \tag{12}$$

Lemma 2.3 *For each* $n > 0, \beta_n < \beta_{n-1}$.

Proof of Lemma 2: From, (8), $\beta_n < \beta_{n-1}$ is equivalent to

$$\frac{d_n + d_{n+1}}{d_n + d_{n+1} + 1} < \frac{d_{n-1} + d_n}{d_{n-1} + d_n + 1}.$$

Clearing of fractions and simplifying gives $d_{n+1} < d_{n-1}$, which follows from (12).

Returning to the proof of Theorem 2, (12) and Lemma 2 imply that

$$d_n \leq \beta_1 d_{n-1} \leq \beta_1^n d_0. \tag{13}$$

For any positive integers m, n with $m > n$, it follows from (13) that

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{i=n}^{m-1} d_i \leq \sum_{i=n}^{m-1} \beta_1^i d_0 \\ &= \beta_1^n d_0 \sum_{j=0}^{m-n-1} \beta_1^j \leq \frac{\beta_1^n}{1-\beta_1} d_0. \end{aligned}$$

Therefore $\{x_n\}$ is Cauchy. Since X is complete, there exists a point $p \in X$ such that $\lim_n x_n = p$.

Using (2) - (4), (8), and the fact that each $\beta_n < \beta_1$, gives

$$\begin{aligned} d(x_{2n+1}, Tp) = d(Sx_{2n}, Tp) < \beta_1 \max\{&d(x_{2n}, p), d(x_{2n}, x_{2n+1}), \\ &d(p, Tp), [d(x_{2n}, Tp) + d(p, x_{2n+1})]/2\}. \end{aligned} \tag{14}$$

Taking the limit of both sides of (14) as $n \to \infty$ one obtains

$$d(p, Tp) \leq \beta_1 d(p, Tp),$$

which implies that $p = Tp$. From Lemma 1, $p \in F(S)$, and (a) is satisfied.

To prove (b), merely observe that, from (5) and the fact that x_0 is arbitrary, we may write $x_{2n+1} = (ST)^{n/2} x$ and $x_{2n+2} = T(ST)^{n/2} x$.

To prove (c), suppose that $p, q \in F(S) \cap F(T)$ with $p \neq q$.

From (3) and (4), $N(p, q) = 2d(p, q)$ and $m(p, q) = d(p, q)$. Thus (2) becomes

$$d(p, q) \leq 2d^2(p, q),$$

which implies (c).

Corollary 2.4 *Let (x, d) be a complete metric space, T a selfmap of X satisfying $(2) - (4)$ with $S = T$.*

Then
(a) T has at least one fixed point.
(b) $\{T^n x\}$ converges to a fixed point of T.
(c) If p and q are distinct fixed points of T, then $d(p, q) \geq 1/2$.

Proof: Set $S = T$ in Theorem 2.

Note that Theorem 1 is a special case of Corollary 1, since (1) is a special case of (2) with $S = T$.

For the balance of this paper we shall need the following notations:

$CB(X) = \{$A : A is a nonempty closed and bounded subset of $X\}$,
$D(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$,
$\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}$,
$H(A, B) = \max\{\sup_{x \in B} D(x, A), \sup_{x \in A} D(x, B))\}$.
For any multivalued mapping, the statement $p \in F(T)$ means that $p \in Tp$. The following is the statement of Theorem 5 of [1].

Theorem 2.5 *Let (X, d) be a complete metric space and let T be a multi-valued mapping from X into $CB(X)$. Let T satisfy the following:*

$$H(Tx, Ty) \leq \Big(\frac{D(x, Ty) + D(y, TX)}{\delta(x, Tx) + \delta(y, Ty + 1}\Big) d(x, y)$$

for all $x, y \in X$. Then T has a fixed point $\dot{x} \in X$.

The following result generalizes Theorem 3.

Theorem 2.6 *Let (X, d) be a complete metric space, $T : X \to CL(X)$ satisfying, for all $x, y \in X$,*

$$H(Sx, Ty) \leq N(x, y) m(x, y), \tag{15}$$

where

$$N(x, y) := [\max\{d(x, y), D(x, Sx) + D(y, Ty), D(x, Ty) + D(y, Sx)] \div [\delta(x, Sx) + \delta(y, Ty) + 1], \tag{16}$$

and

$$m(x, y) = \max\{d(x, y), D(x, Sx), D(y, Ty), [D(x, Ty) + D(y, Sx)]/2\}, \tag{17}$$

Then

(a) S and T have at least one common fixed point $p \in X$.

(b) For n even, $\{(ST)^{n/2}x\}$ and $T(ST)^{n/2}x\}$ converge to a common fixed point for each $x \in X$.

(c) If p and q are distinct common fixed points of S and T, then $d(p,q) \geq 1/2$.

We shall first prove the following Lemma.

Lemma 2.7 *If S and T satisfy the hypotheses of Theorem 4, then every fixed point of S is a fixed point of T, and conversely.*

Proof of Lemma 3: Suppose that p is a fixed point of S. Using (15) and the definition of H,

$$D(p,T) \leq H(p,Tp) \leq H(Sp,Tp) \leq N(p,p)m(p,p).$$

Using (16),

$$\begin{aligned} N(p,p) &= \frac{\max\{d(p,p), D(p,Sp)+D(p,Tp), D(p,Tp)+D(p,Sp)\}}{\delta(p,Sp)+\delta(p,Tp)+1} \\ &\leq \frac{D(p,Tp)}{D(p,Tp)+1} := \beta < 1, \end{aligned}$$

and, from (17),

$$\begin{aligned} m(p,p) &= \max\{d(p,p), D(p,Sp)+D(p,Tp), [d(p,Tp)+d(p,Sp)]/2\} \\ &= D(p,Tp). \end{aligned}$$

Therefore

$$D(p,Tp) \leq \beta D(p,Tp),$$

which implies that p is also a fixed point of T.

In a similar manner it can be shown that, if $p \in Tp$, then $p \in Sp$.

Returning to the proof of Theorem 4, part (a), let $x_0 \in X, x_1 \in Tx_0$.

The following Lemma is an observation of Nadler [2].

Lemma 2.8 *Let $A, B \in CB(X)$, and let $x \in A$. Then, for each $\alpha > 0$, there exists a $y \in B$ such that*

$$d(x,y) \leq H(A,B) + \alpha.$$

Using Lemma 4, for any $0 < h_1 < 1$, choose $x_2 \in Tx_1$ so that

$$\begin{aligned} d(x_1, x_2) &\leq H(Sx_0, Tx_1) + \Big(\frac{1}{h_1} - 1\Big) H(Sx_0, Tx_1) \\ &= \frac{1}{h_1} H(Sx_0, Tx_1). \end{aligned}$$

In a similar manner, for any $0 < h_2 < 1$ choose $x_3 \in Sx_2$ so that

$$d(x_2, x_3) \leq \frac{1}{h_2} H(Sx_2, Tx_1),$$

and, in general, for any $0 < h_{2n} < 1$, choose $x_{2n+1} \in Sx_{2n}$ so that

$$d(x_{2n}, x_{2n+1}) \leq \frac{1}{h_{2n}} H(Sx_{2n}, Tx_{2n-1}), \tag{18}$$

and, for any $0 < h_{2n+1} < 1$, choose $x_{2n+1} \in Tx_{2n+1}$ so that

$$d(x_{2n+1}, x_{2n+2}) \leq \frac{1}{h_{2n+1}} H(Sx_{2n}, Tx_{2n+1}). \tag{19}$$

Without loss of generality we may assume that $H(Sx_{2n}, Tx_{2n-1}) \neq 0$ and $H(Sx_{2n}, Tx_{2n+1}) \neq 0$ for each n. For, if there exist an n such that $(Sx_{2n}, Tx_{2n-1}) = 0$, then $Sx_{2n} = Tx_{2n-1}$, which implies that $x_{2n} \in Sx_{2n}$, since $x_{2n} \in Tx_{2n-1}$, and x_{2n} is a fixed point of S, hence of T by Lemma 3. Similar remarks apply if there exists an n for which $H(Sx_{2n}, Tx_{2n+1}) = 0$. We may also assume that $x_n \neq x_{n+1}$ for each n. For, if there exists an n for which $x_{2n} = x_{2n+1}$, then, since $x_{2n+1} \in Sx_{2n}, x_{2n+1} \in F(S)$, and by Lemma 3, $x_{2n} \in F(T)$. Similarly, $x_{2n+1} = x_{2n+2}$ for any n implies that $x_{2n+1} \in F(T) \cap F(S)$.

The h_n are defined by $h_n = \sqrt{\beta_n}$, where

$$\beta_n := \frac{d_{n-1} + d_n}{d_{n-1} + d_n + 1}. \tag{20}$$

From (16) and (20),

$$\begin{aligned} N(x_{2n}, x_{2n-1}) &= \frac{\max\{d_{2n-1}, D(x_{2n}, Sx_{2n}) + D(x_{2n-1}, Tx_{2n_1}), D(x_{2n}, Tx_{2n-1}) + D(x_{2n-1}, Sx_{2n})\}}{\delta(x_{2n}, Sx_{2n}) + \delta(x_{2n-1}, Tx_{2n-1}) + 1} \\ &\leq \frac{max\{d_{2n-1}, d_{2n} + d_{2n-1}, 0 + d(x_{2n-1}, x_{2n+1})\}}{d_{2n} + d_{2n-1} + 1} \\ &= \frac{d_{2n-1} + d_{2n}}{d_{2n-1} + d_{2n} + 1} = \beta_{2n}. \end{aligned} \tag{21}$$

$$m(x_{2n}, x_{2n-1}) = \max\{d_{2n-1}, D(x_{2n}, Sx_{2n}), D(x_{2n-1}, Tx_{2n-1}), [D(x_{2n}, Tx_{2n-1}) + D(x_{2n-1}, Sx_{2n})]/2\}$$
$$\leq \max\{d_{2n-1}, d_{2n}, d_{2n-1}, [0 + d(x_{2n-1}, x_{2n+1})]/2\}.$$

Therefore

$$m(x_{2n}, x_{2n-1}) \leq \max\{d_{2n-1}, d_{2n}\}. \tag{22}$$

Using (16), (21), and (22) in (19) yields

$$d_{2n} \leq \frac{1}{h_{2n}} H(Sx_{2n}, Tx_{2n-1}) \leq \sqrt{\beta_{2n}} \max\{d_{2n-1}, d_{2n}\}.$$

Since each $x_n \neq x_{n+1}, d_{2n} > 0$, the above inequality implies that

$$d_{2n} \leq \sqrt{\beta_{2n}} d_{2n_1}. \tag{23}$$

A similar computation verifies that

$$d_{2n+1} \leq \sqrt{\beta_{2n+1}} d_{2n}. \tag{24}$$

From inequalities (23) and (24), for all $n > 0$,

$$d_{n+1} \leq \sqrt{\beta_{n+1}} d_n. \tag{25}$$

Therefore $\{d_n\}$ is a monotone decreasing positive sequence, so it has a limit $\ell \geq 0$.

Taking the limit of both sides of (25) as $n \to \infty$, and using (20), it follows that $\ell = 0$.

For any integers $m, n > 0$, using (25) and the triangular inequality,

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} d_k \leq \sum_{k=n}^{m-1} (\beta_{k-1} \cdots \beta_0) d_0 = d_0 \sum_{k=n}^{m-1} a_k,$$

where $a_k := \beta_{k-1} \cdots \beta_0$. Since $\lim_k a_{k+1}/a_k = \lim_k \beta_k = 0$, the series converges, which implies that $\{x_n\}$ is a Cauchy sequence, hence convergent to some point p, since X is complete.

$$D(p, Tp) \leq d(p, x_{2n+1}) + D(x_{2n+1}, Tp) \leq d(p, x_{2n+1}) + H(Sx_{2n}, Tp). \tag{26}$$

Using (16),

$$N(x_{2n}, p) = \max\{d(x_{2n}, p), D(x_{2n}, Sx_{2n}) + D(p, Tp), D(x_{2n}, Tp) + d(p, Sx_{2n})\} \div [\delta(x_{2n}, Sx_{2n}) + \delta(p, Tp) + 1]$$
$$\leq \max\{dx_{2n}, p), d(x_{2n}, x_{2n+1}) + d(p, Tp), d(x_{2n}, Tp) + d(p, x_{2n+1})\} \div [d(x_{2n}, x_{2n+1}) + d(p, Tp) + 1] \tag{27}$$

From (16),

$$\begin{aligned} m(x_{2n},p) = \max\{&d(x_{2n},p), D(x_{2n},Sx_{2n}), D(p,Tp), \\ &[D(x_{2n},Tp)+D(p,Sx_{2n})]/2\} \\ \le \max\{&d(x_{2n},p), d_{2n}, D(p,Tp), \\ &[d(x_{2n},Tp)+d(p,x_{2n+1})]/2\}. \end{aligned} \tag{28}$$

Substituting (27) and (28) into (26), using (15), and taking the limit of both sides as $n \to \infty$, one obtains

$$D(p,Tp) \le) + \frac{d(p,Tp)}{d(p,Tp)+1} D(p,Tp),$$

which implies that $D(p,Tp) = 0$, and hence that $p \in F(T)$. From Lemma 3, $p \in F(S)$.

The proof of part (b) uses the same argument as that of the proof of part (b) in Theorem 2.

(b). Suppose that p and q are distinct common fixed points of S and T. Then

$$\begin{aligned} d(p,q) =& D(p,q) \le D(p,Sp) + D(Sp,Tq) + D(q,Tq) \\ &\le H(Sp,Tq). \end{aligned} \tag{29}$$

Using (16),

$$\begin{aligned} N(p,q) &= \max\left\{\frac{d(p,q), 0, D(p,Tq)+D(q,Sp)}{\delta(p,Sp)+\delta(q,Tq)+1}\right\} \\ &\le \max\left\{\frac{d(p,q), d(p,q)+d(q,p)}{d(p,Sp)+d(q,Tq)+1}\right\} \\ &= 2d(p,q). \end{aligned}$$

Using (17),

$$\begin{aligned} m(p,q) &= \max\{d(p,q), 0, 0, [D(p,Tq)+D(q,Sp)]/2\} \\ &= d(p,q). \end{aligned}$$

Using (15) and substituting it into (29) gives

$$d(p,q) \le 2d^2(p,q),$$

which yields the result.

Theorem 5 of [1] is a special case of Theorem 4 .

On page 3, formula (24) of [1] has an error. The expression

$$\left(1 - \frac{1}{h_1}\right)$$

should read

$$\left(\frac{1}{h_1} - 1\right).$$

Also, formula (27) of [1] is incorrect, since $0 < \beta_n < 1$. However, the remaining argument remains valid with β_n replaced by $\sqrt{\beta_n}$.

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