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ON THE OSCILLATION OF CERTAIN SECOND ORDER DIFFERENTIAL EQUATIONS

RAVI P. AGARWAL AND S. R. GRACE

Abstract. Some new criteria for the oscillation of all solutions of second order equations of the form

$$\big(a(t)|y'(t)|^{\sigma-1}y'(t)\big)' + q(t)|y(t)|^{\sigma}\ \mathrm{sgn}\ y(t) = 0, \quad \sigma > 1,$$

are established. The obtained results extend, unify and corollate many of the known oscillation criteria for second order equations.

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1. Introduction

We consider the second order equation

$$\Big(a(t)|y'(t)|^{\sigma-1}y'(t)\Big)' + q(t)|y(t)|^{\sigma}\ \mathrm{sgn}\ y(t) = 0, \quad \sigma > 1, \tag{1}$$

where $a,\ q : [t_0, \infty) \to \mathbb{R} = (-\infty, \infty)$ are continuous functions and $a(t) > 0$ for $t \geq t_0$.

As usual a nontrivial solution of equation (1) is called oscillatory if it has arbitrarily large zeros, otherwise, it is said to be nonoscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

The subject of oscillation for the nonlinear differential equations of second order has received much attention in the last 40 years; see e.g., the survey papers by Kartsatos [6], and Wong [8,9], and also see Grace and Lalli [4] and the references cited therein.

The purpose of this paper is to establish some new criteria for the oscillation of all solutions of equation (1). The obtained results extend and improve the results of Grace [1–3] obtained for equations of type (1) with $\sigma = 1$. Also, the results presented here complement those obtained by Wong and Agarwal [10,11] for equations related to (1). Further, we shall employ the technique used in this paper to establish some new oscillation criteria for equations of the form

$$(a(t)y'(t))' + p(t)|y'(t)|^{\sigma}y(t) + q(t)y(t) = 0, \quad \sigma > 1, \tag{1*}$$

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and the more general equation

$$\left(a(t)|y'(t)|^{\sigma-1}y'(t)\right)' + p(t)|y'(t)|^{\sigma-1}y'(t) + q(t)f(y(t)) = 0, \quad \sigma > 1, \ (1)^{**}$$

where $a,\ p,\ q : [t_0, \infty) \to \mathbb{R},\ f : \mathbb{R} \to \mathbb{R}$ are continuous, $a(t) > 0$ for $t \geq t_0$ and $yf(y) > 0$ for $y \neq 0$.

2. Main Results

We need the following:

Lemma 1 ([5]). *If A and B are nonnegative, then*

$$A^\lambda - \lambda A B^{\lambda-1} + (\lambda - 1)B^\lambda \geq 0, \quad \lambda > 1,$$

and equality holds if and only if $A = B$.

Theorem 1. *Suppose that there exists a differentiable function $\rho : [t_0, \infty) \to (0, \infty)$ such that $\rho'(t) \geq 0$ for $t \geq t_0$, and*

$$\int^\infty \frac{ds}{(a(s)\rho(s))^{1/\sigma}} = \infty. \tag{2}$$

If

$$\int^\infty Q(s)ds = \infty, \tag{3}$$

where

$$Q(t) = \rho(t)q(t) - \frac{1}{(\sigma+1)^{\sigma+1}} \frac{a(t)(\rho'(t))^{\sigma+1}}{\rho^\sigma(t)}, \quad t \geq t_0,$$

then equation (1) *is oscillatory.*

Proof. Let $y(t)$ be a nonoscillatory solution of equation (1), say $y(t) > 0$ for $t \geq t_0$. Define

$$w(t) = \frac{a(t)\rho(t)|y'(t)|^{\sigma-1}y'(t)}{y^\sigma(t)}, \quad t \geq t_0.$$

Then

$$\begin{aligned} w'(t) &= -\rho(t)q(t) + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{\sigma a(t)\rho(t)|y'(t)|^{\sigma-1}y'^2(t)}{y^{\sigma+1}(t)} \\ &\leq -\rho(t)q(t) + \frac{\rho'(t)}{\rho(t)}|w(t)| - \frac{\sigma}{(a(t)\rho(t))^{1/\sigma}}|w(t)|^{(\sigma+1)/\sigma}, \ \ t \geq t_0. \end{aligned} \tag{4}$$

Set

$$A = \left(\frac{\sigma}{(a(t)\rho(t))^{1/\sigma}}\right)^{\sigma/(\sigma+1)} |w(t)|$$

and

$$B = \left[\frac{\sigma}{\sigma+1}\frac{\rho'(t)}{\rho(t)}\left(\frac{\sigma}{(a(t)\rho(t))^{1/\sigma}}\right)^{-\sigma/(\sigma+1)}\right]^\sigma.$$

By applying Lemma 1 in (4), we get

$$w'(t) \le -\rho(t)q(t) + \frac{1}{(\sigma+1)^{\sigma+1}} \frac{a(t)(\rho'(t))^{\sigma+1}}{\rho^{\sigma}(t)} := -Q(t), \quad t \ge t_0. \tag{5}$$

Now we consider the following three cases:

Case 1. Suppose that $y'(t) \ge 0$ for $t \ge t_1 \ge t_0$. Integrating (5) from t_1 to t, we have

$$0 \le \frac{a(t)\rho(t)(y'(t))^{\sigma}}{y^{\sigma}(t)} \le \frac{a(t_1)\rho(t_1)(y'(t_1))^{\sigma}}{y^{\sigma}(t_1)} - \int_{t_1}^{t} Q(s)ds,$$

and by condition (3), we arrive at a contradiction.

Case 2. Suppose that $y'(t)$ oscillates. Then, there exists a sequence $\{T_n\}$, $T_n \to \infty$ as $n \to \infty$ such that $y'(T_n) = 0$. Choose k such that $T_k \ge t_0$. Without loss of generality we assume that $y'(t) > 0$ for $t \in (T_k, T_{k+1})$. Further, in view of (3) we obtain

$$\int_{T_k}^{T_{k+1}} Q(s)ds > 0. \tag{6}$$

Integrating (5) from T_k to T_{k+1}, we have

$$-\int_{T_k}^{T_{k+1}} Q(s)ds \ge \frac{a(t)\rho(t)|(y'(t))^{\sigma-1}|y'(t)}{y^{\sigma}(t)} \Big|_{t=T_k}^{t=T_{k+1}} = 0,$$

which contradicts (6).

Case 3. Suppose that $y'(t) < 0$ for $t \ge t_1 \ge t_0$. Condition (3) implies that there exists a $t_2 \ge t_1$ such that

$$\int_{t_2}^{t} \rho(s)q(s) \ge 0 \quad \text{for} \quad t \ge t_2.$$

Multiplying equation (1) by $\rho(t)$ and integrating from t_2 to t, we obtain

$$\begin{aligned} a(t)\rho(t)|y'(t)|^{\sigma-1}y'(t) &= a(t_2)\rho(t_2)|y'(t_2)|^{\sigma-1}y'(t_2) \\ &+ \int_{t_2}^{t} a(s)\rho'(s)|y'(s)|^{\sigma-1}y'(s)ds - y^{\sigma}(t)\int_{t_2}^{t}\rho(s)q(s)ds \\ &+ \int_{t_2}^{t}\left[\sigma y'(s)y^{\sigma-1}(s)\int_{t_2}^{s}\rho(\tau)q(\tau)d\tau\right]ds, \quad t \ge t_2. \end{aligned}$$

Now it is easy to see that

$$|y'(t)|^{\sigma-1}y'(t) \le \frac{a(t_1)\rho(t_1)}{a(t)\rho(t)}|y'(t_2)|^{\sigma-1}y'(t_2)$$

or

$$y'(t) \leq \left(\frac{a(t_1)\rho(t_1)}{a(t)\rho(t)}\right)^{1/\sigma} y'(t_2), \quad t \geq t_2. \tag{7}$$

Integrating (7) from t_2 to t and using condition (2) we arrive at a contradiction. $\square$

Next, we present the following oscillation criterion for equation (1).

Theorem 2. *Suppose that there exists a differentiable function* $\rho : [t_0, \infty) \to (0, \infty)$ *such that*

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_{t_0}^{t} \Big[(t-s)^m \rho(s) q(s) - \frac{1}{(\sigma+1)^{\sigma+1}} (t-s)^{-m\sigma} a(s)\rho(s) P^{\sigma+1}(t,s)\Big] ds = \infty \tag{8}$$

for some number $m > 1$, *where*

$$P(t,s) = \left|(t-s)^m \frac{\rho'(s)}{\rho(s)} - m(t-s)^{m-1}\right|, \quad t \geq s \geq t_0. \tag{9}$$

Then equation (1) *is oscillatory.*

Proof. Let $y(t)$ be a nonoscillatory solution of equation (1), say $y(t) \neq 0$ for $t \geq t_0$. Define the function $w(t)$ as in the proof of Theorem 1 and obtain (4). Multiplying (4) by $(t-s)^m$ and integrating from t_0 to $t \geq s \geq t_0$, we have

$$\int_{t_0}^{t} (t-s)^m w'(s) ds \leq -\int_{t_0}^{t} (t-s)^m \rho(s) q(s)\, ds + \int_{t_0}^{t} (t-s)^m \frac{\rho'(s)}{\rho(s)} |w(s)|\, ds - \int_{t_0}^{t} (t-s)^m \frac{\sigma}{(a(s)\rho(s))^{1/\sigma}} |w(s)|^{(\sigma+1)/\sigma} ds.$$

Since

$$\int_{t_0}^{t} (t-s)^m w'(s)\, ds = -(t-t_0)^m w(t_0) + \int_{t_0}^{t} m(t-s)^{m-1} w(s)\, ds$$

one can easily obtain

$$\int_{t_0}^{t}(t-s)^m\rho(s)q(s)ds \le (t-t_0)^m w(t_0)$$
$$+\int_{t_0}^{t}\left|(t-s)^m\frac{\rho'(s)}{\rho(s)}-m(t-s)^{m-1}\right||w(s)|ds$$
$$-\int_{t_0}^{t}(t-s)^m\frac{\sigma}{(a(s)\rho(s))^{1/\sigma}}|w(s)|^{(\sigma+1)/\sigma}ds. \tag{10}$$

Now, we let

$$A=\left(\frac{\sigma}{(a(s)\rho(s))^{1/\sigma}}\,(t-s)^m\right)^{\sigma/(\sigma+1)}|w(s)|$$

and

$$B=\left[\left(\frac{\sigma}{\sigma+1}\right)P(t,s)\left(\frac{\sigma}{(a(s)\rho(s))^{1/\sigma}}(t-s)^m\right)^{-\sigma/(\sigma+1)}\right]^{\sigma}.$$

Applying Lemma 1 in (10), we get

$$\frac{1}{t^m}\int_{t_0}^{t}(t-s)^m\rho(s)q(s)ds \le w(t_0)$$
$$+\frac{1}{t^m}\int_{t_0}^{t}\frac{1}{(\sigma+1)^{\sigma+1}}a(s)\rho(s)P^{\sigma+1}(t,s)(t-s)^{-m\sigma}ds. \tag{11}$$

Taking lim sup as $t\to\infty$ in the above inequality, we arrive at a contradiction. □

Finally, we present the following result which extends our earlier work in [2,3].

Theorem 3. *Suppose that the function $\rho(t)$ defined in Theorem 2 is such that*

$$\limsup_{t\to\infty}\frac{1}{t^m}\int_{t_0}^{t}(t-s)^m\rho(s)q(s)\,ds<\infty \tag{12}$$

for some number $m>1$. If there exists a continuous function $\phi:[t_0,\infty)\to\mathbb{R}$ such that for every $T\ge t_0$ and some constant $m>1$

$$\liminf_{t\to\infty}\frac{1}{t^m}\int_{T}^{t}\Big[(t-s)^m\rho(s)q(s)$$
$$-\frac{1}{(\sigma+1)^{\sigma+1}}(t-s)^{-m\sigma}a(s)\rho(s)P^{\sigma+1}(t,s)\Big]ds\ge\phi(T), \tag{13}$$

where the function $P(t,s)$ *is defined in* (9), *and*

$$\int_T^\infty \frac{\phi_+^{\sigma/(\sigma+1)}(s)}{(a(s)\rho(s))^{1/\sigma}}\,ds = \infty, \tag{14}$$

where $\phi_+(t) = \max\{\phi(t), 0\}$, *then equation* (1) *is oscillatory.*

Proof. Let $y(t)$ be a nonoscillatory solution of equation (1) say $y(t) \neq 0$ for $t \geq t_0$. Define the function $w(t)$ as in Theorem 2 and obtain (10) and (11). Then, for $t > T \geq t_0$ we find

$$\liminf_{t\to\infty} \frac{1}{t^m} \int_T^t \Big[(t-s)^m \rho(s) q(s) - \frac{1}{(\sigma+1)^{\sigma+1}}(t-s)^{-m\sigma} a(s)\rho(s)P^{\sigma+1}(t,s)\Big] ds \leq w(T).$$

Therefore, by condition (13) we have

$$\phi(T) \leq w(T) \quad \text{for every} \quad T \geq t_0 \tag{15}$$

and

$$\phi(t_0) \leq \liminf_{t\to\infty} \frac{1}{t^m} \int_{t_0}^t (t-s)^m \rho(s) q(s)\,ds - \liminf_{t\to\infty} \frac{c}{t^m} \int_{t_0}^t (t-s)^{-m\sigma} a(s)\rho(s)P^{\sigma+1}(t,s)\,ds,$$

where $c = 1/(\sigma+1)^{\sigma+1}$, and by condition (12), we get

$$\liminf_{t\to\infty} \frac{c}{t^m} \int_{t_0}^t (t-s)^{-m\sigma} a(s)\rho(s)P^{\sigma+1}(t,s)\,ds < \infty. \tag{16}$$

Next, we define the functions

$$F(t) = \frac{1}{t^m} \int_{t_0}^t P(t,s)|w(s)|\,ds$$

and

$$G(t) = \frac{\sigma}{t^m} \int_{t_0}^t (t-s)^m \frac{1}{(a(s)\rho(s))^{1/\sigma}} |w(s)|^{(\sigma+1)/\sigma} ds.$$

From (10), we obtain for $t \geq t_0$

$$G(t) - F(t) \leq w(t_0) - \frac{1}{t^m} \int_{t_0}^t (t-s)^m \rho(s) q(s)\,ds. \tag{17}$$

Now, condition (12) together with (17) leads to

$$\limsup_{t\to\infty}[G(t)-F(t)] \le w(t_0) - \liminf_{t\to\infty} \frac{1}{t^m}\int_{t_0}^{t}(t-s)^m \rho(s)q(s)\,ds$$
$$\le w(t_0)-\phi(t_0)<\infty. \tag{18}$$

There exists a sequence $\{T_n\}$, $n=1,2,\dots$ in (t_0,∞) with $\lim_{n\to\infty} T_n=\infty$ so that

$$\limsup_{t\to\infty}[G(t)-F(t)] = \lim_{n\to\infty}[G(T_n)-F(T_n)].$$

By (18), there exists a constant C such that

$$G(T_n)-F(T_n)\le C, \quad n=1,2,\dots. \tag{19}$$

We claim that

$$\int_{t_0}^{\infty}\frac{|w(s)|^{\sigma/(\sigma+1)}}{(a(s)\rho(s))^{1/\sigma}}\,ds<\infty. \tag{20}$$

To prove it, suppose that (20) fails. Then there exists a $t_1>t_0$ such that

$$\int_{t_0}^{t}\frac{|w(s)|^{\sigma/(\sigma+1)}}{(a(s)\rho(s))^{1/\sigma}}\,ds \ge \frac{\alpha}{\sigma} \quad \text{for all} \quad t\ge t_1,$$

where α is an arbitrary positive number.

Therefore, for all $t\ge t_1$

$$G(t)=\frac{\sigma}{t^m}\int_{t_0}^{t}(t-s)^m d\Big(\int_{t_0}^{s}\frac{|w(u)|^{(\sigma+1)/\sigma}}{(a(u)\rho(u))^{1/\sigma}}\,du\Big)$$
$$=-\frac{m\sigma}{t^m}\int_{t_0}^{t}(t-s)^{m-1}\Big(\int_{t_0}^{s}\frac{|w(u)|^{(\sigma+1)/\sigma}}{(a(u)\rho(u))^{1/\sigma}}\,du\Big)ds$$
$$\ge-\frac{m\sigma}{t^m}\int_{t_1}^{t}(t-s)^{m-1}\Big(\int_{t_0}^{s}\frac{|w(u)|^{(\sigma+1)/\sigma}}{(a(u)\rho(u))^{1/\sigma}}\,du\Big)ds$$
$$\ge-\frac{m\alpha}{t^m}\int_{t_1}^{t}(t-s)^{m-1}\,ds=\frac{\alpha(t-t_1)^m}{t^m}.$$

There exist a $t_2\ge t_1$ and a positive constant η such that $(1-t_1/t)^m\ge\eta$ for $t\ge t_2$ and hence $G(t)\ge\alpha\eta$ for $t\ge t_2$. Now, since α is arbitrary

$$\lim_{t\to\infty}G(t)=\infty$$

which ensures that

$$\lim_{n\to\infty}G(T_n)=\infty.$$

Hence (19) gives

$$\lim_{n\to\infty} F(T_n) = \infty. \tag{21}$$

From (19), we see that for n large

$$\frac{F(T_n)}{G(T_n)} - 1 \ge -\frac{C}{G(T_n)} > -\frac{1}{2}.$$

Therefore,

$$\frac{F(T_n)}{G(T_n)} > \frac{1}{2} \quad \text{for all large} \quad n,$$

which by (21) ensures that

$$\lim_{n\to\infty} \frac{F^{\sigma+1}(T_n)}{G^{\sigma}(T_n)} = \infty. \tag{22}$$

Next, by Hölder's inequality, we have for every $n = 1, 2, \ldots$

$$\begin{aligned}
F(T_n) &= \frac{1}{T_n^m} \int_{t_0}^{T_n} P(T_n, s)|w(s)|\, ds \\
&= \int_{t_0}^{T_n} \Big(\frac{\sigma^{\sigma/(\sigma+1)}}{T_n^{m\sigma/(\sigma+1)}} \frac{|w(s)|(T_n - s)^{m\sigma/(\sigma+1)}}{(a(s)\rho(s))^{1/(\sigma+1)}} \Big) \\
&\quad \times \Big(\frac{\sigma^{-\sigma/(\sigma+1)}}{T_n^{m/(\sigma+1)}} \frac{P(T_n, s)(a(s)\rho(s))^{1/(\sigma+1)}}{(T_n - s)^{m\sigma/(\sigma+1)}} \Big) ds \\
&\le \Big(\frac{\sigma}{T_n^m} \int_{t_0}^{T_n} \frac{|w(s)|^{(\sigma+1)/\sigma}(T_n - s)^m}{(a(s)\rho(s))^{1/\sigma}}\, ds \Big)^{\sigma/(\sigma+1)} \\
&\quad \times \Big(\frac{\sigma^{-\sigma}}{T_n^m} \int_{t_0}^{T_n} \frac{a(s)\rho(s) P^{\sigma+1}(T_n, s)}{(T_n - s)^{m\sigma}}\, ds \Big)^{1/(\sigma+1)},
\end{aligned}$$

and hence

$$\frac{F^{\sigma+1}(T_n)}{G^{\sigma}(T_n)} \le \frac{\sigma^{-\sigma}}{T_n^m} \int_{t_0}^{T_n} a(s)\rho(s)\, \frac{P^{\sigma+1}(T_n, s)}{(T_n - s)^{m\sigma}}\, ds.$$

From (22), we see that

$$\lim_{n\to\infty} \frac{1}{T_n^m} \int_{t_0}^{T_n} a(s)\rho(s)\, \frac{P^{\sigma+1}(T_n, s)}{(T_n - s)^{m\sigma}}\, ds = \infty,$$

which gives that

$$\lim_{t\to\infty} \frac{1}{t^m} \int_{t_0}^{t} a(s)\rho(s)\, \frac{P^{\sigma+1}(t, s)}{(t - s)^{m\sigma}}\, ds = \infty$$

which contradicts (16) and hence (20) holds. Now, from (15) we get

$$\int_{t_0}^{\infty} \frac{\phi_+^{\sigma/(\sigma+1)}(s)}{(a(s)\rho(s))^{1/\sigma}}\, ds \le \int_{t_0}^{\infty} \frac{|w(s)|^{\sigma/(\sigma+1)}}{(a(s)\rho(s))^{1/\sigma}}\, ds < \infty,$$

which contradicts (14). $\square$

Example 1. Consider the equation

$$\Big(t^{\lambda}|y'(t)|^{\sigma-1}y'(t)\Big)' + \Big(\frac{t\sin t + (2-\cos t)}{t}\Big)|y(t)|^{\sigma}\text{sgn } y(t) = 0, \quad t \ge t_0 \ge 0, \quad (23)$$

where λ and σ are real numbers, $\sigma > 1$. Here, we let $m = 2$ and $\rho(t) = t$. Now condition (8) leads to

$$\frac{1}{t^2}\int_{t_0}^{t}\Big[(t-s)^2 d(s[2-\cos s]) - \frac{s^{\alpha-\sigma}}{(\sigma+1)^{\sigma+1}}\,\frac{(t-3s)^{\sigma+1}}{(t-s)^{\sigma-1}}\Big]ds$$
$$\ge \frac{1}{t^2}\int_{t_0}^{t}\Big[2(t-s)(s-\gamma) - \frac{s^{\alpha-\sigma}(t-s)^2}{(\sigma+1)^{\sigma+1}}\Big]ds$$

for some constant γ.

It is easy to check that condition (8) of Theorem 2 holds if $\lambda < \sigma - 1$, and hence all solutions of equation (23) are oscillatory.

Example 2. Consider the equation

$$\Big(t^{\alpha}|y'(t)|^{\sigma-1}y'(t)\Big)' + (t^{\gamma}\cos t)|y(t)|^{\sigma}\ \text{sgn } y(t) = 0, \quad t \ge t_0 > 0, \qquad (24)$$

where $\alpha,\ \gamma$ and σ are real numbers, $-2 < \gamma \le -1$, $\sigma > 1$ and $\alpha \le \sigma - 1$. Here, we take $m = 2$ and $\rho(t) = t$. Now condition (12) holds and for arbitrarily small costant $\theta > 0$, there exists a $t_1 \ge t_0$ such that for $T \ge t_1$

$$\liminf_{t\to\infty}\frac{1}{t^2}\int_{T}^{t}\Big[(t-s)^2 s^{\gamma+1}\cos s - \frac{s^{\alpha-\sigma}}{(\sigma+1)^{\sigma+1}}\,\frac{(t-3s)^{\sigma+1}}{(t-s)^{\sigma-1}}\Big]ds$$
$$\ge \liminf_{t\to\infty}\frac{1}{t^2}\int_{T}^{t}\Big[(t-s)^2 s^{\gamma+1}\cos s - \frac{1}{(\sigma+1)^{\sigma+1}}s^{\alpha-\sigma}(t-s)^2\Big]ds$$
$$\ge -T^{\gamma+1}\sin T - \theta.$$

Set $\phi(T) = -T^{\gamma+1}\sin T - \theta$. Then, there exists an integer N such that $(2N + 1)\pi - \pi/4 > t_1$ and if $n \ge N$

$$(2n+1)\pi - \frac{\pi}{4} \le T \le (2n+1)\pi + \frac{\pi}{4}, \quad \phi(T) \ge cT^{\gamma+1},$$

where c is a small constant. Now

$$\int_{t_0}^{\infty} \frac{\phi_+^{\sigma/(\sigma+1)}(s)}{(a(s)\rho(s))^{1/\sigma}} ds \geq \sum_{n=N}^{\infty} c^{\sigma/(\sigma+1)} \int_{(2n+1)\pi-\pi/4}^{(2n+1)\pi+\pi/4} s^{\beta}\, ds,$$

where $\beta = (\gamma+1)\frac{\sigma}{\sigma+1} - \left(\frac{\alpha+1}{\sigma}\right)$. If $\beta \geq -1$ we see that

$$\int_{t_0}^{\infty} \frac{\phi_+^{\sigma/(\sigma+1)}(s)}{(a(s)\rho(s))^{1/\sigma}} ds = \infty$$

and hence by Theorem 3, equation (24) is oscillatory.

3. Some Extentions

The following results deal with the oscillation of all unbounded solutions of equation $(1)^*$.

Theorem 4. *Let $p(t) > 0$ for $t \geq t_0$. If there exists a differentiable function $\rho : [t_0, \infty) \to (0, \infty)$ such that for $t \geq t_1 \geq t_0$, and some constant $m > 1$*

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_{t_1}^{t} (t-s)^m Q(s)\, ds = \infty \tag{25}$$

and

$$\lim_{t\to\infty} \frac{1}{t^m} \int_{t_1}^{t} (t-s)^{((m-1)\sigma-m)/(\sigma-1)} P^{1/(1-\sigma)}(s)\, ds < \infty, \tag{26}$$

where

$$Q(t) = \rho(t)q(t) - \frac{a(t)(\rho'(t))^2}{4\rho(t)} \quad \text{and} \quad P(t) = \frac{p(t)}{a^{\sigma}(t)\rho^{\sigma-1}(t)},$$

then every unbounded solution of equation $(1)^*$ is oscillatory.

Proof. Let $y(t)$ be an unbounded and nonoscillatory solution of equation $(1)^*$. Without loss of generality, we assume that $y(t) \neq 0$ for $t \geq t_0$. Define

$$w(t) = \frac{a(t)\rho(t)y'(t)}{y(t)}, \quad t \geq t_0. \tag{27}$$

Then

$$\begin{aligned} w'(t) &= -\rho(t)q(t) - p(t)\rho(t)|y'(t)|^{\sigma} + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{1}{a(t)\rho(t)}w^2(t) \\ &= -Q(t) - p(t)\rho(t)|y'(t)|^{\sigma} - \left[\frac{1}{\sqrt{a(t)\rho(t)}}w(t) - \frac{\sqrt{a(t)\rho(t)}}{2\rho(t)}\rho'(t)\right]^2 \\ &\leq -Q(t) - P(t)|y(t)|^{\sigma}|w(t)|^{\sigma} \quad \text{for} \quad t \geq t_0. \end{aligned}$$

Since $y(t)$ is unbounded, there exist a $t_1 \geq t_0$ and a positive constant c such that $|y(t)|^\sigma \geq c$ for $t \geq t_1$. Thus, the above inequality takes the form

$$w'(t) \leq -Q(t) - cP(t)|w(t)|^\sigma \quad \text{for} \quad t \geq t_1. \tag{28}$$

The rest of the proof is similar to that of Theorem 2. □

Remark 1. Conditions (25) and (26) can be combined in the form

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_{t_1}^{t} \Big[(t-s)^m Q(s) - k(t-s)^{((m-1)\sigma - m)/(\lambda-1)} P^{1/(1-\lambda)}(s)\Big] ds = \infty$$

for every constant $k > 0$ and some constant $m > 1$.

If condition (25) fails, we have the following criterion:

Theorem 5. *Suppose that*

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_{t_1}^{t} (t-s)^m Q(s) ds < \infty$$

for some constant $m > 1$. If there exists a continuous function $\phi : [t_0, \infty) \to \mathbb{R}$ such that for every constant $k > 0$, every $T \geq t_1 \geq t_0$ and some constant $m > 1$

$$\liminf_{t\to\infty} \frac{1}{t^m} \int_{T}^{t} \Big[(t-s)^m Q(s) - k(t-s)^{((m-1)\sigma - m)/(\sigma-1)} P^{1/(1-\lambda)}(s)\Big] ds \geq \phi(T),$$

where the functions P and Q are as in Theorem 4, and

$$\int_{T}^{\infty} P(s)\phi_+^{1/\sigma}(s)\, ds = \infty,$$

where $\phi_+(s) = \max\{\phi(s), 0\}$, then every unbounded solution of equation $(1)^*$ is oscillatory.

Proof. Let $y(t)$ be a nonoscillatory solution of equation $(1)^*$ and assume that $y(t) \neq 0$ for $t \geq t_0$. As in the proof of Theorem 4, we define the function $w(t)$ as in (27) and obtain (28) for $t \geq t_1$. The rest of the proof is similar to that of Theorem 3. □

Finally, we consider equation $(1)^{**}$ and establish the following result.

Theorem 6. *Let*

$$\frac{f'(y)}{|f^{(\sigma-1)/\sigma}(y)|} \geq k > 0 \quad \textit{for} \quad y \neq 0.$$

If there exists a differentiable function $\rho : [t_0, \infty) \to (0, \infty)$ such that

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_{t_0}^{t} \Big[(t-s)^m \rho(s) q(s) - \gamma(t-s)^{-m\sigma} a(s)\rho(s) P_1^{\sigma+1}(t,s)\Big] ds = \infty$$

for some constant $m > 1$, where $\gamma = \sigma^\sigma/(k^\sigma(\sigma+1)^{\sigma+1})$ and

$$P_1(t,s) = \left|(t-s)^m\left(\frac{\rho'(s)}{\rho(s)} - \frac{\rho(s)p(s)}{a(s)}\right) - m(t-s)^{m-1}\right| \quad \text{for} \quad t \geq s \geq t_0,$$

then equation $(1)^{**}$ is oscillatory.

Proof. Let $y(t)$ be a nonoscillatory solution of equation $(1)^{**}$, say $y(t) \neq 0$ for $t \geq t_0$. Define the function $w(t)$ as in Theorem 2, and obtain

$$\begin{aligned} w'(t) = -\rho(t)q(t) &+ \left(\frac{\rho'(t)}{\rho(t)} - \frac{P(t)\rho(t)}{a(t)}\right)w(t) \\ &- a(t)\rho(t)\frac{f'(y(t))|y'(t)|^{\sigma-1}(y'(t))^2}{f^2(y(t))} \\ \leq -\rho(t)q(t) &+ \left(\frac{\rho'(t)}{\rho(t)} - \frac{P(t)\rho(t)}{a(t)}\right)w(t) \\ &- \frac{k}{(a(t)\rho(t))^{1/\sigma}}|w(t)|^{(\sigma+1)/\sigma} \quad \text{for} \quad t \geq t_0. \end{aligned}$$

The rest of the proof is similar to that of Theorem 2. □

Remark 2. Oscillation criterion similar to Theorem 3 can be stated for equation $(1)^{**}$.

Remark 3. As in [3] and [7] we can replace $(t-s)^m$ for some constant $m > 1$ by the function $H : D = \{(t,s) : t \geq s \geq t_0\}$ such that $H(t,t) = 0$ for $t \geq t_0$, $H(t,s) > 0$ for $(t,s) \in D$ and

$$h(t,s) := -\frac{\partial H(t,s)}{\partial s}$$

is a nonnegative continuous function on D.

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Authors' addresses:

R. P. Agarwal
Department of Mathematics
National University of Singapore
10 Kent Ridge Crescent, 119260
Singapore
E-mail: matravip@nus.edu.sg

S. R. Grace
Department of Engineering Mathematics
Faculty of Engineering Cairo University
Orman, Giza 12221
Egypt
E-mail: srgrace@alpha1-eng.cairo.eun.eg