Georgian Mathematical Journal
Volume 7 (2000), Number 1, 97-116

QUASI-LINEARISATION METHODS FOR A NON-LINEAR HEAT EQUATION WITH FUNCTIONAL DEPENDENCE

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ABSTRACT. We consider a heat equation with the non-linear right-hand side which depends on certain Volterra-type functionals acting on the unknown function and on its gradient. We give some natural sufficient conditions for the existence and uniqueness of solutions to this equation. The solution is obtained as a limit of a fast convergent sequence of successive approximations obtained by the quasi-linearisation method.

1. INTRODUCTION

Suppose $a > 0$, $\tau_0, \tau_1, \dots, \tau_n \in R_+$ and $[-\tau, \tau] = [-\tau_1, \tau_1] \times \dots \times [-\tau_n, \tau_n]$. Define $E = [0, a] \times R^n$, $E_0 = [-\tau_0, 0] \times R^n$ and $B = [-\tau_0, 0] \times [-\tau, \tau]$. If $u : E_0 \cup E \to R$ and $(t, x) \in E$, then we define the Hale-type functional $u_{(t,x)} : B \to R$ by $u_{(t,x)}(s, y) = u(t+s, x+y)$ for $(s, y) \in B$. If $U = (u_1, \dots, u_n) : E_0 \cup E \to R^n$, then $U_{(t,x)} = \Big((u_1)_{(t,x)}, \dots, (u_n)_{(t,x)}\Big)$. Denote by Δ the Laplacian, i.e., $\Delta = D_{x_1x_1} + \dots + D_{x_nx_n}$. Denote by D_x the gradient operator $(D_{x_1}, \dots, D_{x_n})$. Given

$$f : \Omega := E \times R \times C(B, R) \times R^n \times C(B, R^n) \to R \quad \text{and} \quad \phi : E_0 \to R,$$

we consider the Cauchy problem

$$D_t u(t, x) = \Delta u(t, x) + f\Big(t, x, u(t, x), u_{(t,x)}, D_x u(t, x), (D_x u)_{(t,x)}\Big) \quad \text{for} \quad (t, x) \in E \setminus \{0\} \times R^n, \tag{1}$$

$$u(t, x) = \phi(t, x) \qquad \text{for} \quad (t, x) \in E_0\,. \tag{2}$$

After [1] and [2] we define the Green function $H : R^{1+n} \to R$ by

$$H(t, x) = \begin{cases} \frac{1}{(2\sqrt{\pi t})^n} \exp\Big(-\frac{\|x\|^2}{4t}\Big) & \text{for } (t, x) \in (0, \infty) \times R^n, \\ 0 & \text{for } (t, x) \in (-\infty, 0] \times R^n, \end{cases}$$

where $\|x\| = \sqrt{x_1^2 + \dots + x_n^2}$. Define Nemytskii's operator

$$f[u](t, x) := f\Big(t, x, u(t, x), u_{(t,x)}, D_x u(t, x), (D_x u)_{(t,x)}\Big)\,.$$

2000 *Mathematics Subject Classification.* Primary 35K05(10), Secondaries 35A05, 35K15.
Key words and phrases. Quasi-linearisation, iterative method, integral equation.

ISSN 1072-947X / \$8.00 /

Next, taking into consideration $D_x u$, we obtain a system of integral-functional equations which turn out to be equivalent to the differential-functional problem (1), (2).

$$u(t,x) = \mathcal{L}_0[u](t,x), \tag{3}$$

$$D_x u(t,x) = \mathcal{L}'[u](t,x) \tag{4}$$

for $(t,x) \in E_0 \cup E$, where $\mathcal{L}' = (\mathcal{L}_1, \dots, \mathcal{L}_n)$, and

$$\begin{aligned} \mathcal{L}_0[u](t,x) :&= \int_{R^n} H(t, x-y)\phi(0,y)\,dy \\ &+ \int_0^t \int_{R^n} H(t-s, x-y) f[u](s,y)\,dy\,ds, \end{aligned} \tag{5}$$

$$\begin{aligned} \mathcal{L}_i[u](t,x) :&= \int_{R^n} H(t, x-y) D_{y_i}\phi(0,y)\,dy \\ &+ \int_0^t \int_{R^n} D_{x_i} H(t-s, x-y) f[u](s,y)\,dy\,ds \end{aligned} \tag{6}$$

for $(t,x) \in E^+ := E \backslash \{0\} \times R^n$, $(i = 1, \dots, n)$, and

$$\mathcal{L}_0[u](t,x) := \phi(t,x), \quad \mathcal{L}_i[u](t,x) := D_{x_i}\phi(t,x) \quad (i = 1, \dots, n),$$

for $(t,x) \in E_0$.

We are interested in the so-called $C^{0,1}$ solutions to (1), (2), i.e., $u \in C(E_0 \cup E, R)$ satisfying (2) in E_0, and (3) in E with the continuous gradient $D_x u$. In [3] we obtained some existence results by means of the Banach contraction principle and discussed the question of their continuous differentiability in the set E^+, which leads to classical solutions to the Cauchy problem. The crucial condition is as follows: there exist continuous partial derivatives of $f(t,x,p,w,P,W)$ with respect to x, p, w, P, W. This condition is also inevitable in the Chaplygin's method. In fact, the limit of the Chaplygin's sequence of $C^{0,1}$ solutions is nearly a classical solution to (1), (2) at least for a sufficiently small interval $(0,a]$ and under natural constraints on f and $D_x f$.

Many important concepts in the existence theory for parabolic and first-order equations are due to Prof. J. Szarski [4] whose theorems on differential and differential-functional inequalities are applied as an important tool in iterative techniques, in particular, in the Chaplygin's method.

The original Chaplygin's method starts from a pair of lower and upper solutions (say, u^0 and $\bar{u}^0$), which is assumed to be given, and produces two monotone sequences of lower and upper solutions converging to a function that is a unique solution of the differential equation between u^0 and $\bar{u}^0$. This method is very well investigated by many authors. We cite some of them: parabolic equations are considered in [5]–[9], first-order PDEs with functional dependence are analysed in [10]. Our approach is also based on the concept of quasi-linearisation, but there are some differences. Firstly, we obtain fast convergent sequences in both the bounded and the unbounded case. Secondly, we include the functional dependence on the variables which represent the gradient of the solution. Thirdly, the existence at each stage of the classical Chaplygin's method is deduced from the existence theory for linear (or quasi-linear equations), see [1], and next it becomes a monotone iterative method, cf. [12,13]. However we are strongly convinced that one cannot directly introduce any monotone technique in our Chaplygin's iterations, because of the functional dependence on the gradient.

Fast convergent numerical sequences apply to numerical computations in the following way. Linear differential or differential-functional problems appear at each stage of the quasilinearization method. Numerical libraries such as NAG (Numerical Analysis Group) contain many standard procedures for linear problems. Approximate solutions can be somehow interpolated in order to continue the iterative process. Another useful technicue is the theory of semigroups [14], provided that the linear operators generate analytic semigroups.

In [15] we prove a convergence theorem for finite difference schemes that approximate unbouded solutions to parabolic differential-functional dependences by means of a comparison lemma, which is possible in the absence of functionals acting on partial derivatives. Nevertheless, there were some technical problems. The present paper shows new ways to solve parabolic equations with more complex functional dependences such as delays and Volterra-type integrals.

2. Existence and Convergence Results

We will use the symbol C_B to indicate classes of bounded and continuous functions. We denote the supremum norm by $\|\cdot\|_0$. Write

$$\mathcal{X}[\phi] := \left\{ (u,U) \in C_B\left(E_0 \cup E, R^{1+n}\right) \,\Big|\, \left(u_{|_{E_0}}, U_{|_{E_0}}\right) = (\phi, D_x\phi) \right\}.$$

For all $u, v \in C(E_0 \cup E, R)$ and $f : \Omega \to R$ of the variables (t,x,p,w,P,W) with $P = (p_1, \dots, p_n)$ and $W = (w_1, \dots, w_n)$, we denote

$$\begin{aligned}\mathcal{D}f[u;v](t,x) &:= D_p f[u](t,x)\, v(t,x) + D_w f[u](t,x) v_{(t,x)} \\ &+ \sum_{i=1}^{n} \left\{ D_{p_i} f[u](t,x) D_{x_i} v(t,x) + D_{w_i} f[u](t,x) \left(D_{x_i} v\right)_{(t,x)} \right\}.\end{aligned}$$

We define the quasi-linearisation method which starts from a function $u^0 \in C(E_0 \cup E, R)$ such that $(u^0, D_x u^0) \in \mathcal{X}[\phi]$, and, having computed u^ν such that $(u^\nu, D_x u^\nu) \in \mathcal{X}[\phi]$, we define $u^{\nu+1}$ as the solution to the problem

$$D_t u(t,x) = \Delta u(t,x) + f[u^\nu](t,x) + \mathcal{D}f[u^\nu; u - u^\nu](t,x)\,, \tag{7}$$

$$u(t,x) = \phi(t,x) \quad \text{on} \quad E_0. \tag{8}$$

Observe that the right-hand side of (7) consists of first-order terms in the Taylor formula applied to $f[u](t,x)$.

First, we cite after [3] the following existence theorem whose aim is the $C^{0,1}$ solvability of (7), (8) at each stage of the Chaplygin's method.

Theorem 1 ([3]). *Assume that:*

1°. $\phi \in C_B(E_0, R)$, $D_x\phi = (D_{x_1}\phi, \ldots, D_{x_n}\phi) \in C_B(E_0, R^n)$, $f[0] \in C_B(E, R)$.

2°. *There are* $L_i, L_i' \in R_+$ *for* $i = 0, \ldots, n$ *such that*

$$\Big| f(t,x,p,w,P,W) - f(t,x,\bar{p},\bar{w},\bar{P},\bar{W}) \Big| \\ \le L_0 |p-\bar{p}| + L_0' \|w-\bar{w}\|_0 + \sum_{i=1}^{n} \Big(L_i\, |p_i - \bar{p}_i| + L_i' \|w_i - \bar{w}_i\|_0 \Big)$$

for (t,x,p,w,P,W), $(t,x,\bar{p},\bar{w},\bar{P},\bar{W}) \in \Omega$, *where* $P = (p_1, \ldots, p_n)$ *and* $W = (w_1, \ldots, w_n)$.

Then there exists a function $u \in C_B(E_0 \cup E, R)$ *which is a* $C^{0,1}$ *solution to problem* (1), (2).

In fact, the above theorem is a slight modification (strenghthening) of the original version. In the present paper we are interested also how much every single component may influence the properties of the sequence in terms of the maximum norm.

Any function $\sigma : [0, +\infty) \to [0, +\infty)$ will be called a modulus of continuity when it is nondecreasing and $\sigma(t) \to 0$ as $t \to 0$. It is said to be a modulus of continuity for a real (or vector) function g if $\|g(Q) - g(\bar{Q})\| \le \sigma\Big(\|Q - \bar{Q}\|\Big)$ for all Q, $\bar{Q}$ from the domain of g. In particular, the Lipschitz condition with a constant $L \ge 0$ is represented by $\rho(t) = L\,t$, whereas the Hölder condition by $\rho(t) = L\,t^{\kappa}$, $\kappa \in (0,1]$.

Put

$$v^{\nu} = u^{\nu+1} - u^{\nu}, \quad V^{\nu} = (v_1^{\nu}, \ldots, v_n^{\nu}) = D_x v^{\nu} \quad \text{for } \nu = 0, 1, \ldots .$$

Denote by L, L' and $\tilde{L}$ the constants:

$$L := L_0 + L_0', \quad L' := \sum_{i=1}^{n} (L_i + L_i'), \quad \tilde{L} := \sum_{i,j=0}^{n} \sum_{k=0}^{3} L_{ij}^{k}\,. \tag{9}$$

Denote $p_0 := p$, $w_0 := w$. This notation simplifies the formulation of our fundamental assumption

Assumption 1.

1°. There are the derivatives $D_{p_i} f$ and $D_{w_i} f$ defined in Ω for $i = 0, \ldots, n$, and they are continuous.

2°. There are L_i, $L_i' \in R_+$ and $L_{ij}^k \in R_+$ for $i, j = 1, \ldots, n$, $k = 0, 1, 2, 3$, such that

$$|D_{p_i} f(Q)| \le L_i, \quad \|D_{w_i} f(Q)\|_0 \le L_i',$$

$$|D_{p_i} f(Q) - D_{p_i} f(\bar{Q})| \le \sum_{j=0}^{n} \Big\{ L_{ij}^0\, \sigma\,(t; |p_j - \bar{p}_j|) + L_{ij}^1\, \sigma\,(t; \|w_j - \bar{w}_j\|_0) \Big\},$$

$$\|D_{w_i} f(Q) - D_{w_i} f(\bar{Q})\|_0 \le \sum_{j=0}^{n} \Big\{ L_{ij}^2\, \sigma\,(t; |p_j - \bar{p}_j|) + L_{ij}^3\, \sigma\,(t; \|w_j - \bar{w}_j\|_0) \Big\}$$

for $i=0,\dots,n$, $Q,\bar{Q}\in\Omega$, where $Q=(t,x,p,w,P,W)$, $\bar{Q}=(t,x,\bar{p},\bar{w},\bar{P},\bar{W})$, and $\sigma(t;\cdot)$ is a family of moduli of continuity for $t\in(0,a]$.

3°. $\sigma(\cdot;r)$ are continuous on $(0,a]$, and the functions σ_0, $\sigma_1 : [0,+\infty)\to[0,+\infty)$ defined by

$$\sigma_0(r) := \int_0^a \sigma(s;r)ds, \quad \sigma_1(r) \coloneqq \sup_{t\in(0,a]} \int_0^t \frac{1}{\sqrt{t-s}}\,\sigma(s;r)ds \quad \big(r\in[0,+\infty)\big)$$

are moduli of continuity.

4°. The following estimates hold:

$$1 > \max\left\{a,\ \frac{2\sqrt{a}}{\sqrt{\pi}}\right\}(L+L'), \tag{10}$$

$$1 > \frac{\tilde{L}}{1-aL-\frac{2\sqrt{a}}{\sqrt{\pi}}\,L'} \int_0^1 \left\{\sigma_0\left(\zeta\,\|(v^0,V^0)\|_0\right)\left(1-\frac{2\sqrt{a}}{\sqrt{\pi}}\,L'\right) + \sigma_1\left(\zeta\,\|(v^0,V^0)\|_0\right)\frac{1-aL}{\sqrt{\pi}}\right\} d\zeta. \tag{11}$$

5°. $\phi\in C_B(E_0,R)$, $D_x\phi=(D_{x_1}\phi,\dots,D_{x_n}\phi)\in C_B(E_0,R^n)$, $f[0]\in C_B(E,R)$.

Note that inequalities (10) and (11) are satisfied most frequently subject to a sufficient decrease in length of the interval $[0,a]$, with some exceptions which will be commented in a remark.

Theorem 2. *Suppose that Assumption* 1 *is satisfied, then the Chaplygin's sequence* u^ν *defined by* (7), (8) *is well defined, and* (u^ν, D_xu^ν) *are uniformly convergent to* $(u^*, D_xu^*)\in\mathcal{X}[\phi]$, *and* u^* *is a* $C^{0,1}$ *solution to problem* (1), (2). *Moreover, there is* $C\in R_+$ *and* $\{\gamma^\nu\}_\nu$ *such that*

$$\|\left(u^{\nu+1}-u^*,\ D_xu^{\nu+1}-D_xu^*\right)\|_0 \le C\,\gamma^\nu \to 0 \quad \textit{and}$$
$$\gamma^{\nu+1}/\gamma^\nu \le C\ \left(\sigma_0\left(\gamma^\nu\right)+\sigma_1\left(\gamma^\nu\right)\right)\to 0.$$

Corollary 1. *Suppose that the assumptions of Theorem* 2 *are satisfied. Assume that there is* $\alpha\in(0,1]$ *such that*

$$\sqrt{\pi\,a}\sum_{i=1}^n\left\{L_i+\frac{L_i'}{\alpha}\right\}<1$$

and

$$\|D_{w_j}f(Q)\,(u_j)_{(t,x)}\|_0 \le L_j' \max_{\alpha t\le\xi\le t}\|u_j(\xi,\cdot)\|_0 \quad (j=1,\dots,n)$$

for $Q\in\Omega$ *and* $U=(u_1,\dots,u_n)\in C_B\left(E_0\cup E,R^n\right)$ *and* $(t,x)\in E^+$. *If* D_xf *is continuous and bounded in* Ω, *then* u^* *is a classical solution to the Cauchy problem.*

3. Existence and Convergence Results. Unbounded Solutions

Denote by $C_+([-\tau_0,a],R_+)$ the class of all continuous and nondecreasing functions from $[-\tau_0,a]$ into R_+. Given $\psi\in C_+([-\tau_0,a],R_+)$, we denote by $\|\cdot\|_{2,\psi}$ and $\|\cdot\|'_{2,\psi}$ the norms

$$\|z\|_{2,\psi}=\sup\Big\{|z(t,x)|\exp\Big(-\psi(t)\|x\|^2\Big)\,|(t,x)\in E_0\cup E\Big\},$$
$$\|Z\|'_{2,\psi}=\max_{i=1,\dots,n}\sup\Big\{|z_i(t,x)|\Big(1+|x_i|\Big)^{-1}$$
$$\times\exp\Big(-\psi(t)\|x\|^2\Big)\,|(t,x)\in E_0\cup E\Big\},$$

where $Z=(z_1,\dots,z_n)$. Define the spaces

$$E_{2,\psi}=\{z\in C(E_0\cup E,R)\mid\ \ \|z\|_{2,\psi}<+\infty\},$$
$$E'_{2,\psi}=\Big\{Z\in C(E_0\cup E,R^n)\Big|\ \ \|Z\|'_{2,\psi}<+\infty\Big\},$$

where $\psi\in C_+([-\tau_0,a],R_+)$. We write after [3] the fundamental assumption and the theorem in this section which is a slight modification of the theorem from this source. It guarantees the existence of unbounded $C^{0,1}$ solutions at each stage of the Chaplygin's method.

Assumption 2 ([3]). Suppose that

1°. $\psi\in C_+([-\tau_0,a],R_+)$, $f\in C(\Omega,R)$, and there are $L_0,L_1,L_2,L_3\in R_+$ such that

$$\Big|f(t,x,p,w,P,W)-f(t,x,\bar p,\bar w,\bar P,\bar W)\Big|\le L_0|p-\bar p|+L_1e^{-K(s)\circ|x|}\|w-\bar w\|_0$$
$$+L_2\sum_{j=1}^n\frac{|p_j-\bar p_j|}{1+|x_j|}+L_3\sum_{j=1}^n e^{-K(t)\circ|x|}\,\frac{\|w_j-\bar w_j\|_0}{1+|x_j|}$$

for (t,x,p,w,P,W), $(t,x,\bar p,\bar w,\bar P,\bar W)\in\Omega$, where $K=(K_1,\dots,K_n)\in C\Big([-\tau_0,a],R^n_+\Big)$ is such that $K_j(t)\ge 2\psi(t)|\tau_j|$ for $t\in[0,a]$, $j=1,\dots,n$.

2°. $\psi\in C_+([-\tau_0,a],R_+)$ satisfies the inequality

$$\psi(t)\ge\frac{\psi(s)}{1-4(t-s)\psi(s)}\quad\text{for}\quad 0\le s\le t\le a.$$

4°. There are M_ϕ, M'_ϕ, $M_f\in R_+$ such that

$$|\phi(t,x)|\le M_\phi\,\exp\Big(\psi(0)\|x\|^2\Big)\quad for\quad (t,x)\in E_0,\tag{12}$$
$$|D_{x_i}\phi(t,x)|\le M'_\phi(1+|x_i|)\exp\Big(\psi(0)\|x\|^2\Big)$$
$$\text{for}\ \ (t,x)\in E_0,\ \ i=1,\dots,n,\tag{13}$$

and

$$|f[0](t,x)|\le M_f\,\exp\Big(\psi(t)\|x\|^2\Big)\quad\text{for}\quad (t,x)\in E.\tag{14}$$

Theorem 3 ([3]). *Suppose that Assumption 2 is satisfied. Then there is $u \in E_{2,\psi}$ such that $(u, D_x u)$ is a unique fixed point of the operator $\mathcal{L} = (\mathcal{L}_0, \mathcal{L}')$ in the class $E_{2,\psi} \times E'_{2,\psi}$. This function is a $C^{0,1}$ solution to problem* (1), (2).

Set

$$\mathcal{X}_{2,\psi}[\phi] := \Big\{(u, U) \in E_{2,\psi} \times E'_{2,\psi} \,\Big|\, \Big(u_{|E_0}, U_{|E_0}\Big) = (\phi, D_x\phi)\Big\},$$

and define by means of (7), (8) the quasi-linearisation method which starts from a function u^0 such that $(u^0, D_x u^0) \in \mathcal{X}_{2,\psi}[\phi]$. Put

$$L(t) = L_0 + L'_0 e^{\psi(t)\|\tau\|^2}, \quad L'(t) = \sum_{i=1}^{n}(1+\tau_i)\left(L_i + L'_i e^{\psi(t)\|\tau\|^2}\right),$$

$$\tilde{L}(t) = \sum_{i,j=0}^{n}(1+\delta_{0,i}\tau_i)(1+\delta_{0,j}\tau_j)^{\kappa}\Big[L^0_{ij} + L^1_{ij}e^{\kappa\psi(s)\|\tau\|^2} + L^2_{ij}e^{\psi(s)\|\tau\|^2} + L^3_{ij}e^{(1+\kappa)\psi(s)\|\tau\|^2}\Big],$$

$$c_\beta = 2\int_0^1 (1-\zeta^2)^\beta \, d\zeta,$$

$$C(t) = \max\left\{2t\frac{(\psi(t))^{(n+2)/2}}{(\psi(0))^{n/2}}, \frac{2\sqrt{t}}{\sqrt{\pi}}\left(\frac{\psi(t)}{\psi(0)}\right)^{(n+1)/2}\right\},$$

$$\tilde{C}(t) = \max\left\{\frac{2\,t^{\beta+1}}{\beta+1}\frac{(\psi(t))^{(n+2)/2}}{(\psi(0))^{n/2}}, \frac{t^{\beta+1/2}\,c_\beta}{\sqrt{\pi}}\left(\frac{\psi(t)}{\psi(0)}\right)^{(n+1)/2}\right\},$$

for $t \in (0, a]$ and $\beta \geq -1/2$. Define also $D(a)$ by

$$D(a) = \frac{\tilde{L}(a)\{\frac{a^{\beta+1}}{\beta+1}(1-C(a)L(a)) + \tilde{C}(a)(1-L(a)a(\frac{\psi(a)}{\psi(0)})^{n/2})\}}{(\kappa+1)\{1 - L(a)a(\frac{\psi(a)}{\psi(0)})^{n/2} - L'(a)C(a)\}}. \tag{15}$$

Assumption 3. Suppose that condition 1° of Assumption 1 is satisfied, and:

1°. $\psi \in C_+([-\tau_0, a], R_+)$, $f \in C(\Omega, R)$, and there are L_i, $L'_i \in R_+$ and $L^k_{ij} \in R_+$ for $i, j = 1, \ldots, n$, $k = 0, 1, 2, 3$ such that

$$|D_{p_i}f(Q)| \leq L_i/(1+|x_i|), \quad \|D_{w_i}f(Q)\|_0 \leq L'_i e^{-K(t)\circ|x|}/(1+|x_i|),$$

$$|D_{p_i}f(Q) - D_{p_i}f(\bar{Q})| \leq \sum_{j=0}^{n} e^{-\kappa\psi(t)\|x\|^2} \times \Big\{L^0_{ij}\sigma\,(t; |p_j - \bar{p}_j|)\,/((1+|x_i|)(1+|x_j|)^\kappa) + L^1_{ij}\sigma\,(t; \|w_j - \bar{w}_j\|_0)\,e^{-\kappa\,K(t)\circ|x|}/((1+|x_i|)\,(1+|x_j|)^\kappa)\Big\},$$

$$\|D_{w_i}f(Q) - D_{w_i}f(\bar{Q})\|_0 \leq \sum_{j=0}^{n} e^{-\kappa\psi(t)\|x\|^2} \times \Big\{L^2_{ij}\,\sigma\,(t; |p_j - \bar{p}_j|)\,e^{-K(t)\circ|x|}/((1+|x_i|)(1+|x_j|)^\kappa) + L^3_{ij}\sigma\,(t; \|w_j - \bar{w}_j\|_0)\,e^{-(1+\kappa)K(t)\circ|x|}/((1+|x_i|)(1+|x_j|)^\kappa)\Big\},$$

$i=0,\dots,n$, $Q,\ \bar{Q}\in\Omega$, where $Q=(t,x,p,w,P,W)$, $\bar{Q}=(t,x,\bar{p},\bar{w},\bar{P},\bar{W})$, $x_0:=0$, and

$$\sigma(t;r) := r^{\kappa}t^{\beta} \ \text{ for } \ t\in(0,a] \ \text{ and } \ r\in[0,+\infty) \ \text{ with some } \ \kappa\in(0,1] \ \text{ and } \ \beta\geq -1/2;$$

and $K=(K_1,\dots,K_n)\in C([0,a],R_+^n)$ is such that

$$K_i(t)\geq 2\psi(t)\,\tau_i \quad \text{for} \quad t\in[0,a],\ \ i=1,\dots,n.$$

2°. $\psi\in C_+([-\tau_0,a],R_+)$ satisfies the inequality

$$\psi(t)\geq \frac{\psi(s)}{1-4(t-s)\psi(s)} \quad \text{for} \quad 0\leq s\leq t\leq a.$$

3°. $\psi(a)<1/(4a)$ and

$$1>\Big\{L(a),\,L'(a)\Big\}\times\max\Big\{C(a),\,a\left(\frac{\psi(a)}{\psi(0)}\right)^{n/2}\Big\},$$
$$1>\Big(\max\Big\{\|v^0\|_{2,\psi},\,\|V^0\|'_{2,\psi}\Big\}\Big)^{\kappa}\,D(a).$$

4°. There are $M_\phi,\ M'_\phi,\ M_f\in R_+$ such that the initial data obey conditions (12), (13), and (14).

Theorem 4. *Suppose that Assumption* 3 *is satisfied. then the Chaplygin's sequence* u^ν *defined by* (7), (8) *is well defined, and the sequence* (u^ν,D_xu^ν) *is almost uniformly convergent to* $(u^*,D_xu^*)\in\mathcal{X}_{2,\psi}[\phi]$*, and* u^* *is a* $C^{0,1}$ *solution to problem* (1), (2)*. This convergence is uniform with respect to the norm in* $\mathcal{X}_{2,\psi}[\phi]$*, and*

$$\max\Big\{\|v^\nu\|_{2,\psi},\|V^\nu\|'_{2,\psi}\Big\}$$
$$\leq\Big(\max\Big\{\|v^0\|_{2,\psi},\|V^0\|'_{2,\psi}\Big\}\Big)^{(1+\kappa)^\nu}(D(a))^{\frac{(1+\kappa)^\nu-1}{\kappa}}\to 0,$$

where $D(a)$ *is given by* (15).

Assumption 4. Suppose that:

1°. There is $D_xf(Q)$ for $Q\in\Omega$, and $D_{x_j}f[u]\in E_{2,\psi}$.

2°. There is $\alpha\in(0,a]$ such that

$$\|D_{w_j}f(Q)(u_j)_{(t,x)}\|_0$$
$$L'_j\frac{e^{-K(t)\circ|x|}}{1+|x_j|}\max_{\alpha t\leq\xi\leq t}\max_{|\eta-x|\leq\tau}|u_j(\xi,\eta)| \ \ (j=1,\dots,n)$$

for $Q\in\Omega$ and $U\in C_B(E_0\cup E,R^n)$ and $(t,x)\in E^+$.

3°. $\sqrt{a}\,C(a)<1$, where

$$C(a)=\frac{2\sqrt{a}}{\sqrt{\pi}}\sum_{i=1}^{n}L'_ie^{\psi(a)\|\tau\|^2}(1+\tau_i)\|\tau\|_0\sum_{i=1}^{n}\Big[L_i+L'_ie^{\psi(a)\|\tau\|^2}(1+\tau_i)\Big]$$
$$\times\max\left\{C_0(a),\,C_1(a),\,C_2(a),\,C_3(a)\right\},$$

$$C_0(a) = \frac{4}{\pi}\, a \left(\frac{\psi(a)}{\psi(0)}\right)^{(n+2)/2} + \sqrt{\pi} \left(\frac{\psi(a)}{\psi(0)}\right)^{(n+1)/2},$$

$$C_1(a) = \frac{4}{\pi} \left(\frac{\psi(a)}{\psi(0)}\right)^{(n+3)/2}, \quad C_2(a) = \left(2 + \frac{16\,a}{3\sqrt{\pi}}\right) \psi(a) \left(\frac{\psi(a)}{\psi(0)}\right)^{(n+1)/2},$$

$$C_3(a) = \psi(a) \left(\frac{\psi(a)}{\psi(0)}\right)^{(n+2)/2}.$$

Theorem 5. *If Assumptions* 3 *and* 4 *are satisfied, then* $u^* \in E_{2,\psi}$ *is a classical solution to* (1), (2).

4. The Proofs and Remarks

We begin this section with some considerations which hold under the assumptions of Theorems 2 and 4.

The functions $u = u^{\nu+1}$ defined as the unique solutions to (7), (8) and their partial derivatives $D_{x_j} u^{\nu+1}$ $(j = 1, \dots, n)$ satisfy the recurrence relations

$$u^{\nu+1}(t,x) = \mathcal{L}_0[u^\nu](t,x) + \int_0^t \int_{R^n} H(t-s, x-y)\mathcal{D}f[u^\nu; u^{\nu+1} - u^\nu](s,y)dy\,ds, \tag{16}$$

$$u_j^{\nu+1}(t,x) = \mathcal{L}_j[u^\nu](t,x) + \int_0^t \int_{R^n} D_{x_j} H(t-s, x-y) \times \mathcal{D}f[u^\nu; u^{\nu+1} - u^\nu](s,y)dy\,ds \quad (j = 1, \dots, n) \tag{17}$$

for $(t,x) \in E^+ := (0,a] \times R^n$. Formulae (16) and (17) are equivalent to (3) and (4) with $f(\dots)$ replaced by the right-hand side of (7).

It is easy to verify that

$$\int_{R^n} H(t, x-y)\, dy = 1, \tag{18}$$

$$\int_{R^n} |D_{x_j} H(t, x-y)|\, dy = \frac{1}{\sqrt{\pi\, t}} \quad (j = 1, \dots, n) \tag{19}$$

for $(t,x) \in E^+$. By the Hadamard mean-value theorem we get

$$f\left[u^{\nu+1}\right](s,y) - f\left[u^\nu\right](s,y) = \int_0^1 \mathcal{D}f\left[u^\nu + \zeta v^\nu; v^\nu\right](s,y)\, d\zeta \quad (\nu = 0, 1, \dots)$$

for $(s,y) \in E$. The above expression appears in $\mathcal{L}_j[u^{\nu+1}] - \mathcal{L}_j[u^\nu]$, $j = 0, 1, \dots, n$.

Proof of Theorem 2. The existence and uniqueness of the Chaplygin's sequence u^ν such that $(u^\nu, D_x u^\nu) \in \mathcal{X}[\phi]$ follows from Theorem 1.

If $v \in C(E_0 \cup E, R)$ and $v_0 := v$ and $v_l := D_{x_l} v$ for $l = 1, \ldots, n$, then we denote by $\Gamma[v](s,y)$ and $\tilde{\Gamma}[v](s,y)$ the expressions

$$\Gamma[v](s,y) = \sum_{i=0}^{n} \left(L_i |v_i(s,y)| + L_i' \|(v_i)_{(s,y)}\|_0 \right),$$

$$\begin{aligned}\tilde{\Gamma}[v](s,y) = \sum_{i,j=0}^{n} \Bigg(& L_{ij}^0 |v_i(s,y)| \int_0^1 \sigma\left(s; \zeta |v_j(s,y)|\right) d\zeta \\ & + L_{ij}^1 |v_i(s,y)| \int_0^1 \sigma\left(s; \zeta \|(v_j)_{(s,y)}\|_0\right) d\zeta \Bigg) \\ + \sum_{i,j=0}^{n} \Bigg(& L_{ij}^2 \|(v_i)_{(s,y)}\|_0 \int_0^1 \sigma\left(s; \zeta |v_j(s,y)|\right) d\zeta \\ & + L_{ij}^3 \|(v_i)_{(s,y)}\| \int_0^1 \sigma\left(s; \zeta \|(v_j)_{(s,y)}\|_0\right) d\zeta \Bigg)\end{aligned}$$

for $(s,y) \in E$.

If we take (16), (17) for ν and next the same equations for $\nu + 1$, then subtracting these equations and applying Assumption 1 yield the estimates

$$|v_0^{\nu+1}(t,x)| \le \int_0^t \int_{R^n} H(t-s, x-y) \left\{ \Gamma\left[v^{\nu+1}\right](s,y) + \tilde{\Gamma}\left[v^\nu\right](s,y) \right\} dy\, ds \tag{20}$$

and

$$|v_l^{\nu+1}(t,x)| \le \int_0^t \int_{R^n} |D_{x_l} H(t-s, x-y)| \left\{ \Gamma\left[v^{\nu+1}\right](s,y) + \tilde{\Gamma}\left[v^\nu\right](s,y) \right\} dy\, ds \quad (l = 1, \ldots, n), \tag{21}$$

where $v_0^\nu := v^\nu$ and $v_l^\nu := D_{x_l} v^\nu$ for $l = 1, \ldots, n$. Taking supremum norms in the right-hand sides of (20) and (21), we get

$$\begin{aligned}|v_0^{\nu+1}(t,x)| \le \int_0^t \Bigg\{ & L\|v^{\nu+1}\|_0 + L'\|V^{\nu+1}\|_0 \\ & + \tilde{L}\|(v^\nu, V^\nu)\|_0 \int_0^1 \sigma\left(s; \zeta \|(v^\nu, V^\nu)\|_0\right) d\zeta \Bigg\} ds,\end{aligned} \tag{22}$$

$$\begin{aligned}|v_l^{\nu+1}(t,x)| \le \int_0^t \frac{1}{\sqrt{\pi(t-s)}} \Bigg\{ & L\|v^{\nu+1}\|_0 + L'\|V^{\nu+1}\|_0 \\ & + \tilde{L}\|(v^\nu, V^\nu)\|_0 \int_0^1 \sigma(s; \zeta \|(v^\nu, V^\nu)\|_0)\, d\zeta \Bigg\} ds \quad (l = 1, \ldots, n),\end{aligned} \tag{23}$$

where L, L', $\tilde{L}$ are defined by (9). We set

$$\gamma_0^0 = \|v^0\|_0, \quad \gamma_1^0 = \|V^0\|_0, \quad \gamma^0 = \|(v^0, V^0)\|_0, \quad \beta^0 = L\,\gamma_0^0 + L'\,\gamma_1^0,$$

$$c_0^\nu = \tilde{L}\,\gamma^\nu \int_0^1 \sigma_0\left(\zeta\,\gamma^\nu\right)\,d\zeta\,, \quad c_1^\nu = \frac{\tilde{L}\,\gamma^\nu}{\sqrt{\pi}} \int_0^1 \sigma_1\left(\zeta\,\gamma^\nu\right)\,d\zeta\,.$$

Denote

$$\beta^{\nu+1} = \frac{Lc_0^\nu + L'c_1^\nu}{1 - aL - \frac{2\sqrt{a}}{\sqrt{\pi}}L'}, \tag{24}$$

$$\gamma_0^{\nu+1} = \frac{c_0^\nu\left(1 - \frac{2\sqrt{a}}{\sqrt{\pi}}L'\right) + c_1^\nu aL'}{1 - aL - \frac{2\sqrt{a}}{\sqrt{\pi}}L'}, \tag{25}$$

$$\gamma_1^{\nu+1} = \frac{\frac{2\sqrt{a}}{\sqrt{\pi}}Lc_0^\nu + c_1^\nu(1 - aL)}{1 - aL - \frac{2\sqrt{a}}{\sqrt{\pi}}L'}, \tag{26}$$

$$\begin{aligned} \gamma^{\nu+1} = \frac{\tilde{L}\gamma^\nu}{1 - aL - \frac{2\sqrt{a}}{\sqrt{\pi}}L'} \int_0^1 &\left\{\sigma_0\left(\zeta\,\gamma^\nu\right)\left(1 - \frac{2\sqrt{a}}{\sqrt{\pi}}L'\right)\right. \\ &\left. + \sigma_1\left(\zeta\gamma^\nu\right)\frac{1 - aL}{\sqrt{\pi}}\right\} d\zeta \end{aligned} \tag{27}$$

for $\nu = 0, 1, \ldots$. The recurrence formula given by (27) determines the convergence rate because we easily deduce the assertion

$$\lim_{\nu\to\infty} \gamma^{\nu+1}/\gamma^\nu = \lim_{\nu\to\infty} \int_0^1 \left\{C_0\,\sigma_0\left(\zeta\,\gamma^\nu\right) + C_1\,\sigma_1\left(\zeta\,\gamma^\nu\right)\right\}\,d\zeta = 0$$

with some constants C_0, C_1. Hence, by (27) and by conditions 3° and 4° of Assumption 2, the sequence $\{\gamma^\nu\}_\nu$ is decreasing and convergent to 0 as $\nu \to \infty$.

One can prove by induction on $\nu = 0, 1, \ldots$ that the following relations hold:

$$\begin{aligned} \beta^{\nu+1} &= \left(L\,a + L'\,\frac{2\,\sqrt{a}}{\sqrt{\pi}}\right)\,\beta^{\nu+1} + L\,c_0^\nu + L'\,c_1^\nu\,, \\ \|v^{\nu+1}\|_0 &\le \gamma_0^{\nu+1} \le a\,\beta^{\nu+1} + c_0^\nu\,, \\ \|V^{\nu+1}\|_0 &\le \gamma_1^{\nu+1} \le \frac{2\,\sqrt{a}}{\sqrt{\pi}}\,\beta^{\nu+1} + c_1^\nu\,, \\ \|(v^\nu, V^\nu)\|_0 &\le \gamma^\nu \end{aligned}$$

for $\nu = 0, 1, \ldots$, and hence

$$\begin{aligned} \sum_{\nu=\nu_0}^\infty \|(v^{\nu+1}, V^{\nu+1})\|_0 &\le \sum_{\nu=\nu_0}^\infty \gamma^{\nu+1} \le \sum_{\nu=\nu_0}^\infty \gamma^\nu \Xi(\gamma^\nu) \\ &\le \sum_{\nu=\nu_0}^\infty \gamma^{\nu_0}\left(\Xi\left(\gamma^{\nu_0}\right)\right)^{\nu-\nu_0+1} \le \gamma^{\nu_0}\,\frac{\Xi\left(\gamma^{\nu_0}\right)}{1\,-\,\Xi\left(\gamma^{\nu_0}\right)}, \end{aligned} \tag{28}$$

where

$$\Xi(\gamma^\nu) = \frac{\tilde{L}}{1 - aL - \frac{2\sqrt{a}}{\sqrt{\pi}} L'} \int_0^1 \left\{ \sigma_0 \left(\zeta \gamma^\nu\right) \left(1 - \frac{2\sqrt{a}}{\sqrt{\pi}} L'\right) \right.$$
$$\left. + \, \sigma_1 \left(\zeta \gamma^\nu\right) \frac{1 - aL'}{\sqrt{\pi}} \right\} d\zeta$$

for $\nu_0 = 0, 1, \ldots$. Consequently, the sequence $(u^\nu, D_x u^\nu)$ is uniformly convergent to a pair $(u^*, D_x u^*)$, and $\|(u^{\nu+1} - u^*, D_x u^{\nu+1} - D_x u^*)\|_0$ is bounded by the right-hand side of (28). This completes the proof. $\square$

Remark 1. It is seen that the sequence $\{u^\nu\}_\nu$ converges as faster than $\{U^\nu\}_\nu$ in a neighbourhood of $t = 0$ as much as the coefficient 'a' is less than $\frac{2\sqrt{a}}{\sqrt{\pi}}$ there. If $\sigma(t; r) = \sigma(r)$ (i.e., it is independent of 't'), then

$$\|(v^{\nu+1}, V^{\nu+1})\|_0 \leq \gamma^{\nu+1} = \frac{\max\{a, \frac{2\sqrt{a}}{\sqrt{\pi}}\}}{1 - aL - \frac{2\sqrt{a}}{\sqrt{\pi}} L'} \tilde{L} \gamma^\nu \int_0^1 \sigma\left(\zeta \gamma^\nu\right) d\zeta$$

for $\nu = 0, 1, \ldots$. If $\sigma(r) = r$, i.e., the Lipschitz conditions for $D_p f$, $D_w f$, $D_P f$, $D_W f$ hold, then we obtain the convergence with the so-called Newton's rate because the above estimate turns out to be equivalent to the inequality

$$\|(v^{\nu+1}, V^{\nu+1})\|_0 \leq \gamma^{\nu+1} = \frac{\max\{a, \frac{2\sqrt{a}}{\sqrt{\pi}}\}}{1 - aL - \frac{2\sqrt{a}}{\sqrt{\pi}} L'} \tilde{L} \left(\gamma^\nu\right)^2 / 2,$$

whose solution γ^ν can be written explicitly as

$$\gamma^\nu = \left(\frac{\max\{a, \frac{2\sqrt{a}}{\sqrt{\pi}}\}}{1 - aL - \frac{2\sqrt{a}}{\sqrt{\pi}} L'} \tilde{L}/2 \right)^{2^\nu - 1} \left(\gamma^0\right)^{2^\nu},$$

and, to make the sequence converge, it is enough to assume that

$$1 > \frac{\max\{a, \frac{2\sqrt{a}}{\sqrt{\pi}}\}}{1 - aL - \frac{2\sqrt{a}}{\sqrt{\pi}} L'} \tilde{L} \gamma^0 / 2.$$

Remark 2. We give some nontrivial examples of $\sigma(t; r)$ which represent the Hölder continuity of $D_p f$, $D_w f$, $D_{p_i} f$, $D_{w_i} f$ with singular Hölder coefficients at $t = 0^+$. Let $\kappa \in (0, 1]$. Put $\sigma(t; r) := r^\kappa / \sqrt{t}$ for $r \in [0, +\infty)$ and $t \in (0, a]$. Then $\sigma_0(r) = 2\sqrt{a}\, r^\kappa$ and $\sigma_1(r) = \pi\, r^\kappa$ for $r \in [0, +\infty)$, and condition (11) looks as follows:

$$1 > \frac{\tilde{L}}{1 - aL - \frac{2\sqrt{a}}{\sqrt{\pi}} L'} \left\{ 2\sqrt{a} \frac{\left(\|(v^0, V^0)\|_0\right)^\kappa}{\kappa + 1} \left(1 - \frac{2\sqrt{a}}{\sqrt{\pi}} L'\right) \right.$$
$$\left. + \, \pi \frac{\left(\|(v^0, V^0)\|_0\right)^\kappa}{\kappa + 1} \frac{1 - aL}{\sqrt{\pi}} \right\}.$$

The above inequality can be fulfilled when $\tilde{L}\ (\|(v^0, V^0)\|_0)^\kappa$ is small enough, i.e., when either the initial approximant is close to the exact solution or the sum $\tilde{L}$ of Hölder coefficients is sufficiently small. This is an extremal case when the estimate cannot be essentially improved by means of any decrease in length of the interval $[0, a]$.

Remark 3. The most natural initial guess in the Chaplygin's method can be defined by means of the function ϕ in the following way: $u^0(t,x) = \phi(0,x)$ for $(t,x) \in E$. In this particular case, we can estimate $\|v^0\|_0$ and $\|V^0\|_0$ based on the inequalities (cf. (4), (6), (16), (17), (18), (19))

$$|v^0(t,x)| \le M_\phi + a\left\{M_f + L\|v^0\|_0 + L'\|V^0\|_0\right\},$$
$$|v_i^0(t,x)| \le M'_\phi + \frac{2\sqrt{a}}{\sqrt{\pi}}\left\{M_f + L\|v^0\|_0 + L'\|V^0\|_0\right\} \quad (i = 1, \ldots, n),$$

where

$$M_f = \sup_{Q\in\Omega}|f(Q)|, \quad M_\phi = \sup_{y,\bar{y}\in R^n}|\phi(0,y) - \phi(0,\bar{y})|,$$
$$M'_\phi = \sup_{y,\bar{y}\in R^n}\|D_x\phi(0,y) - D_x\phi(0,\bar{y})\|_0 .$$

From these inequalities we derive the estimates

$$\|v^0\|_0 \le \frac{M_\phi(1 - \frac{2\sqrt{a}}{\sqrt{\pi}}L') + aL'M'_\phi + aM_f}{1 - aL - \frac{2\sqrt{a}}{\sqrt{\pi}}L'},$$
$$\|V^0\|_0 \le \frac{M_\phi\frac{2\sqrt{a}}{\sqrt{\pi}}L + M'_\phi(1 - aL) + M_f\frac{2\sqrt{a}}{\sqrt{\pi}}}{1 - aL - \frac{2\sqrt{a}}{\sqrt{\pi}}L'},$$
$$\|(v^0, V^0)\|_0 \le \delta_0 := \frac{M_\phi(1 - \frac{2\sqrt{a}}{\sqrt{\pi}}L') + M'_\phi(1 - aL) + M_f\max\{a, \frac{2\sqrt{a}}{\sqrt{\pi}}\}}{1 - aL - \frac{2\sqrt{a}}{\sqrt{\pi}}L'}.$$

Condition (11) in Assumption 2 can be replaced by

$$1 > \frac{\tilde{L}}{1 - aL - \frac{2\sqrt{a}}{\sqrt{\pi}}L'}\int_0^1\left\{\sigma_0(\zeta\delta_0)\left(1 - \frac{2\sqrt{a}}{\sqrt{\pi}}L'\right) + \sigma_1(\zeta\delta_0)\frac{1 - aL}{\sqrt{\pi}}\right\}d\zeta.$$

Proof of Theorem 4. The existence and uniqueness of the Chaplygin's sequence u^ν such that $(u^\nu, D_xu^\nu) \in \mathcal{X}_{2,\psi}[\phi]$ follows from an obvious strengthening of Theorem 3.

If $v \in C(E_0 \cup E, R)$ and $v_0 := v$ and $v_l := D_{x_l}v$ for $l = 1, \ldots, n$, then we denote by $\Lambda[v](s,y)$ and $\tilde{\Lambda}[v](s,y)$ the expressions

$$\Lambda[v](s,y) = \sum_{i=0}^{n}\left(\frac{L_i\,|v_i(s,y)|}{1 + |y_i|} + \frac{L'_i\|(v_i)_{(s,y)}\|_0}{1 + |y_i|}e^{-K(s)\circ|y|}\right),$$

$$\tilde{\Lambda}[v](s,y) = \sum_{i,j=0}^{n} e^{-\kappa\psi(s)\|y\|^2}\Bigg(\frac{L^0_{ij}|v_i(s,y)|}{(1+|y_i|)(1+|y_j|)^\kappa}\int_0^1 \sigma\Big(s;\zeta|v_j(s,y)|\Big)d\zeta$$
$$+\frac{L^1_{ij}|v_i(s,y)|\exp\left(-\kappa K(s)\circ|y|\right)}{(1+|y_i|)(1+|y_j|)^\kappa}\int_0^1 \sigma\left(s;\zeta\|(v_j)_{(s,y)}\|_0\right)d\zeta\Bigg)$$
$$+\sum_{i,j=0}^{n} e^{-\kappa\psi(s)\|y\|^2}\Bigg(\frac{L^2_{ij}\|(v_i)_{(s,y)}\|_0\exp\left(-K(s)\circ|y|\right)}{(1+|y_i|)(1+|y_j|)^\kappa}$$
$$\times\int_0^1 \sigma\left(s;\zeta|v_j(s,y)|\right)d\zeta+\frac{L^3_{ij}\|(v_i)_{(s,y)}\|\exp\left(-(1+\kappa)K(s)\circ|y|\right)}{(1+|y_i|)(1+|y_j|)^\kappa}$$
$$\times\int_0^1 \sigma\left(s;\zeta\|(v_j)_{(s,y)}\|_0\right)d\zeta\Bigg)$$

for $l=1,\dots,n$ and $(s,y)\in E$, where we take $y_0 := 0$ for convenience, and $\sigma(s;r) = s^\beta r^\kappa$.

If we take (16), (17) for ν and next the same equations for $\nu+1$, then subtracting these equations and applying Assumption 2 yield the estimates

$$|v_0^{\nu+1}(t,x)| \le \int_0^t\int_{R^n} H(t-s,x-y)\left\{\Lambda\left[v^{\nu+1}\right](s,y)+\tilde{\Lambda}\left[v^\nu\right](s,y)\right\}dy\,ds \tag{29}$$

and

$$|v_l^{\nu+1}(t,x)|$$
$$\le \int_0^t\int_{R^n} |D_{x_l}H(t-s,x-y)|\left\{\Lambda\left[v^{\nu+1}\right](s,y)+\tilde{\Lambda}\left[v^\nu\right](s,y)\right\}dy\,ds \tag{30}$$

for $l=1,\dots,n$. We recall that $v_0^\nu := v^\nu$ and $v_l^\nu := D_{x_l}v^\nu$ for $l=1,\dots,n$.

Under Assumption 3 one can prove the estimates

$$\int_{R^n} H(t-s,x-y)e^{\psi(s)\|y\|^2}dy \le \frac{\exp\left(\psi(t)\|x\|^2\right)}{(1-4(t-s)\psi(s))^{n/2}}, \tag{31}$$

$$\int_{R^n} |D_{x_i}H(t-s,x-y)|e^{\psi(s)\|y\|^2}dy$$
$$\le \exp\left(\psi(t)\|x\|^2\right)(1-4(t-s)\psi(s))^{-(n+1)/2}$$
$$\times\left(\frac{2\psi(s)|x_i|}{\sqrt{1-4(t-s)\psi(s)}}+\frac{1}{\sqrt{\pi(t-s)}}\right), \tag{32}$$

$$1/(1-4(t-s)\psi(s)) \le \psi(t)/\psi(s) \le \psi(t)/\psi(0) \tag{33}$$

for $0 < s < t \le a$ and $x \in R^n$. Observe that for $\beta \ge -1/2$ we have

$$\begin{aligned}\int_0^t \frac{s^\beta}{\sqrt{t-s}}ds &= 2t^{\beta+1/2}\int_0^1 \left(1-\xi^2\right)^\beta d\xi \\ &\le 2t^{\beta+1/2}\int_0^1 \left(1-\xi^2\right)^{-1/2} d\xi = \pi t^{\beta+1/2} \qquad (34)\end{aligned}$$

for $t \in (0, a]$. Finally, we derive

$$\begin{aligned}&e^{-K(s)\circ|y|-\psi(s)\|y\|^2}\|(v_j^\nu)_{(s,y)}\|_0/(1+|y_j|) \\ &\le \max\left\{\|v^\nu\|_{2,\psi},\, \|V^\nu\|'_{2,\psi}\right\} e^{\psi(s)\|\tau\|^2}(1+\delta_{0,j}\tau_j) \qquad (35)\end{aligned}$$

for $j = 0, \dots, n$ and $(s, y) \in E$. Denote

$$\tilde\gamma_0^\nu = \|v^\nu\|_{2,\psi}, \;\; \tilde\gamma_1^\nu = \|V^\nu\|'_{2,\psi}, \;\; \tilde\gamma^\nu = \max\left\{\|v^\nu\|_{2,\psi},\, \|V^\nu\|'_{2,\psi}\right\} \qquad (36)$$

for $\nu = 0, 1, \dots$. Now, we regard (35) and (36) as the main tool in the following estimation of $|v^\nu(t, x)|$ and $|v_i^\nu(t, x)|$ derived from (29), (30):

$$\begin{aligned}|v_0^{\nu+1}(t,x)| \le &\int_0^t \int_{R^n} H(t-s, x-y)\, \exp\left(\psi(s)\, \|y\|^2\right) \\ &\times \Big\{\left(L_0 + L_0' e^{\psi(s)\|\tau\|^2}\right)\tilde\gamma_0^{\nu+1} + \sum_{i=1}^n (1+\tau_i)\left(L_i + L_i' e^{\psi(s)\,\|\tau\|^2}\right), \tilde\gamma_1^{\nu+1} \\ &+ \sum_{i,j=0}^n \tilde\gamma^\nu \int_0^1 \sigma\left(s; \zeta\, \tilde\gamma^\nu\right)\, d\zeta\, (1+\delta_{0,i}\tau_i)\, (1+\delta_{0,j}\tau_j)^\kappa \\ &\times \left[L_{ij}^0 + L_{ij}^1\, e^{\kappa\,\psi(s)\,\|\tau\|^2} + L_{ij}^2\, e^{\psi(s)\,\|\tau\|^2} + L_{ij}^3\, e^{(1+\kappa)\,\psi(s)\,\|\tau\|^2}\right]\Big\} dy\, ds\end{aligned}$$

and

$$\begin{aligned}|v_l^{\nu+1}(t,x)| \le &\int_0^t \int_{R^n} |D_{x_l} H(t-s, x-y)|\, \exp\left(\psi(s)\, \|y\|^2\right) \\ &\times \Big\{\left(L_0 + L_0' e^{\psi(s)\|\tau\|^2}\right)\tilde\gamma_0^{\nu+1} + \sum_{i=1}^n (1+\tau_i)\left(L_i + L_i' e^{\psi(s)\|\tau\|^2}\right)\tilde\gamma_1^{\nu+1} \\ &+ \sum_{i,j=0}^n \tilde\gamma^\nu \int_0^1 \sigma\left(s; \zeta\, \tilde\gamma^\nu\right)\, d\zeta\, (1+\delta_{0,i}\tau_i)\, (1+\delta_{0,j}\tau_j)^\kappa \\ &\times \left[L_{ij}^0 + L_{ij}^1\, e^{\kappa\,\psi(s)\,\|\tau\|^2} + L_{ij}^2\, e^{\psi(s)\,\|\tau\|^2} + L_{ij}^3\, e^{(1+\kappa)\,\psi(s)\,\|\tau\|^2}\right]\Big\} dy\, ds \\ &\qquad (l = 1, \dots, n).\end{aligned}$$

Then, by virtue of (31)–(34) and (36), we get

$$|v_0^{\nu+1}(t,x)| \le \exp\Big(\psi(t)\,\|x\|^2\Big)\left(\frac{\psi(t)}{\psi(0)}\right)^{n/2} \times \int_0^t \Big\{L(t)\tilde\gamma_0^{\nu+1} + L'(t)\tilde\gamma_1^{\nu+1} + \tilde L(t)\tilde\gamma^\nu \int_0^1 \sigma\left(s;\zeta\tilde\gamma^\nu\right)d\zeta\Big\}ds,$$

$$|v_l^{\nu+1}(t,x)| \le \exp\Big(\psi(t)\|x\|^2\Big)\int_0^t\Big(2|x_l|\,\frac{(\psi(t))^{(n+2)/2}}{(\psi(0))^{n/2}} + \frac{(\psi(t)/\psi(0))^{(n+1)/2}}{\sqrt{\pi\,(t-s)}}\Big) \times\Big\{L(t)\tilde\gamma_0^{\nu+1} + L'(t)\tilde\gamma_1^{\nu+1} + \tilde L(t)\tilde\gamma^\nu\int_0^1\sigma\left(s;\zeta\tilde\gamma^\nu\right)d\zeta\Big\}ds \quad (l=1,\dots,n).$$

We are ready to summarise the above considerations in the recurrence inequalities

$$\tilde\gamma_0^{\nu+1} \le \left(\frac{\psi(a)}{\psi(0)}\right)^{n/2} a\Big\{L(a)\tilde\gamma_0^{\nu+1} + L'(a)\,\tilde\gamma_1^{\nu+1}\Big\} + \tilde L(a)\ \left(\tilde\gamma^\nu\right)^{1+\kappa}\ \frac{a^{\beta+1}}{(\beta+1)(\kappa+1)},$$

$$\tilde\gamma_1^{\nu+1} \le C(a)\Big\{L(a)\tilde\gamma_0^{\nu+1} + L'(a)\tilde\gamma_1^{\nu+1}\Big\} + \tilde C(a)\frac{1}{\kappa+1}\tilde L(a)\left(\tilde\gamma^\nu\right)^{1+\kappa},$$

or, equivalently, in

$$\tilde\gamma_0^{\nu+1} \le \frac{\tilde L(a)(\tilde\gamma^\nu)^{1+\kappa}\Big\{\frac{a^{\beta+1}}{\beta+1}(1-C(a)L(a)) + L'(a)\tilde C(a)a(\frac{\psi(a)}{\psi(0)})^{n/2}\Big\}}{(\kappa+1)\Big\{1-L(a)a(\frac{\psi(a)}{\psi(0)})^{n/2} - L'(a)C(a)\Big\}}$$

$$\tilde\gamma_1^{\nu+1} \le \frac{\tilde L(a)(\tilde\gamma^\nu)^{1+\kappa}\Big\{L(a)C(a)\frac{a^{\beta+1}}{\beta+1} + \tilde C(a)\Big(1-L(a)a(\frac{\psi(a)}{\psi(0)})^{n/2}\Big)\Big\}}{(\kappa+1)\Big\{1-L(a)a(\frac{\psi(a)}{\psi(0)})^{n/2} - L'(a)C(a)\Big\}},$$

and

$$\tilde\gamma^{\nu+1} \le \frac{\tilde L(a)(\tilde\gamma^\nu)^{1+\kappa}\Big\{\frac{a^{\beta+1}}{\beta+1}(1-C(a)\,L(a)) + \tilde C(a)\Big(1-L(a)a(\frac{\psi(a)}{\psi(0)})^{n/2}\Big)\Big\}}{(\kappa+1)\Big\{1-L(a)a(\frac{\psi(a)}{\psi(0)})^{n/2} - L'(a)C(a)\Big\}}.$$

This completes the proof. □

Proof of Corollary 1 *and Theorem* 5. Theorems 2 and 4 produce $C^{0,1}$ solutions, bounded and unbounded, respectively. Their assumptions state a sufficient regularity of the derivatives D_pf, D_wf, D_Pf and D_Wf. The assumptions of Corollary 1 and Theorem 5 provide similar regularity conditions on D_xf, which complete the assumptions of the adequate theorems from [3] on the existence of classical solutions. We omit the details. □

5. Complementary Notes

Our results on the existence and convergence of the Chaplygin's quasi-linearisation method are formulated in terms of local versions. However it is possible to extend these theorems to the whole set E. We discuss this question for bounded solutions (the case of unbounded solutions seems to be more complex). Define seminorms $\|\cdot\|_0(t)$ for $t \in [0,a]$ as the supremum norm of functions restricted to $[-\tau_0, t] \times R^n$. While proceeding as in the proof of Theorem 2, one can observe that instead of inequalities (22), (23) it is possible to get

$$\|v^{\nu+1}\|_0(t) \le \int_0^t \Big\{ L\|v^{\nu+1}\|_0(s) + L'\|V^{\nu+1}\|_0(s) + \tilde{L}\|(v^\nu, V^\nu)\|_0(s) \int_0^1 \sigma\left(s; \zeta \|(v^\nu, V^\nu)\|_0(s)\right) d\zeta \Big\} ds, \tag{37}$$

$$\|V^{\nu+1}\|_0(t) \le \int_0^t \frac{1}{\sqrt{\pi(t-s)}} \Big\{ L\|v^{\nu+1}\|_0(s) + L'\|V^{\nu+1}\|_0(s) + \tilde{L}\|(v^\nu, V^\nu)\|_0(s) \int_0^1 \sigma\left(s; \zeta \|(v^\nu, V^\nu)\|_0(s)\right) d\zeta \Big\} ds \quad (l = 1, \dots, n), \tag{38}$$

where L, L', $\tilde{L}$ are defined by (9). Define w^ν, $g^\nu : [0,a] \to R$ for $\nu = 0, 1, \dots$ as

$$w^\nu(t) = L\,\|v^\nu\|_0(t) + L'\,\|V^\nu\|_0(t), \quad t \in [0,a], \quad \nu = 0, 1, \dots,$$

$$g^\nu(s) = \tilde{L}\,\|(v^\nu, V^\nu)\|_0(s) \int_0^1 \sigma\left(s; \zeta\,\|(v^\nu, V^\nu)\|_0(s)\right)\, d\zeta\,, \quad s \in (0,a], \quad \nu = 0, 1, \dots .$$

To solve (37), (38) with respect to $v^{\nu+1}$, $V^{\nu+1}$, it suffices to do so for the following integral inequality with a weak singularity

$$w^{\nu+1}(t) \le \int_0^t \left(L + L' \frac{1}{\sqrt{\pi\,(t-s)}} \right) \left(w^{\nu+1}(s) + g^\nu(s) \right) ds, \quad t \in [0,a], \quad \nu = 0, 1, \dots . \tag{39}$$

This equality can be solved by means of a natural comparison problem, which is a simple consequence of the following lemma. First, given a function $g : [0,a] \to R$, we consider the problems

$$w(t) = \int_0^t \left(L + L' \frac{1}{\sqrt{\pi\,(t-s)}} \right) (w(s) + g(s))\, ds \tag{40}$$

and

$$w(t) \le \int_0^t \left(L \,+\, L' \frac{1}{\sqrt{\pi\,(t-s)}}\right) (w(s) \,+\, g(s))\; ds. \tag{41}$$

Lemma 1. *If* $g \in C([0,a], R)$*, then there is a unique solution* $w \in C([0,a], R)$ *to problem* (40)*. Moreover, if* $\tilde{w} : [0,a] \to R$ *is a solution to* (41)*, then* $\tilde{w} \le w$*.*

Proof. The last part of the lemma is obvious. The question of the uniqueness is clear, too. Concerning the first part, we choose a constant $T \in R_+$ such that $T \ge g(t)$ on $[0,a]$, and consider the problem

$$w(t) = \int_0^t \left(L \,+\, L' \frac{1}{\sqrt{\pi\,(t-s)}}\right) (w(s) \,+\, T)\; ds. \tag{42}$$

We look for a solution w of the form

$$w(t) = T \sum_{j=0}^{\infty} w_j t^{j/2}.$$

Denote

$$I_j(t) := \int_0^t \frac{s^{j/2}}{\sqrt{t-s}}\, ds, \quad j = 0, 1, \dots\,, \quad t \in [0,a].$$

One can prove that

$$I_{2k}(t) = \pi t^{k+1/2} \frac{(2k-1)!!}{(2k)!!}, \;\; I_{2k+1}(t) = 2 t^{k+1} \frac{(2k)!!}{(2k+1)!!}, \;\; k = 0, 1, \dots\,.$$

Substitution of the above relations into (40) results in the recurrence formulas: $w_0 = 0,\; w_1 = 2$, and

$$w_{2l} = w_{2l-2} \frac{L}{l} + 2\, L'\, w_{2l-1} \frac{(2l-2)!!}{(2l-1)!!} \quad (l = 1, 2, \dots),$$

$$w_{2l+1} = w_{2l-1} \frac{L}{l + 1/2} + \pi\, L'\, w_{2l} \frac{(2l-1)!!}{(2l)!!} \quad (l = 1, 2, \dots).$$

Since $\frac{w_{j+1}}{w_j} \to 0$ as $j \to \infty$, the function w is well defined on $[0,a]$.

Now, we employ a kind of a monotone iterative method, see [12]. We define a sequence $W^\nu \in C([0,a], R)$ in the following way. Take $W^0 = w$ where w is a solution to (42). Define

$$W^{\nu+1}(t) = \int_0^t \left(L + L' \frac{1}{\sqrt{\pi\,(t-s)}}\right) (W^\nu(s) \,+\, g(s))\; ds \quad (\nu = 0, 1, \dots).$$

It is easy to prove by induction on ν that $W^\nu(t) \ge W^{\nu+1} \ge 0$ for $\nu = 0, 1, \dots\,,\; t \in [0,a]$. Hence the sequence $\{W^\nu\}_\nu$ is monotone and convergent to the unique solution to (40). □

Remark 4. It follows from the above lemma applied to inequality (39) that

$$w^{\nu+1}(t) \;=\; O(1)\, w^{\nu}(t)\, \{\sigma_0\, (w^{\nu}(t)) + \sigma_1\, (w^{\nu}(t))\} \quad (\nu = 0, 1, \dots).$$

Consequently, the sequence $\|\, (v^{\nu}, V^{\nu})\, \|_0$ converges very fast to 0 provided that $\|\, (v^0, V^0)\, \|_0$ is chosen small enough.

Remark 5. The existence and convergence results of the present paper can be extended to the weakly coupled systems of differential-functional equations

$$\begin{aligned} D_t u_k(t,x) &= \sum_{j,l=1}^{n} a_{jl}^{(k)} D_{x_j x_l} u_k(t,x) \\ &\qquad + f^{(k)}\Big(t, x, u(t,x), u_{(t,x)}, D_x u(t,x), (D_x u)_{(t,x)}\Big) \quad (k = 1, \dots, m), \\ u_k(t,x) &= \phi_k(t,x) \;\text{ on }\; E_0 \;\; (k = 1, \dots, m), \end{aligned}$$

and their further generalisation

$$\begin{aligned} D_t u_k(t,x) &= \sum_{j,l=1}^{n} a_{jl}^{(k)} D_{x_j x_l} u_k(t,x) \\ &\qquad + f^{(k)}\Big(t, x, u(t,x), V_{(t,x)} u, D_x u(t,x), V_{(t,x)}(D_x u)\Big) \quad (k = 1, \dots, m), \\ u_k(t,x) &= \phi_k(t,x) \;\text{ on } E_0 \;\; (k = 1, \dots, m), \end{aligned}$$

where $u = (u_1, \dots, u_m) : E_0 \cup E \to R^m$, $\phi_k : E_0 \to R$ and the real coefficients $a_{jl}^{(k)}$ are such that the matrices $A^{(k)} = \left[a_{jl}^{(k)}\right]_{j,l=1,\dots,n}$ are positive and symmetric; the coefficients of these matrices may also depend on (t,x) as Hölder continuous functions. The functionals $V_{(t,x)} u$ are some generalisations of the Hale-type functionals $u_{(t,x)}$.

Acknowledgements

The research was supported by the Polish Research Committee (KBN), Grant No. 2 P03A 022 13.

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(Received 22.09.1998)

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