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INTERFACE CRACK PROBLEM FOR ELECTROELASTIC BODY

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Abstract. A two-dimensional crack problem in the Comninou formulation is investigated for a piecewise homogeneous plane. Applying a special integral representation formula for the displacement vector the problem is reduced to a system of singular integral equations. The system is analysed and its solvability is proved using the potential method and the theory of singular integral equations.

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Introduction

The basic ideas about an interface crack as modeled in this paper are due to Comninou. In [1, 2] he considered a mixed interface problem in the isotropic case. The basic idea of a new approach is that the crack faces are always in contact at the very tips of the crack. The Comninou formulation of the interface crack is free of contradiction, and it is possible to satisfy the conditions of a positive gap between the crack faces on the open part of the crack and compressive normal tractions in the contact zones. It is also known that solutions in the Comninou formulation are unique. The first problem considered by Comninou [1] was that of an interface crack in a tension field. She reduced the formulation to a single integral equation and succeeded in extracting the results of physical interest for an extreme mismatch between the two materials by a numerical method. Gautesen and Dundurs [3] have recently shown that the Comninou integral equation can be solved exactly, and that it is possible to give a simple asymptotic formula for the size of contact zones, a mode ll stress intensity factor, and the stress concentration factor for finite but possibly very large tension transmitted by the bond at the tips of the crack. Gautesen and Dundurs [4] show that the governing integral equation for this more general case can also be solved exactly and they derive accurate results covering the whole range of mismatch in the elastic constants of the two materials. It should be noted that no effects of friction in the contact zones are considered. Natroshvili and Zazashvili [5] consider the interface crack problem for anisotropic bodies in the Comninou formulation.

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In the present paper the ideas of [1, 2] are applied and the two-dimensional interface crack problem is investigated for electroelastic bodies. Applying a special integral representation formula for the displacement vector, the problem in question is reduced to a system of singular integral equations with the index equal to -2. The analysis of this system is given.

We consider in $\mathbb{R}^2$ two transversally isotropic electroelastic half-planes $x_3 < 0$ and $x_3 > 0$ filled up by different materials, with the boundary ox_1-axes. In addition, this solid structure contains an interface crack of finite length. The crack interval (cut) is divided into three subintervals. Special type conditions are given on these contact intervals. The crack faces are always in contact at the very tips of the crack.

1. Some Previous Results

Denote by $D^{(1)}$ half-plane $x_3 > 0$ and by $D^{(0)}$ half-plane $x_3 < 0$. We assume that the domains $D^{(j)}$, $j = 0, 1$, are filled up by electroelastic materials with elastic, piezoelastic and dielectric constants $c_{11}^{(j)}$, $c_{44}^{(j)}$, $c_{33}^{(j)}$, $c_{13}^{(j)}$, $e_{13}^{(j)}$, $e_{33}^{(j)}$, $\epsilon_{11}^{(j)}$, $\epsilon_{33}^{(j)}$, $j = 0, 1$. (Sometimes we will omit the superscript (j) when this causes no confusion.) The boundary ($0x_1$-axis) of this piecewise homogeneous plane will be denoted by l and the normal vector on the boundary l by n.

The basic equation of the plane theory of electroelasticity has the form [6]

$$C^{(j)}(\partial x)U^{(j)} = 0, \tag{1}$$

where $C^{(j)}(\partial x) = \|C_{km}^{(j)}\|_{3,3}$,

$$\begin{aligned}
&C_{11}^{(j)} = c_{11}^{(j)} \frac{\partial^2}{\partial x_1^2} + c_{44}^{(j)} \frac{\partial^2}{\partial x_3^2}, \qquad && C_{12}^{(j)} = (c_{13}^{(j)} + c_{44}^{(j)}) \frac{\partial^2}{\partial x_1 \partial x_3},\\
&C_{13}^{(j)} = (e_{13}^{(j)} + e_{15}^{(j)}) \frac{\partial^2}{\partial x_1 \partial x_3}, && C_{22}^{(j)} = c_{44}^{(j)} \frac{\partial^2}{\partial x_1^2} + c_{33}^{(j)} \frac{\partial^2}{\partial x_3^2},\\
&C_{23}^{(j)} = e_{15}^{(j)} \frac{\partial^2}{\partial x_3^2} + e_{33}^{(j)} \frac{\partial^2}{\partial x_3^2}, && C_{33}^{(j)} = -\epsilon_{11}^{(j)} \frac{\partial^2}{\partial x_1^2} - \epsilon_{33}^{(j)} \frac{\partial^2}{\partial x_3^2},\\
&U^{(j)} = U^{(j)}(u_1^{(j)}, u_3^{(j)}, u_4^{(j)})^T,
\end{aligned}$$

$u_1^{(j)}$, $u_3^{(j)}$, are the components of the displacement vector, $u_4^{(j)}$ is an electrostatic potential. Here and in what follows T denotes transposition.

The components of the electromechanical stress vector

$$T^{(j)}(\partial x, n)U^{(j)} = \Big((T^{(j)}U^{(j)})_1, (T^{(j)}U^{(j)})_3, (T^{(j)}U^{(j)})_4\Big),$$

acting on an arc element with the unit normal vector $n(n_1, n_3)$ are calculated by the formula

$$\begin{aligned}
(Tu)_k &= \tau_{k1}n_1 + \tau_{k3}n_3, \quad k = 1, 3, 4,\\
\tau_{13} = \tau_{31} &= c_{44}\Big(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\Big) + e_{15} \frac{\partial u_4}{\partial x_1},
\end{aligned}$$

$$\tau_{33} = c_{13}\,\frac{\partial u_1}{\partial x_1} + c_{33}\,\frac{\partial u_3}{\partial x_3} + e_{33}\,\frac{\partial u_4}{\partial x_3},$$
$$\tau_{43} = -\epsilon_{33}\,\frac{\partial u_4}{\partial x_3} + e_{13}\,\frac{\partial u_1}{\partial x_1} + e_{33}\,\frac{\partial u_3}{\partial x_3}.$$

The Basic Contact Problem. In this section we present an explicit solution of the basic contact problem of electroelasticity for the piecewise homogeneous plane introduced above. This problem can be formulated as follows. Find regular solutions $u^{(j)}$, $j = 0, 1$, to system (1) in the domains D^j, satisfying on the interface l the contact conditions

$$[u^{(1)}]^+ - [u^{(0)}]^- = f(x_1), \quad [T^{(1)}u^{(1)}]^+ - [T^{(0)}u^{(0)}]^- = F(x_1), \qquad (2)$$
$$-\infty < x_1 < +\infty,$$

where $f \in C^{1,\alpha}(l)$ and $F \in C^{\alpha}(l)$ are given functions with the following asymptotics at infinity:

$$f(x_1) = c + O(|x_1|^{-\epsilon}), \quad F(x_1) = O(|x_1|^{-1-\eta}), \quad \eta > 0, \ \ \epsilon > 0.$$

Here c is some real constant. In addition, we assume that $\int\limits_{-\infty}^{+\infty} F(x)dx = 0$.

We seek a solution of the problem in the form

$$u^{(j)}(z) = \frac{1}{\pi}\,\mathrm{Im}\sum_{k=1}^{3} E^{(j)}_{(k)} \int\limits_{-\infty}^{+\infty} \frac{g^{(j)}(\xi)d\xi}{\xi - z^{(j)}_k},$$
$$z^{(j)}_k = x_1 + \alpha^{(j)}_k x_3, \quad z \in D^{(j)}, \quad z = x_1 + ix_3, \qquad (3)$$

where $g^{(j)}$ is the unknown Hölder continuous real vector density, $E^{(j)}_{(k)} = -iA^{(j)}_{(k)}h^{(j)}$. For the expression of the matrix $A_{(k)}$ see [7]. The elements of the real symmetric matrix $h^{(j)}$ are defined as follows:

$$h = |h_{pq}|\,\frac{b_3}{A_1 h_{11}}, \quad h_{11} = c_{11}c_{44}\Big(\sqrt{a_1a_2a_3}\Big)^{-1}, \quad \alpha_k = i\sqrt{a}_k, \quad k = 1, 2, 3,$$
$$h_{22} = c_{11}c_{44}C - \alpha B + c_{44}c_{33}A, \quad h_{12} = h_{13} = 0,$$
$$-h_{33} = c_{11}\epsilon_{11}C + B\Big[c_{11}\epsilon_{33} + c_{44}\epsilon_{11} + (e_{13} + e_{15})^2\Big] + c_{44}\epsilon_{33}A,$$
$$h_{23} = c_{11}e_{15}C - B\Big[e_{13}(c_{13} + c_{44}) + c_{13}e_{15} - c_{11}e_{33}\Big] + c_{44}e_{33}A,$$
$$\alpha = c_{13}^2 - c_{11}c_{33} + 2c_{13}c_{44}, \quad A_1 = \frac{b_0}{c_{11}}C + \alpha_0 B + \frac{b_3}{c_{44}}A, \qquad (4)$$
$$b_0 = c_{11}(c_{44}\epsilon_{11} + e_{15}^2), \quad b_3 = c_{44}(c_{33}\epsilon_{33} + e_{33}^2),$$
$$C = B\Big(\sqrt{a}_1 + \sqrt{a}_2 + \sqrt{a}_3\Big)\Big(\sqrt{a_1a_2a_3}\Big)^{-1}, \quad A = B\Big(\sqrt{a_1a_2} + \sqrt{a_1a_3} + \sqrt{a_2a_3}\Big),$$
$$B^{-1} = \Big(\sqrt{a}_1 + \sqrt{a}_3\Big)\Big(\sqrt{a}_2 + \sqrt{a}_1\Big)\Big(\sqrt{a}_2 + \sqrt{a}_3\Big),$$
$$\alpha_0 = c_{44}\epsilon_{33} + c_{33}\epsilon_{11} + 2e_{15}e_{33}.$$

Upon acting the operator $T^{(j)}(\partial x, n)$ to the vector $u^{(j)}$ we get

$$T^{(j)}u^{(j)}(z) = -\frac{1}{\pi}\,\mathrm{Im}\sum_{k=1}^{3} R^{(j)}_{(k)} h^{(j)} i \int_{-\infty}^{+\infty} \frac{g^{(j)\prime}(\xi)d\xi}{\xi - z^{(j)}_k}, \quad z \in D^{(j)}, \quad j = 0, 1, \tag{5}$$

where

$$\begin{aligned}
R^{(j)}_{(k)} &= (h^{(j)} - \kappa^{(j)}_N)A^{(j)}_{(k)},\\
\kappa^{(j)}_N &= \|\kappa^{(j)}_{km}\|_{3,3}, \quad \kappa_{mm} = 0, \quad \kappa_{23} = \kappa_{32} = 0,\\
A_1\kappa_{21} &= -A_1\kappa_{12} = A_1\kappa_{1N}\\
&= b_3A + B(c_{33}\eta - e_{33}\omega - c_{13}\alpha_0) - c_{13}b_0c_{11}^{-1}C,\\
A_1\kappa_{31} &= -A_1\kappa_{13} = A_1\kappa_{0N}\\
&= e_{15}c_{44}^{-1}b_3A + B(e_{33}\eta + \epsilon_{33}\omega - e_{13}\alpha_0) - e_{13}b_0c_{11}^{-1}C,\\
\eta &= \epsilon_{11}(c_{13} + c_{44}) + e_{15}(e_{15} + e_{13}), \omega = c_{44}e_{13} - c_{13}e_{15}.
\end{aligned} \tag{6}$$

Taking into account the boundary condition (2) and the relation

$$\left(\int_l \frac{g(\xi)\,d\xi}{\xi - z_k}\right)^{\pm} = \pm i\pi g + \int_l \frac{g(\xi)\,d\xi}{\xi - x_1},$$

for the unknown density we obtain the system of singular integral equations

$$\begin{gathered}
g^{(1)} + g^{(0)} = f(x_1),\\
-\kappa^{(1)}_N \frac{dg^{(1)}(x_1)}{dx_1} - \kappa^{(0)}_N \frac{dg^{(0)}(x_1)}{dx_1} + \frac{1}{\pi}\int_{-\infty}^{+\infty} h^{(1)} \frac{g^{(1)\prime}(\xi)d\xi}{\xi - x_1}\\
-\frac{1}{\pi}\int_{-\infty}^{+\infty} h^{(0)} \frac{g^{(0)\prime}(\xi)d\xi}{\xi - x_1} = F(x_1), \quad x_1 \in l.
\end{gathered} \tag{7}$$

From (7) we deduce that

$$DG(x_1) = C + \tilde{F}_0(x_1) + (\kappa^{(1)}_N + ih^{(1)})f_0(x_1), \quad C \in \mathbb{R}, \quad C = const, \tag{8}$$

$$\begin{aligned}
& g^{(0)}(x_1) + \frac{i}{\pi}\int_{-\infty}^{+\infty} \frac{g^{(0)}(\xi)d\xi}{\xi - x_1} = G(x_1),\\
& f(x_1) + \frac{i}{\pi}\int_{-\infty}^{+\infty} \frac{f(\xi)(\xi)d\xi}{\xi - x_1} = f_0(x_1),\\
& \tilde{F} + \frac{i}{\pi}\int_{-\infty}^{+\infty} \frac{\tilde{F}(\xi)d\xi}{\xi - x_1} = \tilde{F}_0(x_1), \quad \tilde{F} = \int_0^{x_1} F(t)\,dt,\\
& D = \kappa^{(1)}_N + ih^{(1)} - \kappa^{(0)}_N + ih^{(0)}.
\end{aligned} \tag{9}$$

By direct calculations we have

$$\det D = iD_0 = \frac{i}{D_{11}}\Big[(D_{13}^2 - D_{11}D_{33})(D_{11}D_{22} - D_{12}^2) + (D_{12}D_{13} - D_{11}D_{23})^2\Big], \tag{10}$$

where

$$\begin{aligned}
&D_{11} = \sum_{j=0}^{1}\frac{b_3^{(j)}}{A_1^{(j)}} > 0, \quad D_{22} = \sum_{j=0}^{1}\frac{b_3^{(j)}h_{22}^{(j)}}{A_1^{(j)}h_{11}^{(j)}} > 0,\\
&D_{33} = \sum_{j=0}^{1}\frac{b_3^{(j)}h_{33}^{(j)}}{A_1^{(j)}h_{11}^{(j)}} < 0, \quad D_{23} = \sum_{j=0}^{1}\frac{b_3^{(j)}h_{23}^{(j)}}{A_1^{(j)}h_{11}^{(j)}},\\
&D_{13}^2 - D_{11}D_{33} > 0, \quad D_{21} = -D_{12} = \kappa_{1N}^{(1)} - \kappa_{1N}^{(0)},\\
&D_{31} = -D_{13} = \kappa_{0N}^{(1)} - \kappa_{0N}^{(0)}, \quad D_{23} = D_{32},\\
&D_{22}D_{11} - D_{12}^2 = \sum_{j=0}^{1}\frac{1}{A_1^{(j)}}\Big[\frac{b_0^{(j)}}{c_{11}^{(j)}}\Big(c_{11}^{(j)}c_{33}^{(j)} - (c_{13}^{(j)})^2\Big)C^{(j)} + c_{44}^{(j)}\Delta^{(j)}B^{(j)}\Big]\\
&\qquad + \frac{b_3^{(0)}b_3^{(1)}}{A_1^{(1)}A_1^{(0)}}\sum_{j=0}^{1}\frac{h_{22}^{(j)}}{h_{11}^{(j)}} + 2\kappa_{1N}^{(1)}\kappa_{1N}^{(0)} > 0,
\end{aligned} \tag{11}$$

$$c_{33}^{(j)}\Delta^{(j)} = \frac{b_3^{(j)}}{c_{44}^{(j)}}\Big(c_{11}^{(j)}c_{33}^{(j)} - (c_{13}^{(j)})^2\Big) + \Big(e_{33}c_{13}^{(j)} - c_{33}^{(j)}e_{13}^{(j)}\Big)^2 > 0. \tag{12}$$

Taking into account the inequalities

$$D_{11} > 0,\ \ D_{33} < 0,\ \ c_{11}^{(j)}c_{33}^{(j)} - c_{13}^{(j)2} > 0,\ \ h_{22} > 0,\ \ C > 0,\ \ B > 0,\ \ h_{33} < 0,$$

from (10) we get $D_0 > 0$.

By solving the matrix equation (8) we obtain

$$G(x_1) = D^{-1}\Big[C + \tilde{F}_0 + (\kappa_N^{(j)} + ih^{(1)})f_0\Big].$$

Equating the real parts of both sides, from the last equality we find $g^{(0)}$ and $g^{(1)}$.

Substituting the densities $g^{(j)}$ obtained into (3) and taking into account the formulas [8], [9]

$$\int_{-\infty}^{+\infty}\frac{d\xi}{\xi - z_k^{(j)}} = (-1)^{j+1}i\pi,$$

$$\int_{-\infty}^{+\infty}\frac{d\xi}{(\xi - z_k^{(j)})(t-\xi)} = \frac{i(-1)^{j+1}\pi}{t - z_k^{(j)}}, \ \ j = 0, 1,$$

we get solutions of the formulated problem

$$u^{(j)} = C + \frac{1}{\pi}\,\mathrm{Im}\sum_{k=1}^{3}\Bigg[N_{(k)}^{(j)}\int\limits_{-\infty}^{+\infty} F(\xi)\ln(\xi - z_k^{(j)})d\xi$$
$$+\int\limits_{-\infty}^{+\infty} M_{(k)}^{(j)}\frac{f(\xi)d\xi}{\xi - z_k^{(j)}}\Bigg], \quad z \in D^{(j)}, \tag{13}$$

where

$$N_{(k)}^{(0)} = -E_{(k)}^{(0)}D^{-1}, \quad N_{(k)}^{(1)} = E_{(k)}^{(1)}\bar{D}^{-1},$$
$$M_{(k)}^{(0)} = E_{(k)}^{(0)}D^{-1}(\kappa_N^{(1)} + ih^{(1)}),$$
$$M_{(k)}^{(1)} = -E_{(k)}^{(1)}\bar{D}^{-1}(\kappa_N^{(0)} - ih^{(0)}).$$

2. Formulation and Solution of the Interface Crack Problem

Consider a crack of length $2L$ lying along the segment $[-L, L]$ on the contact line in the interface of two piecewise homogeneous electroelastic half-planes $D^{(1)}$ and $D^{(0)}$.We follow the mathematical model of M. Comninou developed in [1], [2] and assume that under the action of $(T, S, D)^T$ applied at infinity, in a direction normal to the interface, the crack opens in the interval $(-a, b)$,where a and b are unknowns, which are to be determined in the process of solution. The crack interval $(-L, L)$ is divided into three subintervals $(-L, -a)$, $(-a, b)$ and (b, L). T, S, D are given real constant numbers. In addition, we assume that the conditions

$$\tau_{13}^{\infty} = S, \quad \tau_{33}^{\infty} = T, \quad \tau_{43}^{\infty} = D \tag{14}$$

are fulfilled at infinity.

Interface Crack Problem. Find regular solutions $u^{(j)}$, $j = 0, 1$, to system (1) in $D^{(j)}$, satisfying the following boundary and contact conditions on the contact line l:

$$\begin{aligned}
&1.\ (u^{(1)})^+ = (u^{(0)})^-, \quad (T^{(1)}u^{(1)})^+ = (T^{(0)}u^{(0)})^-, \quad |x_1| > L,\\
&2.\ (u_k^{(1)})^+ = (u_k^{(0)})^-, \quad k = 3, 4,\\
&\quad (\tau_{k3}^{(1)})^+ = (\tau_{k3}^{(0)})^-, \quad k = 1, 3, 4, \quad x_1 \in (-L, a)\bigcup(b, L),\\
&3.\ (\tau_{k3}^{(1)})^+ = 0, \quad (\tau_{k3}^{(0)})^- = 0, \quad k = 1, 3, 4,\\
&\quad x_1 \in (-a, b), \quad 0 < a < L, \quad 0 < b < l,
\end{aligned} \tag{15}$$

where the symbols $[\cdot]^+$, $[\cdot]^-$ denote the limits on l from $D^{(1)}$ and $D^{(0)}$, respectively.

By a regular solution to system (1) is understood a vector $u^{(j)}$ such that:

1. $u^{(j)} \in \bar{C}(D^{(j)})\bigcap C^2(D^{(j)})$.

2. The corresponding stress components τ_{k3} are continuously extendable on the whole x_1-axis except the points $-L, -a, b, L$ in whose vicinity they have integrable singularities.

3. For sufficiently large $|x| = \sqrt{x_1^2 + x_3^2}$

$$u^{(j)} - u^{(j)}_\infty = O(1),$$
$$\frac{\partial}{\partial x_k}\left[u^{(j)} - u^{(j)}_\infty\right] = O(|x|^{-2}),$$

where

$$u^{(j)}_\infty = \frac{c^{(j)}_{44}}{b^{(j)}_3} \begin{pmatrix} \dfrac{b^{(j)}_3}{c^{(j)2}_{44}} S \\ \epsilon^{(j)}_{33} T + e^{(j)}_{33} D \\ e^{(j)}_{33} T - c^{(j)}_{33} D \end{pmatrix} x_3. \tag{16}$$

Remark. We note that if, in addition, the displacement fields in $D^{(j)}$ vanish at infinity, then the homogeneous interface crack problem possesses a trivial solution only.

Using the solution of the basic contact problem (13) and taking into account conditions (15), we easily conclude that $(TU)^+ = (TU)^- = 0$ on the whole contact line l. Therefore, due to formula (13), we seek solutions $u^{(j)}$, $j = 0, 1$, in the domains $D^{(j)}$ in the form

$$u^{(j)}(z) = C + u^{(j)}_\infty + \frac{1}{\pi} \operatorname{Im} \sum_{k=1}^{3} M^{(j)}_{(k)} \begin{pmatrix} \int\limits_{-L}^{L} (u^{(1)}_1 - u^{(0)}_1) \dfrac{dt}{t - z^{(j)}_k} \\ \int\limits_{-a}^{b} (u^{(1)}_3 - u^{(0)}_3) \dfrac{dt}{t - z^{(j)}_k} \\ \int\limits_{-a}^{b} (u^{(1)}_4 - u^{(0)}_4) \dfrac{dt}{t - z^{(j)}_k} \end{pmatrix}. \tag{17}$$

Here C is again an arbitrary constant vector, $u^{(j)}_\infty$ is given by (16). Here and throughout this section we use the notations $u^{(0)}(t) = [u^{(0)}(t)]^-$, $u^{(1)}(t) = [u^{(1)}(t)]^+$ for $-\infty < t < +\infty$.

Clearly, the difference $u^{(1)}_1 - u^{(0)}_1$ is unknown in the interval $(-L, L)$, while the differences $u^{(1)}_k - u^{(0)}_k$, $k = 3, 4$, are unknowns in the interval $(-a, b)$.

It is evident that the conditions

$$u^{(0)}_1(\pm L) = u^{(1)}_1(\pm L), \quad u^{(0)}_k(b) = u^{(1)}_k(b), \quad u^{(0)}_k(-a) = u^{(1)}_k(-a), \quad k = 3, 4,$$

are the sufficient ones for the above vectors $u^{(j)}$ to be continuously extendable on the whole contact line l [9].

From (17) it follows that

$$T^{(j)}u^{(j)}(z)=\begin{pmatrix} S \\ T \\ D \end{pmatrix}-\frac{1}{\pi}\operatorname{Im}\sum_{k=1}^{3}Q_{(k)}^{(j)}\begin{pmatrix} \int\limits_{-L}^{L}\dfrac{B_1(t)dt}{t-z_k^{(j)}} \\ \int\limits_{-a}^{b}\dfrac{B_3(t)dt}{t-z_k^{(j)}} \\ \int\limits_{-a}^{b}\dfrac{B_4(t)dt}{t-z_k^{(j)}} \end{pmatrix}, \tag{18}$$

where

$$Q_{(k)}^{(0)}=-iR_{(k)}^{(0)}h^{(0)}\bar{D}^{-1}(\kappa_N^{(1)}+ih^{(1)}),\quad Q_{(k)}^{(1)}=iR_{(k)}^{(1)}h^{(1)}\bar{D}^{-1}(\kappa_N^{(0)}+ih^{(0)}),$$

$$u_1^{(1)}-u_1^{(0)}=-\int\limits_{-L}^{t}B_1(t)dt,\quad u_k^{(1)}-u_k^{(0)}=-\int\limits_{-a}^{t}B_k(t)dt,\quad k=3,4.$$

We must require in addition that

$$\int\limits_{-L}^{L}B_1(t)dt=0,\qquad \int\limits_{-a}^{b}B_k(t)dt=0,\quad k=3,4, \tag{19}$$

as these conditions ensure that the vectors $u^{(j)}$, $j=0,1$, are single-valued.

We note that

$$B_k(t)=-\frac{d}{dt}\left(u_k^{(1)}-u_k^{(0)}\right),\quad k=1,3,4. \tag{20}$$

Thus we can assume that B_k, $k=3,4$, is continuous in $(-a,b)$ and bounded at the ends $-a,b$. It is important to note that $B_k(-a)=0$, $B_k(b)=0$, $k=3,4$.

We can easily show that the first conditions in (15) are automatically satisfied. The second and the third conditions in (15) will be completely fulfilled if

$$\tau_{13}^{(1)+}=0,\quad \tau_{33}^{(1)+}=0,\quad \tau_{43}^{(1)+}=0.$$

These boundary conditions together with formula (18) lead to a system of integral equations

$$\begin{pmatrix} K_{21} \\ K_{31} \end{pmatrix}B_1(x_1)+\frac{1}{\pi}\int\limits_{-a}^{b}\begin{pmatrix} K_{22} & K_{23} \\ K_{23} & K_{33} \end{pmatrix}\begin{pmatrix} B_3(t) \\ B_4(t) \end{pmatrix}\frac{dt}{t-x_1}$$
$$=\begin{pmatrix} T \\ D \end{pmatrix},\quad x\in(-a,b), \tag{21}$$

$$\left[H(x_1-b)-H(x_1+a)\right]\left(K_{21}B_3(x_1)+K_{31}B_4(x_1)\right)$$
$$+\frac{K_{11}}{\pi}\int\limits_{-L}^{L}\frac{B_1(t)dt}{t-x_1}=S,\quad x_1\in(-L,L), \tag{22}$$

where $H(x)$ is a Heaviside unit step function and

$$D_0 \begin{pmatrix} K_{21} \\ K_{31} \end{pmatrix} = \sum_{j=0}^{1} (-1)^j \Bigg[\frac{b_3^{(j)} \Delta_0^{(j)} \kappa_{1N}^{(0)} \kappa_{1N}^{(1)}}{A_1^{(j)} h_{11}^{(j)}} \begin{pmatrix} (\kappa_{1N})^{-1} \\ (-\kappa_{0N})^{-1} \end{pmatrix}$$

$$+ \begin{pmatrix} -l_{22}^{(j)} \\ l_{23}^{(j)} \end{pmatrix} \frac{\nu_{12}^{(0)} \nu_{12}^{(1)}}{\nu_{12}^{(j)}} + \begin{pmatrix} l_{23}^{(j)} \\ -l_{33}^{(j)} \end{pmatrix} \frac{\nu_{11}^{(0)} \nu_{11}^{(1)}}{\nu_{11}^{(j)}} \Bigg],$$

$$\nu_{1k} = \frac{b_3}{A_1 h_{11}} \alpha_{1k}, \quad k = 1, 2, \quad \alpha_{11}^{(j)} = h_{23}^{(j)} \kappa_{1N}^{(j)} - h_{22}^{(j)} \kappa_{0N}^{(j)},$$

$$\alpha_{12}^{(j)} = h_{33}^{(j)} \kappa_{1N}^{(j)} - h_{23}^{(j)} \kappa_{0N}^{(j)},$$

$$D_0 \begin{pmatrix} K_{22} & K_{23} \\ K_{23} & K_{33} \end{pmatrix} = \frac{b_3^{(0)} b_3^{(1)}}{A_1^{(0)} h_{11}^{(0)} A_1^{(1)} h_{11}^{(1)}} \Bigg[\sum_{j=0}^{1} \Bigg[\begin{pmatrix} h_{22}^{(j)} & h_{23}^{(j)} \\ h_{23}^{(j)} & h_{33}^{(j)} \end{pmatrix} \frac{1}{\Delta_0^{(0)} \Delta_0^{(1)}} \Delta_0^{(j)}$$

$$+ \begin{pmatrix} l_{22}^{(j)} & l_{23}^{(j)} \\ l_{23}^{(j)} & l_{33}^{(j)} \end{pmatrix} \frac{b_3^{(0)} b_3^{(1)} A_1^{(j)} h_{11}^{(j)}}{b_3^{(j)2}} \Bigg] - \begin{pmatrix} 2\alpha_{12}^{(0)} \alpha_{12}^{(1)}, & \alpha_{12}^{(0)} \alpha_{11}^{(1)} + \alpha_{11}^{(0)} \alpha_{12}^{(1)} \\ \alpha_{12}^{(0)} \alpha_{11}^{(1)} + \alpha_{11}^{(0)} \alpha_{12}^{(1)}, & 2\alpha_{11}^{(0)} \alpha_{11}^{(1)} \end{pmatrix} \Bigg], \tag{23}$$

$$D_0 K_{11} = \sum_{j=0}^{1} \left[\frac{b_3^{(0)} b_3^{(1)} \Delta_0^{(j)}}{A_1^{(0)} A_1^{(1)} h_{11}^{(j)}} - l_{22}^{(j)} \frac{l_{33}^{(0)} l_{33}^{(1)}}{l_{33}^{(j)}} \right] + 2 l_{23}^{(0)} l_{23}^{(1)},$$

$$l_{22} = \frac{b_3}{A_1 a_{11} b_{44}} (b_{44} B + a_{44} C) > 0, \quad l_{23} = \frac{b_3}{A_1 a_{11} b_{44}} (a_{34} C + b_{33} B),$$

$$l_{33} = -\frac{b_3}{A_1 a_{11} b_{44}} [a_{44} A + (b_{11} + 2A_{13}) B + a_{33} C] < 0,$$

$$\Delta_0 = (c_{11} c_{33} - c_{13}^2) \frac{b_0}{c_{11}} B + c_{44} \Delta A > 0,$$

$a_{11}, \ldots, b_{11}, \ldots, b_{14}$ are the real constant values which are combinations of the known piezoelectric, elastic and dielectric constants. For the expression of these values see [10]. Then (22) can be viewed as a Cauchy singular equation for the unknown function $B_1(t)$ and solved formally by treating B_k, $k = 3, 4$, as known. The solution is found in Muskhelishvili's book [9].

We follow [4] and transform in (22) the independent variables

$$x_1 = L \frac{s - \gamma}{1 - s\gamma}, \quad t = L \frac{r - \gamma}{1 - r\gamma}, \qquad \gamma = \frac{L(a - b)}{L^2 - ab + \sqrt{(L^2 - a^2)(L^2 - b^2)}}. \tag{24}$$

Next we introduce the new unknown functions A_k, rather than B_k:

$$K_{11} B_1(t) = A_1(r)(1 - r\gamma)^2, \quad d_0 B_k(t) = A_k(r)(1 - r\gamma)^2, \quad k = 3, 4,$$

$$d_0 = (K_{22} K_{33} - K_{23}^2) = -\frac{b_3^{(0)} b_3^{(1)} (b_3^{(1)} \Delta_0^0 + b_3^{(0)} \Delta_0^{(1)})}{D_0 A_1^{(0)} h_{11}^{(0)} A_1^{(1)} h_{11}^{(1)}} \neq 0.$$

Condition (19) now reads as

$$\int_{-1}^{1} A_1(r)dr = 0, \quad \int_{-c}^{c} A_k(r)dr = 0, \quad k = 3, 4,$$
$$A_k(-c) = A_k(c) = 0, \quad k = 3, 4, \tag{25}$$

where

$$c = \frac{a\sqrt{L^2 - b^2} + b\sqrt{L^2 - a^2}}{L[\sqrt{L^2 - b^2} + \sqrt{L^2 - a^2}]} < 1.$$

Then, applying the equality

$$\int_{-a}^{a} \frac{(1 - r\gamma)A(r)dr}{(1 - s\gamma)(r - s)} = \int_{-a}^{a} \frac{A(r)dr}{r - s} - \frac{\gamma}{1 - s\gamma} \int_{-a}^{a} A(r)\, dr,$$

the integral equations (21) and (22) become

$$\begin{pmatrix} K_{33} & -K_{23} \\ -K_{23} & K_{22} \end{pmatrix} \begin{pmatrix} K_{21} \\ K_{31} \end{pmatrix} \frac{1}{K_{11}} A_1(s) + \frac{1}{\pi} \int_{-c}^{c} \begin{pmatrix} A_3(r) \\ A_4(r) \end{pmatrix} \frac{dr}{r - s}$$
$$= \begin{pmatrix} K_{33} & -K_{23} \\ -K_{23} & K_{22} \end{pmatrix} \begin{pmatrix} T \\ D \end{pmatrix} \frac{1}{(1 - s\gamma)^2}, \quad |s| < c, \tag{26}$$

$$\frac{1}{\pi} \int_{-1}^{1} \frac{A_1(r)dr}{r - s} = \frac{S}{(1 - s\gamma)^2}$$
$$+ d_0^{(-1)} H(c^2 - s^2)(K_{21}A_3(s) + K_{31}A_4(s)), \quad |s| < 1. \tag{27}$$

Further, we apply the general theory of singular integral equations ([9], Chapter 5) and solve the integral equation (27) for A_1 and impose the corresponding auxiliary condition (25). This yields

$$A_1(s) = -\frac{\chi(s)}{\pi d_0} \int_{-1}^{1} \Big[\frac{S d_0}{(1 - r\gamma)^2} + H(c^2 - s^2)\Big(K_{21}A_3(s)$$
$$+ K_{31}A_4(s)\Big)\Big] \frac{dr}{(r - s)\chi(r)}, \quad |s| < 1, \tag{28}$$

where $\chi(s) = \sqrt{(1 - s^2)^{-1}}$.

On account of conditions (25) we can simplify (28) as

$$A_1(s) = S\chi(s) \frac{d}{d\gamma} \frac{\sqrt{1 - \gamma^2}}{1 - s\gamma}$$
$$- \frac{\chi(s)}{\pi d_0} \int_{-c}^{c} \Big(K_{21}A_3(s) + K_{31}A_4(s)\Big)\sqrt{1 - r^2}\, \frac{dr}{r - s}. \tag{29}$$

After the substitution of (29) into (26) we find that

$$\frac{1}{\pi}\int_{-c}^{c}\left[\begin{pmatrix} K_{22}K_{11}, & K_{11}K_{23} \\ K_{11}K_{23}, & K_{11}K_{33}\end{pmatrix}\sqrt{1-s^2}\right.$$
$$\left.-\begin{pmatrix} K_{21}^2 & , K_{21}K_{31} \\ K_{21}K_{31}, & K_{31}^2\end{pmatrix}\sqrt{1-r^2}\right]\begin{pmatrix} A_3(r) \\ A_4(r)\end{pmatrix}\frac{dr}{r-s}$$
$$= K_{11}d_0\left[\frac{\sqrt{1-s^2}}{(1-s\gamma)^2}-\frac{S}{K_{11}}\frac{d}{d\gamma}\frac{\sqrt{1-\gamma^2}}{1-s\gamma}\begin{pmatrix} K_{21} \\ K_{31}\end{pmatrix}\right.$$
$$\left.+\frac{\sqrt{1-s^2}}{(1-s\gamma)^2}\begin{pmatrix} T \\ D\end{pmatrix}\right] = \phi(s), \quad |s|<c. \tag{30}$$

Thus we have obtained the system of integral equations (30) together with conditions (25) to find the unknowns A_k, $k=3,4$, in the class of functions bounded at both ends $\pm c$.

Let us separate the so-ccalled characteristic part of the equation (30)

$$\frac{1}{\pi}\int_{-c}^{c}\begin{pmatrix} A_3(r) \\ A_4(r)\end{pmatrix}\frac{dr}{r-s}+\frac{1}{\pi}\int_{-c}^{c}K(s,r)\begin{pmatrix} A_3(r) \\ A_4(r)\end{pmatrix}dr = \Phi(s), \quad |s|<c, \tag{31}$$

where

$$q=\frac{1}{\delta}\begin{pmatrix} K_{11}K_{33}-K_{31}^2, & K_{21}K_{31}-K_{11}K_{23} \\ K_{21}K_{31}-K_{11}K_{23}, & K_{11}K_{22}-K_{21}^2\end{pmatrix},$$

$$K(s,r)=\sum_{j=0}^{1}(-1)^j\begin{pmatrix} -K_{21}\frac{\alpha_{11}^{(j)}}{\Delta_0^{(j)}} & -K_{31}\frac{\alpha_{11}^{(1)}}{\Delta_0^{(j)}} \\ K_{21}\frac{\alpha_{12}^{(j)}}{\Delta_0^{(j)}} & K_{31}\frac{\alpha_{12}^{(j)}}{\Delta_0^{(j)}}\end{pmatrix}\frac{s+r}{\sqrt{1-s^2}(\sqrt{1-r^2}+\sqrt{1-s^2})},$$

$$\delta=-\frac{K_{11}b_3^{(0)}b_3^{(1)}\Delta_0^{(1)}\Delta_0^{(0)}}{D_0A_1^{(0)}h_{11}^{(0)}A_1^{(1)}h_{11}^{(1)}}, \quad \Phi=q\frac{\phi}{\sqrt{1-s^2}}.$$

Obviously, $K(s,r)\in C^{\frac{1}{2}}([-c,c]\times[-c,c])$ and $\Phi\in C^{\frac{1}{2}}([-c,c])$, i.e.,the kernel K and the right-hand side function Φ are Hölder-continuous functions with the exponent $\frac{1}{2}$.

We seek solutions of equation (31) in the class of functions which are bounded at both ends $-c$ and c. Due to [9] we denote this class by $h(-c,c)$. It is evident that if equation (31) has a solution, then this solution automatically satisfies conditions (25) since the function Φ is Hölder-continuous [9]. It can easily be seen that the index of equation (31) in the class $h(-c,c)$ is equal to -2. Therefore the homogeneous equation adjoint to (31) has only two linearly independent solutions σ_k, $k=3,4$, in the adjoint class (in the class of functions unbounded at both ends $-c$ and c) and the condition for system (31) to be solvable is written as

$$\int_{-c}^{c} \Phi(r)\sigma_k(r)\,dr = 0. \tag{32}$$

The unknown parameters c and γ (i.e., a and b) are to be defined by conditions (25) and (32).

The solutions A_k, $k = 3, 4$, to equation (31) of the class $h(-c, c)$ meet the conditions $A_k(-c) = A_k(c)$, $k = 3, 4$. Therefore the function $A_1(s)$ has a singularity of the type $O((1 - s^2)^{-1/2})$ at the points $s = \pm 1$. Consequently the function $B_1(x_1)$ has singularity of the type $O((L^2 - x_1^2)^{-1/2})$ at the points $x_1 = \pm L$, while the functions $B_k(x_1)$, $k = 3, 4$, are Hölder continuous on the segment $[-a, b]$ and satisfy the conditions $B_k(-a) = B_k(a) = 0$.

From (17) and (18) it follows that

$$\tau_{13}^{(1)+} = S - \frac{K_{11}}{\pi}\int_{-L}^{L} \frac{B_1(t)dt}{t - x_1}, \quad |x_1| > L, \tag{33}$$

$$\begin{pmatrix} \tau_{33}^{(1)+} \\ \tau_{43}^{(1)+} \end{pmatrix} = \begin{pmatrix} T \\ D \end{pmatrix} - \frac{1}{\pi}\int_{-a}^{b} \begin{pmatrix} K_{22} & K_{23} \\ K_{23} & K_{33} \end{pmatrix} \begin{pmatrix} B_3(t) \\ B_4(t) \end{pmatrix} \frac{dt}{t - x_1}, \quad |x_1| > L, \tag{34}$$

$$\begin{pmatrix} \tau_{33}^{(1)+} \\ \tau_{43}^{(1)+} \end{pmatrix} = \begin{pmatrix} T \\ D \end{pmatrix} - \begin{pmatrix} K_{21} \\ K_{31} \end{pmatrix} B_1(t)$$
$$-\frac{1}{\pi}\int_{-a}^{b} \begin{pmatrix} K_{22} & K_{23} \\ K_{23} & K_{33} \end{pmatrix} \begin{pmatrix} B_3(t) \\ B_4(t) \end{pmatrix} \frac{dt}{t - x_1}, \quad x_1 \in (-L, -a) \cup (L, b). \tag{35}$$

Let us consider the integral

$$I(x_1) = \frac{K_{11}}{\pi}\int_{-L}^{L} \frac{B_1(t)dt}{t - x_1}.$$

Performing here the change of the variables x_1 and t by (24) for $|x_1| > L$, we derive [5]

$$I\Big(L\frac{s - \gamma}{1 - s\gamma}\Big) = S + \frac{\operatorname{sign} s}{\sqrt{s^2 - 1}}\Big[\frac{S(\gamma - s)}{\sqrt{1 - \gamma^2}}$$
$$+ \frac{(1 - s\gamma)^2}{\pi d_0}\int_{-c}^{c} \frac{K_{21}A_3(\xi) + K_{31}A_4(\xi)}{\xi - s}\sqrt{1 - \xi^2}d\xi\Big], \tag{36}$$

where $|s| > 1$, $s \neq \frac{1}{\gamma}$.

Further, by applying (22) we have the relation

$$I(x_1) = S + [H(x_1 + a) - H(x_1 - b)][K_{21}A_3(x_1)$$
$$+ K_{31}A_4(x_1)], \quad |x_1| < L. \tag{37}$$

These relations together with (33) imply that the tangent components of the electromecanical stress vectors have a singularity of the type $O((x_1^2-L^2)^{-1/2})$ when the point approaches the end points $\pm L$ from the outside of the cut. Note that the tangent components vanish when the point approaches the end points $\pm L$ along the cut line from the inside.

Formulas (34) and (35) show that the normal components of the stress vectors and the normal conponent of the induction vector have a singularity of the type $O(|L^2-x_1^2|)^{-1/2})$ when the point approaches the end points $\pm L$.

The tangent stress intensity coefficients at the end points $x_1=\pm L$ of the cut can be calculated as follows:

$$K_2(L)=\lim_{x_1\to L^+}\sqrt{2(x_1-L)}(\tau_{13}^{(1)})^{\pm}$$

$$=(1-\gamma)\sqrt{L(1-\gamma^2)}\lim_{s\to 1^-}\sqrt{1-s^2}A_1(s),$$

$$K_2(-L)=-(1+\gamma)\sqrt{L(1-\gamma^2)}\lim_{s\to -1^+}\sqrt{1-s^2}A_1(s).$$

The normal stress intensity coefficients at the end points $\pm L$ vanish.

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