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OSCILLATION OF HIGHER ORDER DELAY DIFFERENTIAL EQUATIONS OF NEUTRAL TYPE

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Abstract. The properties of solutions of higher order neutral differential equations with distributed type deviating arguments are obtained.

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1. Introduction

There are some results on the oscillation theory for higher order functional differential equations. For instance, see [1–5] and the references therein. But nobody studied the oscillation properties of solutions of higher order neutral differential equations with distributed type deviating arguments.

In this paper we study the properties of solutions of higher order neutral differential equations with distributed type deviating arguments of the form

$$[x(t)+\mu(t)x(t-\rho)]^{(n)}+\int\limits_a^b p(t,\xi)x(g(t,\xi))d\sigma(\xi)=0 \quad (b>a), \quad t\geq t_0, \quad (E)$$

where $n\geq 2$ is an even integer.

Suppose that the following conditions (H) hold:

(H1) $\mu\in C^n([t_0,\infty);[0,\infty))$, $0\leq\mu(t)\leq 1$, $\rho=const>0$;

(H2) $p\in C([t_0,\infty)\times[a,b];[0,\infty))$, $g\in C([t_0,\infty)\times[a,b];[0,\infty))$, $g(t,\xi)\leq t$, $\xi\in[a,b]$, $g(t,\xi)$ is a nondecreasing function with respect to t and ξ, respectively, and

$$\lim_{t\to+\infty}\inf_{\xi\in[a,b]}\{g(t,\xi)\}+\infty;$$

(H3) $\sigma\in([a,b];R)$ and $\sigma(\xi)$ is nondecreasing in ξ; the integral in (E) is the Stieltjes integral.

As is customary, a nontrivial solution of (E) is said to be oscillatory if it has arbitrarily large zeros. Otherwise, it is said to be nonoscillatory.

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2. Some Lemmas

Consider the differential inequalities

$$[x(t)+\mu(t)x(t-\rho)]^{(n)}+\int_a^b p(t,\xi)x(g(t,\xi))d\sigma(\xi)\le 0\ \ (b>a),\ \ t\ge t_0, \tag{1}$$

and

$$[x(t)+\mu(t)x(t-\rho)]^{(n)}+\int_a^b p(t,\xi)x(g(t,\xi))d\sigma(\xi)\ge 0\ \ (b>a),\ \ t\ge t_0, \tag{1*}$$

where $\mu(t)$, ρ, $p(t,\xi)$, $g(t,\xi)$ are the same as (H).

Lemma 1 ([6]). *Let $y(t)$ be an n times differentiable function on $[0,\infty)$ of constant sign, $y^{(n)}(t)\not\equiv 0$ on $[t_0,\infty)$, which satisfies $y^{(n)}(t)y(t)\le 0$. Then:*

(i) *There exists $t_1\ge t_0$ such that the functions $y^{(i)}(t)$, $i=1,2,\ldots,n-1$, are of constant sign on $[t_1,\infty)$.*

(ii) *There exists a number $h\in\{1,3,5,\ldots,n-1\}$ when n is even, or $h\in\{0,2,4,\ldots,n-1\}$ when n is odd, such that*

$$y(t)y^{(i)}(t)>0\quad for\quad i=0,1,\ldots,h,\ \ t\ge t_1;$$
$$(-1)^{n+i+1}y(t)y^{(i)}(t)>0\quad for\quad i=h+1,\ldots,n,\ \ t\ge t_1.$$

Lemma 2. *Suppose that $x(t)$ is an eventually positive solution of inequality* (1). *Let*

$$z(t)=x(t)+\mu(t)x(t-\rho). \tag{2}$$

Then there exists a number $t_1\ge 0$ such that

$$z(t)>0,\ \ z'(t)>0,\ \ z^{(n-1)}(t)>0\ \ and\ \ z^{(n)}(t)\le 0,\ \ t\ge t_1. \tag{3}$$

Proof. Since $x(t)$ is an eventually positive solution of inequality (1), we have

$$\lim_{t\to+\infty}\min_{\xi\in[a,b]} g(t,\xi)=+\infty.$$

We obtain that there exists $t_1\ge t_0$ such that $x(t)>0$, $x(t-\rho)>0$ and $x(g(t,\xi))>0$, $t\ge t_1$, $\xi\in[a,b]$. Noting that $0\le\mu(t)\le 1$, we have $z(t)>0, t\ge t_1$, and

$$z^{(n)}(t)\le -\int_a^b p(t,\xi)x(g(t,\xi))d\sigma(\xi)\le 0,\quad t\ge t_1.$$

By Lemma 1 we obtain that there exists $t_2\ge t_1$ such that $z'(t)>0$, $z^{(n-1)}(t)>0$, $t\ge t_2$. □

Lemma 3. *If*

$$\int^{+\infty}\int_a^b p(s,\xi)[1-\mu(g(s,\xi))]d\sigma(\xi)ds = +\infty, \tag{4}$$

then inequality (1) *has no eventually positive solutions and inequality* (1^*) *has no eventually negative solutions.*

Proof. Suppose that $x(t)$ is an eventually positive solution of (1). The case of $x(t)$ being an eventually negative solution of (1^*) can be proved by the same argument. Let $z(t)$ be defined by (2). Then there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(t-\rho) > 0$ and $x(g(t,\xi)) > 0$, $t \geq t_1$, $\xi \in [a,b]$. By Lemma 2 we see that (3) holds. Thus we have

$$z^{(n)}(t) + \int_a^b p(t,\xi)x(g(t,\xi))d\sigma(\xi)$$

$$= z^{(n)}(t) + \int_a^b p(t,\xi)[z(g(t,\xi)) - \mu(g(t,\xi))x(g(t,\xi)-\rho)]d\sigma(\xi) \leq 0, \quad t \geq t_1.$$

It is easy to see that $z(t) \geq x(t)$, and by Lemma 2 we have $z'(t) > 0$, $t \geq t_1$. Then we obtain

$$z^{(n)}(t) + \int_a^b p(t,\xi)[1-\mu(g(t,\xi))]z(g(t,\xi))d\sigma(\xi) \leq 0, \quad t \geq t_1. \tag{5}$$

Noting that

$$\lim_{t\to+\infty} \inf_{\xi\in[a,b]} \{g(t,\xi)\} = +\infty,$$

we can choose $m > t_1, t_2 \geq t_1$ such that

$$z(m) > 0, \quad g(t,\xi) > m, \quad t \geq t_2, \quad \xi \in [a,b].$$

Therefore

$$z(g(t,\xi)) \geq z(m), \quad t \geq t_2, \quad \xi \in [a,b]. \tag{6}$$

Thus we get

$$z^{(n)}(t) + z(m)\int_a^b p(t,\xi)[1-\mu(g(t,\xi))]d\sigma(\xi) \leq 0, \quad t \geq t_2. \tag{7}$$

Integrating (7) from t_2 to t, we have

$$z^{(n-1)}(t) - z^{(n-1)}(t_2) + z(m)\int_{t_2}^t\int_a^b p(s,\xi)[1-\mu(g(s,\xi))]d\sigma(\xi)ds \leq 0, \tag{8}$$

that is,

$$\int_{t_2}^{t}\int_{a}^{b} p(s,\xi)[1-\mu(g(s,\xi))]d\sigma(\xi)ds \le \frac{z^{(n-1)}(t_2)-z^{(n-1)}(t)}{z(m)},$$

which contradicts condition (4). □

Lemma 4. *Suppose that* $\mu(t)\equiv\mu$ *is a positive constant,* $p(t,\xi)$ *is periodic in* t *with period* ρ. *If*

$$g(t-c,\xi)g(t,\xi)-c \ \ \text{for any number} \ \ c>0; \tag{9}$$

$$\int^{+\infty}\int_{a}^{b} p(s,\xi)d\sigma(\xi)ds=+\infty, \tag{10}$$

then inequality (1) *has no eventually positive solutions and inequality* (1^*) *has no eventually negative solutions.*

Proof. Suppose that $x(t)$ is an eventually positive solution of (1). The case of $x(t)$ being an eventually negative solution of (1^*) can be proved by the same argument. Then there exists $t_1\ge t_0$ such that $x(t)>0, x(t-\rho)>0$ and $x(g(t,\xi))>0$, $t\ge t_1$, $\xi\in[a,b]$.

Set

$$z(t)=x(t)+\mu x(t-\rho), t\ge t_1.$$

By the arguments as in the proof of Lemma 2 we obtain

$$z(t)>0,\ \ z^{'}(t)>0,\ \ z^{(n-1)}(t)>0 \quad \text{and} \quad z^{(n)}(t)\le 0,\ \ t\ge t_1.$$

Let

$$w(t)z(t)+\mu z(t-\rho)x(t)+2\mu x(t-\rho)+\mu^2x(t-2\rho),\quad t\ge t_1. \tag{11}$$

Then there exists $t_2\ge t_1$ such that

$$w(t)>0,\ \ w(t-\rho)>0,\ \ w^{(n-1)}(t)>0 \text{ and } w^{(n-1)}(t-\rho)>0,\ \ t\ge t_2.$$

Therefore

$$\begin{aligned} w^{(n)}(t)x^{(n)}(t)&+2\mu x^{(n)}(t-\rho)+\mu^2x^{(n)}(t-2\rho)\\ &\le -\int_{a}^{b} p(t,\xi)x(g(t,\xi))d\sigma(\xi)\\ &-\mu\int_{a}^{b} p(t-\rho,\xi)x(g(t-\rho,\xi))d\sigma(\xi),\ \ t\ge t_2. \end{aligned} \tag{12}$$

Noting that $p(t,\xi)$ is periodic in t with period ρ, combining (9), (11) and (12), we obtain that there exists $t_3\ge t_2$ such that

$$w^{(n)}(t)+\mu w^{(n)}(t-\rho)+\int_{a}^{b} p(t,\xi)w(g(t,\xi))d\sigma(\xi)$$

$$\begin{aligned}
&\leq -\int_a^b p(t,\xi)x(g(t,\xi))d\sigma(\xi) - 2\mu\int_a^b p(t-\rho,\xi)x(g(t-\rho,\xi))d\sigma(\xi)\\
&-\mu^2\int_a^b p(t-2\rho,\xi)x(g(t-2\rho,\xi))d\sigma(\xi)\\
&+\int_a^b p(t,\xi)[x(g(t,\xi))+2\mu x(g(t,\xi)-\rho)+\mu^2 x(g(t,\xi)-2\rho)]d\sigma(\xi)\\
&-\int_a^b p(t,\xi)x(g(t,\xi))d\sigma(\xi) - 2\mu\int_a^b p(t-\rho,\xi)x(g(t-\rho,\xi))d\sigma(\xi)\\
&-\mu^2\int_a^b p(t-2\rho,\xi)x(g(t-2\rho,\xi))d\sigma(\xi)\\
&+\int_a^b p(t,\xi)[x(g(t,\xi))+2\mu x(g(t-\rho,\xi))+\mu^2 x(g(t-2\rho,\xi))]d\sigma(\xi)\\
&=0, \quad t\geq t_3.
\end{aligned} \tag{13}$$

By the arguments as in the proof of Lemma 3 we can choose $m^*>t_3$, $T>t_3$ such that

$$w(g(t,\xi))\geq w(m^*), \quad t\geq T, \quad \xi\in[a,b].$$

Integerating (13) from T to t, we have

$$\begin{aligned}
&w^{(n-1)}(t)-w^{(n-1)}(T)+\mu w^{(n-1)}(t-\rho)-\mu w^{(n-1)}(T-\rho)\\
&\qquad +w(m^*)\int_T^t\int_a^b p(s,\xi)d\sigma(\xi)ds\\
&\leq w^{(n-1)}(t)-w^{(n-1)}(T)+\mu w^{(n-1)}(t-\rho)-\mu w^{(n-1)}(T-\rho)\\
&\qquad +\int_T^t\int_a^b p(s,\xi)w(g(s,\xi))d\sigma(\xi)ds\leq 0, \quad t\geq T,
\end{aligned}$$

that is,

$$\begin{aligned}
&\int_T^t\int_a^b p(s,\xi)d\sigma(\xi)ds\\
&\leq \frac{w^{(n-1)}(T)-w^{(n-1)}(t)+\mu w^{(n-1)}(T-\rho)-\mu w^{(n-1)}(t-\rho)}{w(m^*)}, \quad t\geq T,
\end{aligned}$$

which contradicts condition (10). □

3. Oscillation of Equation (E)

Using Lemma 3 we have

Theorem 1. *Assume that* (4) *holds. Then every solution of equation* (E) *oscillates.*

By Lemma 4 immediately we have

Theorem 2. *Assume that the conditions of Lemma* 4 *hold. Then every solution of equation* (E) *oscillates.*

Example 1. Consider the fourth order neutral delay differential equation

$$[x(t)+(1-e^{-3t})x(t-\pi)]^{(4)}+\int_{-1}^{0} e^{2(t+2\xi)}x\Big(\frac{t}{3}+\xi\Big)d\xi=0,\ \ t>\pi. \tag{14}$$

We can choose $\mu(t)=1-e^{-3t}$, $p(t,\xi)=e^{2(t+2\xi)}$, $g(t,\xi)=\frac{t}{3}$, $\sigma(\xi)=\xi$. Note that

$$\int^{+\infty}\int_a^b p(s,\xi)[1-\mu(g(s,\xi))]d\sigma(\xi)ds=\int^{+\infty}\int_{-1}^{0} e^{s+\xi}d\xi ds=+\infty,$$

i.e., all conditions of Theorem 1 are satisfied. Therefore we can see that all solutions of equation (14) are oscillatory.

Example 2. Consider the second order neutral delay differential equation

$$\Big[x(t)+\frac{1}{2}x(t-\pi)\Big]''+\int_{-\pi}^{-\pi/2}[1+\cos(2s+\xi)]x(s+\xi)d\xi=0,\quad t>\pi. \tag{15}$$

It is easy to see that all conditions of Theorem 2 are satisfied. Thus all solutions of equation (15) are oscillatory.

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