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ON THE DYNAMICS OF THE GENERALIZED EMDEN–FOWLER EQUATION

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Abstract. We present some recent results dealing with the qualitative behavior of solutions of the quasilinear second order differential equation

$$\left[a(t)\Phi_p(x^{'})\right]^{'} - b(t)\Phi_q(x) = 0 \tag{$*$}$$

where a, b are positive continuous real functions, and $\Phi_j(u) = |u|^{j-2}u$, $j > 1$. Following a classification of solutions, proposed in the linear case in [3], we divide all the solutions of $(*)$ with respect to their asymptotic behavior into two classes. The existence and uniqueness is considered: a topological approach is employed and a fixed point result for operators associated to boundary value problems on a half-line is used. In addition, some asymptotic estimates are presented and the convergence of solutions to zero as $t \to \infty$ is studied.

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1. Introduction

It is well-known (see, e.g., [1, 2]) that the linear equation

$$(a(t)x')' = b(t)x, \tag{L}$$

where the functions a, b are continuous and positive for $t \geq 0$, is nonoscillatory. In addition (see, e.g., [3]), the set of nonidentically zero solutions of (L) may be divided into two classes

$$\mathbb{M}^+ = \Big\{x \text{ solution of (L)}: \quad \exists t_x \geq 0: \quad x(t)x'(t) > 0 \text{ for } t > t_x\Big\},$$
$$\mathbb{M}^- = \Big\{x \text{ solution of (L)}: \quad x(t)x'(t) < 0 \text{ for } t \geq 0\Big\}.$$

Solutions in the class $\mathbb{M}^+$ are eventually either positive increasing or negative decreasing and solutions in the class $\mathbb{M}^-$ are either positive decreasing or negative increasing. Both classes are nonempty: $\mathbb{M}^-$ contains the so-called principal solutions at infinity and solutions with positive initial conditions are in the class $\mathbb{M}^+$ [1, 2]. From Theorems 3–4′ in [3] and by taking into account that the set

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of solutions of (L) is a linear space, it is easy to show that the condition

$$\int_0^\infty \Big(\frac{1}{a(t)} + b(t)\Big)\, dt = \infty \tag{C}$$

is necessary and sufficient in order that for any $(t_0, y_0) \in [0,\infty) \times \mathbb{R}\backslash\{0\}$ there exists a unique solution y of (L) in the class $\mathbb{M}^-$ such that $y(t_0) = y_0$. It is worth to note that the above classes $\mathbb{M}^+$, $\mathbb{M}^-$ have been considered in other context, see e.g. [4–8]. In addition they are strictly related with the concepts of dominate and recessive solutions, see e.g. [9, 10].

In this paper we will discuss how far the above results, stated for (L), can be extended to the nonlinear equation

$$\Big(a(t)\Phi_p(x')\Big)' = b(t)\Phi_q(x) \tag{Q}$$

where p, q are positive constants greater than 1 and $\Phi_m(u) = |u|^{m-2}u$. Equation (Q) arises in the theory of elliptic partial differential equations with p-Laplacian dealing with reaction-diffusion problems (see, e.g., [11, 12]). In this context (Q) is called *quasilinear equation* (see also [9, 10, 13, 14]). Observe that (Q) includes, as a special case, the *half-linear equation* (see, e.g. [15])

$$\Big(a(t)\Phi_p(x')\Big)' = b(t)\Phi_p(x). \tag{H}$$

For $a(t) \equiv 1$ and $p = 2$ (Q) is the Emden–Fowler equation and the properties of its solutions are investigated with a sufficient completeness (see [16–20]). The general case (Q) is considered in [21–26], where the Emden–Fowler system is investigated.

Both equations (Q) and (H) have been studied very deeply in the last years: we refer the reader to the recent papers [9, 10, 15, 27, 28] and references contained therein. We refer also the papers [13, 14] which have been appeared during the preparation of the final version of this paper.

More precisely, the aim of this paper is to study the existence of solutions of (Q) in the classes $\mathbb{M}^+, \mathbb{M}^-$ (which may be defined in a similar way for (Q) and (H)), and to give an extension of condition (C) assuring the uniqueness in $\mathbb{M}^-$. Some asymptotic estimates of certain kinds of solutions of (Q) are also presented.

Our results complement some recent ones given in [21, 22, 24], in which sufficient conditions are established concerning the existence of particular kinds of solutions of (Q). They are closely related to those recently obtained in [9, 10], in which the case $b < 0$ is considered, and in [13, 14], where the case $b > 0$ is studied. On the contrary of [13, 14], where cases $p < q$ and $p > q$ are investigated separately, our approach used in Theorems 2–4 includes also the half-linear equation (H).

A topological approach based on the Tychonov fixed point theorem is used: such a tool seems the most useful approach in order to avoid some difficulties (such as compactness, for instance) which often occur in the study of fixed points for operators associated with boundary conditions in noncompact intervals [29].

We close the introduction with a notation. Denote for $p > 1$, the conjugate number $p^* = \frac{p}{p-1}$, or equivalently, $\frac{1}{p} + \frac{1}{p^*} = 1$.

2. Preliminaries

As usual, by a solution of (Q) we always mean a continuously differentiable function x such that $a\Phi_p(x')$ has a continuous derivative satisfying (Q). Following Kiguradze [18, Definitions 12.4 and 12.5] a nontrivial solution x of (Q) is said to be *singular solution of the first kind of* (Q) if there exists $T < \infty$ such that

$$x(t) \equiv 0 \quad \text{for} \quad t \geq T,$$

and a solution x of (Q) is said to be *singular solution of the second kind of* (Q) if there exists $T < \infty$ such that $\lim_{t \to T-} |x(t)| = \infty$.

For brevity, denote by S_1 (S_2) the set of all singular solutions of the first (second) kind.

Observe that the problem whether $S_1 = \varnothing$ is closely related with the one of the uniqueness of solutions with respect to the initial conditions.

Similarly to the linear case, according to a result of Bihari [27], all eventually nonzero solutions x of (Q) defined on (α_x, ∞), $\alpha_x \geq 0$, may be *a-priori* divided, with respect to their asymptotic behavior, into two classes:

$$\mathbb{M}^+ = \Big\{x \text{ solution of (Q)}: \quad \exists t_x \geq \alpha_x : \quad x(t)x'(t) > 0 \text{ for } t > t_x\Big\},$$
$$\mathbb{M}^- = \big\{x \text{ solution of(Q)}: \quad x(t)x'(t) < 0 \text{ for } t > \alpha_x \geq 0\Big\}.$$

Existence of solutions in the above classes $\mathbb{M}^+$, $\mathbb{M}^-$, as well as the existence of singular solutions depends on which of the following cases occurs: 1) $p < q$, 2) $p > q$, 3) $p = q$.

Results of [21, 22, 24] imply the following statement.

Theorem 1.
(a) *If* $p = q$ *then* $S_1 = S_2 = \varnothing$, $\mathbb{M}^- \neq \varnothing$ *and* $\mathbb{M}^+ \neq \varnothing$.
(b) *If* $p < q$ *then* $S_1 = \varnothing$, $S_2 \neq \varnothing$ *and* $\mathbb{M}^- \neq \varnothing$.
(c) *If* $p > q$ *then* $S_1 \neq \varnothing$, $S_2 = \varnothing$ *and* $\mathbb{M}^+ \neq \varnothing$.

Proof. Equation (Q) can be written as a system of the Emden–Fowler type for the vector $(x, y) = (x, a\Phi_p(x'))$

$$x' = a_1(t)|y|^{\lambda_1} \operatorname{sgn} y, \quad y' = a_2(t)|x|^{\lambda_2} \operatorname{sgn} x, \tag{1}$$

where $\lambda_1 = \frac{1}{p-1}$, $\lambda_2 = q - 1$, $a_1(t) = 1/\Phi_{p^*}(a(t))$, $a_2(t) = b(t)$.

Using results of Mirzov [24, Theorems 9.1, 9.2] on the nonexistence of singular solutions of system (1), we obtain for (Q) that $S_1 = \varnothing$ when $p \leq q$ and $S_2 = \varnothing$ when $p \geq q$.

The existence of singular solutions of (1) was proved by Chanturia [21, Theorems 1, 3]. From there it follows for (Q) that $S_2 \neq \varnothing$ if $p < q$ and $S_1 \neq \varnothing$ if $p > q$.

Now assume $p \leq q$. Using a result of Chanturia [22, Theorem 1], we get that there exists a solution x of (Q) such that

$$|x(0)| > 0, \quad x(t)x'(t) \leq 0 \quad \text{for} \quad t > 0.$$

Since $S_1 = \varnothing$, we obtain $x \in \mathbb{M}^-$.

Now assume $p \geq q$. It remains to show that $\mathbb{M}^+ \neq \varnothing$. Let x be a solution of (Q) satisfying the initial condition $x(0)x'(0) > 0$. Then we can assume that x is defined on $(0, \infty)$ since $S_2 = \varnothing$. Consider the function F_x given by

$$F_x(t) = a(t)\Phi_p(x'(t))x(t).$$

Then $F_x(0) > 0$ and

$$\begin{aligned}\frac{d}{dt}F_x(t) &= \Big[a(t)\Phi_p(x'(t))\Big]' x(t) + a(t)\Phi_q(x'(t))x'(t)\\ &= b(t)\Phi_q(x(t))x(t) + a(t)\Phi_p(x'(t))x'(t) \geq 0.\end{aligned}$$

Thus F_x is a nondecreasing function, which implies $x(t)x'(t) > 0$. Taking into account that, in this case, $S_2 = \varnothing$, the assertion follows. □

The natural problems which arise from Theorem 1 are to study the existence of solutions in the class $\mathbb{M}^-$ when $p > q$ and in the class $\mathbb{M}^+$ when $p < q$. Clearly, solutions in $\mathbb{M}^-$ can have either a nonzero limit or the zero limit as t tends to infinity. In virtue of the above mentioned result of Chanturia [21], the crucial case is concerned with the existence of solutions approaching zero. Similarly solutions in $\mathbb{M}^+$ can be either bounded or unbounded in a neighbourhood of infinity. Also in this case, in virtue of the result of Chanturia [21], the crucial problem is concerned with the existence of unbounded solutions as t tends to infinity. This study will be done in the following sections.

3. Existence in $\mathbb{M}^-$

It is known that (see Theorem 4 from [16] and Corollary 17.3 from [20]) if $p = 2$, $a(t) \equiv 1$, $1 < q < 2$ and

$$\liminf_{t \to \infty} t^2 b(t) > 0,$$

then $\mathbb{M}^- = \varnothing$. Consequently, when $p > q$, the class $\mathbb{M}^-$ can be empty.

Concerning the existence of solutions in the class $\mathbb{M}^-$ approaching zero as t tends to infinity, the following holds:

Theorem 2. *Assume*

$$\int_0^\infty \Phi_{p^*}\left(\frac{1}{a(t)}\right) dt < \infty$$

and

$$\int_0^\infty b(t)\Phi_q\Big[\int_t^\infty \Phi_{p^*}\Big(\frac{1}{a(s)}\Big)\,ds\Big]\,dt<\infty. \tag{2}$$

Then there exists at least one solution x of (Q) *in the class $\mathbb{M}^-$ such that* $\lim_{t\to\infty} x(t)=0$ *and*

$$\lim_{t\to\infty}\frac{x(t)}{\int_t^\infty \Phi_{p^*}(\frac{1}{a(s)})\,ds}=\ell_x,\quad 0<\ell_x<\infty. \tag{3}$$

Proof. Let t_0 be large enough for

$$\int_{t_0}^\infty b(t)\Phi_q\Big[\int_t^\infty \Phi_{p^*}\Big(\frac{1}{a(s)}\Big)\,ds\Big]\,dt<1-\Phi_p\Big(\frac{1}{2}\Big) \tag{4}$$

and denote with $C[t_0,\infty)$ the Fréchet space of all continuous functions on $[t_0,\infty)$ endowed with the topology of uniform convergence on compact subintervals of $[t_0,\infty)$. Let Ω be the nonempty subset of $C[t_0,\infty)$ given by

$$\Omega=\Big\{u\in C[t_0,\infty):\ \ \frac{1}{2}\int_t^\infty \Phi_{p^*}\Big(\frac{1}{a(s)}\Big)\,ds\le u(t)\le\int_t^\infty \Phi_{p^*}\Big(\frac{1}{a(s)}\Big)\,ds\Big\}.$$

Clearly Ω is bounded, closed and convex. Now consider the operator $T:\Omega\to C[t_0,\infty)$ which assigns to any $u\in\Omega$ the continuous function $T(u)=y_u$ given by

$$y_u(t)=T(u)(t)=\int_t^\infty \Phi_{p^*}\Big[\frac{1}{a(s)}\Big(1-\int_{t_0}^s b(\tau)\Phi_q(u(\tau))\,d\tau\Big)\Big]\,ds.$$

In virtue of (4) we have

$$\begin{aligned}T(u)(t)&\ge\int_t^\infty \Phi_{p^*}\Big[\frac{1}{a(s)}\Big(1-\int_{t_0}^\infty b(\tau)\Phi_q\Big[\int_\tau^\infty \Phi_{p^*}\Big(\frac{1}{a(r)}\Big)\,dr\Big]\,d\tau\Big)\Big]\,ds\\&\ge\int_t^\infty \Phi_{p^*}\Big[\frac{1}{a(s)}\Big(\Phi_p\Big(\frac{1}{2}\Big)\Big)\Big]\,ds=\frac{1}{2}\int_t^\infty \Phi_{p^*}\Big[\frac{1}{a(s)}\Big]\,ds.\end{aligned}$$

Because

$$T(u)(t)\le\int_t^\infty \Phi_{p^*}\Big[\frac{1}{a(s)}\Big]\,ds,$$

we obtain

$$T(\Omega)\subseteq\Omega. \tag{5}$$

In order to apply to the operator T the Tychonov fixed point theorem, it is sufficient to prove that T is continuous in Ω and $T(\Omega)$ is relatively compact in $C[t_0,\infty)$.

Let $\{u_j\}$, $j \in N$, be a sequence in Ω which is convergent to v in $C[t_0, \infty)$, $v \in \Omega$. Because it holds for $t \geq t_0$

$$\Phi_{p^*}\left[\frac{1}{a(t)}\Big(1 - \int_{t_0}^{t} b(\tau)\Phi_q(u(\tau))\, d\tau\Big)\right] \leq \Phi_{p^*}\Big(\frac{1}{a(t)}\Big),$$

the Lebesgue's dominated convergence theorem gives

$$\lim_{j\to\infty} \int_t^\infty \Phi_{p^*}\left[\frac{1}{a(s)}\Big(1 - \int_{t_0}^{s} b(\tau)\Phi_q(u_j(\tau))\, d\tau\Big)\right] ds$$
$$= \int_t^\infty \Phi_{p^*}\left[\frac{1}{a(s)}\Big(1 - \int_{t_0}^{s} b(\tau)\Phi_q(v(\tau))\, d\tau\Big)\right] ds$$

that is the continuity of T in Ω.

It remains to prove that $T(\Omega)$ is relatively compact in $C[t_0, \infty)$, that is functions in $T(\Omega)$ are equibounded and equicontinuous on every compact subinterval of $[t_0, \infty)$. The equiboundedness easily follows taking into account that $T(\Omega) \subset \Omega$ and Ω is a bounded subset of $C[t_0, \infty)$. In order to prove the equicontinuity, note that for any $u \in \Omega$ it holds

$$\Big(T(u)(t)\Big)' = -\Phi_{p^*}\left[\frac{1}{a(t)}\Big(1 - \int_{t_0}^{t} b(\tau)\Phi_q(u(\tau))\, d\tau\Big)\right].$$

Hence

$$0 > \Big(T(u)(t)\Big)' \geq -\Phi_{p^*}\Big(\frac{1}{a(t)}\Big)$$

and this shows that the functions in $T(\Omega)$ are equicontinuous on every compact subinterval of $[t_0, \infty)$.

By Tychonov fixed point theorem there exists $x \in \Omega$ such that

$$x(t) = \int_t^\infty \Phi_{p^*}\left[\frac{1}{a(s)}\Big(1 - \int_{t_0}^{s} b(\tau)\Phi_q(x(\tau))\, d\tau\Big)\right] ds.$$

It is easily to show that x is a solution of (Q) in $[t_0, \infty)$, $x'(t)x(t) < 0$ and x tends to zero as $t \to \infty$.

Then x is a solution of (Q) and from (5) it follows for large t

$$\frac{1}{2} \leq \frac{x(t)}{\int_t^\infty \Phi_{p^*}(\frac{1}{a(s)})\, ds} \leq 1.$$

Because x belongs to the class $\mathbb{M}^-$, from (Q) the function $a(t)\Phi_p(x'(t))$ is eventually monotone. Hence the function

$$\Phi_{p^*}\Big(a(t)\Phi_p(x'(t))\Big) = \frac{x'(t)}{\Phi_{p^*}(\frac{1}{a(t)})}$$

is eventually monotone too. The assertion follows applying the l'Hospital rule to the limit in (3). □

4. Existence in $\mathbb{M}^+$

It is known that (see [17, Theorem 4] and [20, Corollary 17.4]) if $p = 2$, $a(t) \equiv 1$, $q > 2$ and

$$\liminf_{t\to\infty} t^q b(t) > 0,$$

then $\mathbb{M}^+ = \varnothing$. Consequently, when $p < q$ the class $\mathbb{M}^+$ can be empty.

Theorem 3 below, which is implied by Kvinikadze's results [25], contains conditions ensuring that $\mathbb{M}^+$ is nonempty.

Following Kiguradze and Kvinikadze [19], an eventually positive solution $x \in \mathbb{M}^+$ is said to be *strongly increasing* if

$$\lim_{t\to\infty} x(t) = \infty, \quad \lim_{t\to\infty} \Big[a(t)\Phi_p(x'(t))\Big] = \infty$$

and an eventually positive solution $x \in \mathbb{M}^+$ is said to be *weakly increasing* if at least one of the limits

$$\lim_{t\to\infty} x(t), \quad \lim_{t\to\infty} \Big[a(t)\Phi_p(x'(t))\Big]$$

exists finitely.

Theorem 3. *Let $p < q$ and*

$$\int_0^\infty \Phi_{p^*}\Big(\frac{1}{a(t)}\Big)\Phi_{p^*}\Big(\int_0^t b(\tau)\,d\tau\Big)\,dt < \infty \tag{6}$$

or

$$\int_0^\infty b(t)\Phi_q\Big[\int_0^t \Phi_{p^*}\Big(\frac{1}{a(\tau)}\Big)\,d\tau\Big]\,dt < \infty. \tag{7}$$

Then $\mathbb{M}^+$ contains a one parametric family of strongly increasing solutions and a one-parameter family of weakly increasing solutions.

Proof. As said above, equation (Q) is equivalent to system (1) where

$$\lambda_1 = \frac{1}{p-1}, \quad \lambda_2 = q-1, \quad a_1(t) = \Phi_{p^*}\Big(\frac{1}{a(t)}\Big), \quad a_2(t) = b(t).$$

Therefore if condition (6) is fulfilled, then

$$\int_0^\infty a_1(t)\Big[\int_0^t a_2(\tau)\,d\tau\Big]^{\lambda_1}\,dt < \infty \tag{8}$$

and if condition (7) is fulfilled, then

$$\int_0^\infty a_2(t)\Big[\int_0^t a_1(\tau)\,d\tau\Big]^{\lambda_2}\,dt < \infty. \tag{9}$$

Kvinikadze [25, Theorem 1 and its Corollary] proved that if condition (8) (condition (9)) is fulfilled, then for sufficiently large $t_0 > 0$, system (1) has a one-parameter family of solutions satisfying the condition

$$x(t) > 0, \quad y(t) > 0 \ \text{ for } \ t \geq t_0, \quad \lim_{t\to\infty} x(t) = \infty, \quad \lim_{t\to\infty} y(t) = \infty,$$

and a one-parameter family of solutions satisfying the condition

$$x(t) > 0, \quad y(t) > 0 \ \text{ for } \ t \geq t_0, \quad \lim_{t\to\infty} x(t) < \infty \quad \Big(\lim_{t\to\infty} y(t) < \infty \Big).$$

Now the assertion follows. □

Remark 1. Let x be a weakly increasing solution of equation (Q). Then it is easy to show the following:

(i) it holds

$$\int_0^\infty \left[\Phi_{p^*}\left(\frac{1}{a(t)} \right) + b(t) \right] dt < \infty$$

if and only if

$$0 < \lim_{t\to\infty} x(t) < \infty, \quad 0 < \lim_{t\to\infty} \Big[a(t)\Phi_p(x'(t)) \Big] < \infty;$$

(ii) if

$$\int_0^\infty b(t)\, dt = \infty,$$

then

$$0 < \lim_{t\to\infty} x(t) < \infty, \quad 0 < \lim_{t\to\infty} \left[\frac{a(t)\Phi_p(x'(t))}{\int_0^t b(\tau)\, d\tau} \right] < \infty;$$

(iii) if

$$\int_0^\infty \Phi_{p^*}\left(\frac{1}{a(t)} \right) dt = \infty,$$

then

$$0 < \lim_{t\to\infty} \frac{x(t)}{\int_0^t \Phi_{p^*}\big(\frac{1}{a(s)}\big)\, ds} < \infty, \quad 0 < \lim_{t\to\infty} \Big[a(t)\Phi_p(x'(t)) \Big] < \infty.$$

To check (i), let $t_0 \geq 0$ be such that $x(t) > 0$, $x'(t) > 0$ for $t \geq t_0$ and assume $\lim_{t\to\infty} x(t) = x_\infty < \infty$. Then for $t > t_0$

$$\Phi_q(x(t_0)) \int_{t_0}^t b(s)\, ds < a(t)\Phi_p(x'(t)) - a(t_0)\Phi_p(x'(t_0) < \Phi_q(x_\infty) \int_{t_0}^t b(s)\, ds$$

and, as $t \to \infty$, the assertion follows. If $\lim_{t\to\infty} a(t)\Phi_p(x'(t)) = x_\infty^1 < \infty$, the argument is similar. The claims (ii) and (iii) follow by the similar way and using the l'Hospital rule.

Remark 2. Let $p \geq q$ and x be a solution of (Q) with the initial conditions

$$x(t_0) > 0, \quad x'(t_0) > 0.$$

Then, as is said above, $x \in \mathbb{M}^+$. By writing (Q) as a system (1) and using [24, Theorem 17.2] we immediately obtain that:
(i) if (6) or (7) holds, then x is weakly increasing;
(ii) if (6) and (7) are violated, then x is strongly increasing.

5. Uniqueness in $\mathbb{M}^-$

When $p \leq q$ Theorem 1 states that the class $\mathbb{M}^-$ is nonempty. More precisely, using two results of Chanturia [21, 22] and Mirzov [24], it is possible to prove that for any $(t_0, x_0) \in [0, \infty) \times \mathbb{R}\backslash\{0\}$ there exists at least one solution x of (Q) in the class $\mathbb{M}^-$ such that $x(t_0) = x_0$. As already claimed, in the linear case, the assumption (C) is necessary and sufficient for uniqueness of such a solution.

In this section we will show that also for (Q) the uniqueness in $\mathbb{M}^-$ (when the initial value of the solution is given) is assured by a natural extension of condition (C). The following lemmas will be useful.

Lemma 1. *If*

$$\int_0^\infty \Phi_{p^*}\left(\frac{1}{a(t)}\right) dt = \infty,$$

then for every solution x of (Q) *in the class $\mathbb{M}^-$ it holds*

$$\lim_{t\to\infty} a(t)\Phi_p(x'(t)) = 0.$$

Proof. Assume there exists a solution x of (Q) in the class $\mathbb{M}^-$ such that

$$\lim_{t\to\infty} a(t)\Phi_p(x'(t)) = \lambda_x < 0.$$

(The case $\lambda_x > 0$ is handled in a similar way.) Hence x is positive and the function $a(\cdot)\Phi_p(x'(\cdot))$ is negative increasing. Consequently,

$$x'(t) < \Phi_{p^*}(\lambda_x)\Phi_{p^*}\left(\frac{1}{a(t)}\right).$$

Integrating on (t_0, t), $0 \leq t_0 < t$, we obtain

$$x(t) < x(t_0) + \Phi_{p^*}(\lambda_x)\int_{t_0}^{t} \Phi_{p^*}\left(\frac{1}{a(s)}\right) ds,$$

which contradicts the fact that x is positive as $t \to \infty$. $\square$

Lemma 2. *If*

$$\int_0^\infty b(\tau)d\tau = \infty,$$

then every solution x of (Q) *in the class $\mathbb{M}^-$ tends to zero as $t \to \infty$.*

Proof. Let x be a solution of (Q) in the class $\mathbb{M}^-$. Without loss of generality we can assume $x(t) > 0$, $x'(t) < 0$ for $t \geq 0$. Integrating (Q) in $(0, t)$, we obtain

$$a(t)\Phi_p(x'(t)) - a(0)\Phi_p(x'(0)) = \int_0^t b(\tau)\Phi_q(x(\tau))\,d\tau. \tag{10}$$

Because the function $a(\cdot)\Phi_p(x'(\cdot))$ is negative increasing, from (10) we get

$$\int_0^\infty b(\tau)\Phi_q(x(\tau))\,d\tau < \infty. \tag{11}$$

Assume $\lim_{t\to\infty} x(t) = \ell_x > 0$. Then $\Phi_q(x(t)) > \Phi_q(\ell_x) > 0$. Thus (11) yields

$$\Phi_q(\ell_x)\int_0^\infty b(\tau)\,d\tau < \infty,$$

which is a contradiction. □

Theorem 4. *A necessary and sufficient condition for the uniqueness of a solution x of* (Q) *in the class $\mathbb{M}^-$ such that $x(t_0) = x_0$ for any $(t_0, x_0) \in [0,\infty) \times \mathbb{R}\backslash\{0\}$, is*

$$\int_0^\infty \left(\Phi_{p^*}\left(\frac{1}{a(t)}\right) + b(t)\right) dt = \infty. \tag{12}$$

Proof. Necessity. Assume (12) does not hold, i.e.,

$$\int_0^\infty b(\tau)\,d\tau < \infty, \quad \int_0^\infty \Phi_{p^*}\left(\frac{1}{a(t)}\right) dt < \infty,$$

and let t_0 be large enough for

$$\left[\Phi_{p^*}\left(\int_{t_0}^\infty b(\tau)\,d\tau\right)\right]\left[\int_{t_0}^\infty \Phi_{p^*}\left(\frac{1}{a(t)}\right) dt\right] < \frac{1}{\Phi_{p^*}(2)}. \tag{13}$$

Consider the solutions x_1, x_2 of (Q) with the initial values

$$x_1(t_0) = 1, \quad x_1'(t_0) = -\Phi_{p^*}\left(\frac{c_1}{a(t_0)}\int_{t_0}^\infty b(\tau)\,d\tau\right), \tag{14}$$

$$x_1(t_0) = 1, \quad x_1'(t_0) = -\Phi_{p^*}\left(\frac{c_2}{a(t_0)}\int_{t_0}^\infty b(\tau)\,d\tau\right), \tag{15}$$

where c_i are positive constants such that $c_1 \neq c_2$ and

$$1 \leq c_i \leq 2. \tag{16}$$

Let us show that $x_i \in \mathbb{M}^-$, $i = 1, 2$. Reasoning as in the final part of the proof of Theorem 2, it is easy to show that solutions x_i are positive decreasing on $[0, t_0]$. In order to prove that $x_i \in \mathbb{M}^-$, it will be sufficient to show that $x_i(t)x_i'(t) < 0$ for any $t \geq t_0$. Clearly solutions x_i are positive decreasing in a right neighbourhood of t_0. Assume there exists $t_i > t_0$ such that $x_i(t_i)x_i'(t_i) = 0$, $x_i(t) > 0$, $x_i'(t) < 0$ for $t_0 \leq t < t_i$. Integrating (Q) on (t_0, t_i) we have

$$a(t_i)\Phi_p(x_i'(t_i)) - a(t_0)\Phi_p(x_i'(t_0)) = \int_{t_0}^{t_i} b(\tau)\Phi_q(x_i(\tau))\,d\tau. \tag{17}$$

If $x_i'(t_i) = 0$, from (14) and (17) we obtain

$$c_i \int_{t_0}^{\infty} b(\tau)\,d\tau = \int_{t_0}^{t_i} b(\tau)\Phi_q(x_i(\tau))\,d\tau \leq \Phi_q(x_i(t_0)) \int_{t_0}^{t_i} b(\tau)\,d\tau = \int_{t_0}^{t_i} b(\tau)\,d\tau$$

which implies

$$\int_{t_i}^{\infty} b(\tau)\,d\tau \leq 0,$$

that is a contradiction. Now suppose $x_i(t_i) = 0$. For $t \in (t_0, t_i)$ from

$$a(t)\Phi_p(x_i'(t)) \geq a(t_0)\Phi_p(x_i'(t_0)) = -c_i \int_{t_0}^{\infty} b(\tau)\,dt,$$

we obtain

$$x_i'(t) \geq -\Phi_{p^*}(c_i)\Big[\Phi_{p^*}\Big(\int_{t_0}^{\infty} b(\tau)\,d\tau\Big)\Big]\,\Big[\Phi_{p^*}\Big(\frac{1}{a(t)}\Big)\Big]$$

or

$$x_i(t_i) - x(t_0) = -1 \geq -\Phi_{p^*}(c_i)\Big[\Phi_{p^*}\Big(\int_{t_0}^{\infty} b(\tau)\,d\tau\Big)\Big]\,\Big[\int_{t_0}^{t_i} \Phi_{p^*}\Big(\frac{1}{a(t)}\Big)\,dt\Big].$$

Thus, by virtue of (16),

$$1 \leq \Phi_{p^*}(2)\Big[\Phi_{p^*}\Big(\int_{t_0}^{\infty} b(\tau)\,d\tau\Big)\Big]\,\Big[\int_{t_0}^{\infty} \Phi_{p^*}\Big(\frac{1}{a(t)}\Big)\,dt\Big]$$

which contradicts (13) and the necessity of (12) is proved.

Sufficiency. Let us show that for any $(t_0, x_0) \in [0, \infty) \times \mathbb{R}\backslash\{0\}$, there exists at most one solution x of (Q) in the class $\mathbb{M}^-$ such that $x(t_0) = x_0$ when the condition

$$\int_0^{\infty} b(t)\,dt = \infty \tag{18}$$

or

$$\int_0^{\infty} \Phi_{p^*}\Big(\frac{1}{a(t)}\Big)\,dt = \infty \tag{19}$$

is satisfied.

Let x, y be two solutions of (Q) in the class $\mathbb{M}^-$ such that $x(t_0) = y(t_0)$, $x'(t_0) > y'(t_0)$. Consider the function d given by

$$d(t) = x(t) - y(t).$$

It holds $d(t_0) = 0, d'(t_0) > 0$. We claim that d does not have positive points of maximum greater than t_0, i.e.,

$$d(t) > 0, \quad d'(t) > 0 \quad \text{for} \quad t > t_0. \tag{20}$$

Assume there exists $t_1 > t_0$ such that $d(t_1) > 0$, $d'(t_1) = 0$ and $d'(t) > 0$ in a suitable left-neighbourhood I of t_1. Without loss of generality suppose that $d(t) > 0$ for $t \in I$. Now consider the function G given by

$$G(t) = a(t)\Big[\Phi_p(x'(t)) - \Phi_p(y'(t))\Big]. \tag{21}$$

Hence $G(t_1) = 0$. Taking into account that Φ_p is increasing and $d'(t) > 0$, we have $G(t) > 0$, $t \in I$. In addition, from

$$G'(t) = b(t)\Big[\Phi_q(x(t)) - \Phi_q(y(t))\Big],$$

we obtain $G'(t) > 0$, $t \in I$, which gives a contradiction, because $G(t_1) = 0$. Hence the function d is increasing.

Now assume (18) holds. Then, by Lemma 2, we get $d(\infty) = 0$, which is a contradiction.

Assume (19) holds. Since $d'(t) > 0$ for $t > t_0$, the function G satisfies $G(t) > 0$, $G'(t) > 0$ for $t > t_0$ and, by Lemma 1, $\lim_{t\to\infty} G(t) = 0$, which is a contradiction. □

As a consequence of the above uniqueness result we get:

Corollary 1. *Consider the equation* (Q) *with* $p \le q$. *For any* $(t_0, x_0) \in [0,\infty) \times \mathbb{R}\backslash\{0\}$, *there exists a unique solution* x *of* (Q) *in the class* $\mathbb{M}^-$ *such that* $x(t_0) = x_0$ *if and only if* (12) *is satisfied.*

Proof. The existence follows from the quoted result of Chanturia [22], and the uniqueness from Theorem 2. □

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