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UNCONDITIONAL CONVERGENCE OF RANDOM SERIES AND THE GEOMETRY OF BANACH SPACES

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Abstract. The a.s. unconditionally convergent random series are investigated. The connection of the a.s. unconditionally convergence with the geometry of spaces is established.

1. Introduction

Let X be a real Banach space, $(\Omega, \mathcal{A}, P)$ be a probability space. By a random element in X we mean a separably valued Borel measurable mapping Ω to X. Let (ξ_k) be a sequence of random elements in X.

Definition. A random series $\sum_{k=1}^{\infty} \xi_k$ is called a.s. unconditionally convergent in X if there exists a set $\Omega_0 \in \mathcal{A}$ with probability 1 such that the series $\sum_{k=1}^{\infty} \xi_k(\omega)$ converges unconditionally in X for any $\omega \in \Omega_0$ (i.e. for every permutation π of the integers the series $\sum_{k=1}^{\infty} \xi_{\pi(k)}(\omega)$ is convergent for all $\omega \in \Omega_0$).

We consider the unconditional convergence in the norm topology of a Banach space.

It is easily seen that under the proposed definition, the equivalence between the a.s. unconditional and absolute convergence of random series in the finite dimensional case remains valid, but according to the well-known Dvoretzky–Rogers theorem (see, for instance, [1], p. 80) this is not true in the infinite dimensional case.

It is clear that if the series $\sum_{k=1}^{\infty} \xi_k$ converges a.s. unconditionally, then every its permutation is a.s. convergent. The converse assertion is not true even in the one-dimensional case. The corresponding example is the series $\sum_{k=1}^{\infty} \frac{1}{k}\varepsilon_k$, where (ε_k) is a sequence of independent Bernoulli random variables (i.e., $P[\varepsilon_k = -1] = P[\varepsilon_k = 1] = \frac{1}{2}$, $k = 1, 2, \dots$). It is obvious that every permutation of the series

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$\sum\limits_{k=1}^{\infty} \frac{1}{k}\varepsilon_k$ is a.s. convergent since $\sum\limits_{k=1}^{\infty} \frac{1}{k^2} < \infty$; at the same time this series is not a.s. unconditionally (absolutely) convergent since $\sum\limits_{k=1}^{\infty} |\frac{1}{k}\varepsilon_k| = \sum\limits_{k=1}^{\infty} \frac{1}{k} = \infty$.

A sequence (ξ_k) of random elements with values in a Banach space X is called a Gaussian sequence if every its finite collection $(\xi_{k_1}, \xi_{k_2}, \dots, \xi_{k_n})$, $n \geq 1$, is a Gaussian random element with values in the Banach space X^n, the Cartesian product of n Banach spaces X. A series $\sum\limits_{k=1}^{\infty} \xi_k$ is called a Gaussian series if the sequence (ξ_k) is a Gaussian sequence. In particular, a series $\sum\limits_{k=1}^{\infty} \xi_k$ will be a Gaussian series if ξ_k, $k = 1, 2, \dots$, are independent Gaussian random elements.

In the sequel we shall assume all the considered Banach spaces to be separable.

The unconditional convergence of Gaussian series has been studied by the author in [2]–[8].

Let E_1 and E_2 be isomorphic Banach spaces. Denote by $d(E_1, E_2)$ the infimum of the number set $\{\|T\| \cdot \|T^{-1}\|, T : E_1 \to E_2$ is an isomorphism$\}$. It is abvious that $d(E_1, E_2) \geq 1$. If E_1 and E_2 are isometric, then $d(E_1, E_2) = 1$. Denote R^n with the maximum-norm by l_∞^n. We shall say that a Banach space X contains l_∞^n's uniformly if for each $\varepsilon > 0$ and any integer n there exists an n-dimensional subspace X_n of X such that $d(X_n, l_\infty^n) \leq 1 + \varepsilon$. The examples of such spaces are c_0, l_∞, $C([0,1])$. There exist reflexive spaces containing l_∞^n's uniformly (for instance, $\oplus_{l_2} l_\infty^n$). The examples of spaces which do not contain l_∞^n's uniformly are l_p, $L_p([0,1])$, $1 \leq p < \infty$.

We follow Pietsch [9], [10] and denote by $l_1(X)$ the set of all sequences (x_k) from a Banach space X, for which the following conditions are satisfied:

a) $\displaystyle\sum_{k=1}^{\infty} \Big|\langle x^*, x_k\rangle\Big| < \infty \quad \text{for all} \quad x^* \in X^*,$

b) $\displaystyle\lim_{n\to\infty} \sup_{\|x^*\|_{X^*} \leq 1} \sum_{k=n}^{\infty} \Big|\langle x^*, x_k\rangle\Big| = 0,$

where X^* is the dual space with norm $\|\cdot\|_{X^*}$.

Let us state without proof some properties of the space $l_1(X)$:

(i) $l_1(X)$ is a Banach space with norm $\|(x_k)\|_{l_1(X)} = \displaystyle\sup_{\|X^*\|_{X^*} \leq 1} \sum_{k=1}^{\infty} |\langle x^*, x_k\rangle|$.

(ii) The natural imbedding of the space $l_1(X)$ to the space X^N (the Cartesian product of countably many Banach spaces X) is continuous.

(iii) The dual space of $l_1(X)$ is the vector space of all sequences (x_k^*) from X^* for which $\sum\limits_{k=1}^{\infty} |\langle x_k^*, x_k\rangle| < \infty$ for all $(x_k^*) \in l_1(X)$.

(iv) If X is a separable space, then $l_1(X)$ is a separable space, too.

(v) If $(\alpha_k) \in l_\infty$ and $(x_k) \in l_1(X)$, then $(\alpha_k x_k) \in l_1(X)$.

(vi) If $(\alpha_k) \in c_0$ and $\sum\limits_{k=1}^{\infty} |\langle x^*, x_k\rangle| < \infty$ for all $x^* \in X^*$, then $(\alpha_k x_k) \in l_1(X)$.

(vii) If a series $\sum\limits_{k=1}^{\infty} x_k$ converges unconditionally in a Banach space X, then the sequence (x_k) belongs to the space $l_1(X)$. The converse assertion is also true.

Let X be a Banach space. Then in order that the series $\sum_{k=1}^{\infty} x_k$ in X with $\sum_{k=1}^{\infty} |\langle x^*, x_k\rangle| < \infty$ for each $x^* \in X^*$ be unconditionally convergent, it is both necessary and sufficient that X contain no copy of c_0 [11]. In particular, if X is a reflexive Banach space, then a series $\sum_{k=1}^{\infty} x_k$ converges unconditionally in X if and only if the series $\sum_{k=1}^{\infty} x_k$ converges weakly absolutely (i.e., satisfies the condition a)).

In the spaces not containing l_∞^n's uniformly we can give a structural description of the unconditionally convergent series.

Theorem 1.1 ([3], [5]). *In a Banach space X the following assertions are equivalent:*

1°. *X does not contain l_∞^n's uniformly.*

2°. *A series $\sum_{k=1}^{\infty} a_k$ converges unconditionally in X if and only if there exist a number $p \geq 2$, depending only on X, a sequence of non-negative numbers (α_k), $\sum_{k=1}^{\infty} \alpha_k^p < \infty$, and a linear bounded operator A from l_p to X, such that*

$$a_k = \alpha_k A e_k, \quad k = 1, 2, \dots,$$

where (e_k) is the sequence of unit vectors in l_p.

In the spaces containing l_∞^n's uniformly we have a different situation.

Theorem 1.2 ([12], [13]). *In a Banach space X the following assertions are equivalent:*

1°. *X contains l_∞^n's uniformly.*

2°. *For every sequence (α_k) converging to 0 there exists an unconditional convergent series $\sum_{k=1}^{\infty} a_k$ in X such that $\|a_k\| = |\alpha_k|$, $k = 1, 2, \dots$.*

2. Unconditional Convergence of Random Series in Banach Spaces

In this section we prove the result which extends the result of [6] and the result given in [14], p. 315.

Theorem 2.1. *Let X be a Banach space, (ζ_k) be a sequence of real random variables such that $P[\,|\zeta_k| \in (m_1, m_2)] \geq \delta > 0$ for some $m_2 > m_1 > 0$, $\delta > 0$ and for every $k = 1, 2, \dots$. If a series $\sum_{k=1}^{\infty} a_k \zeta_k$, $a_k \in X$, converges a.s. unconditionally, then the series $\sum_{k=1}^{\infty} a_k$ converges unconditionally in X.*

For the proof of Theorem 2.1 we need

Lemma 2.1. *Let a sequence* (ζ_k) *satisfy the conditions of Theorem* 2.1 *and let* (α_k) *be a sequence of real numbers. Then the a.s. absolute convergence of the series* $\sum_{k=1}^{\infty} \alpha_k\zeta_k$ *implies the absolute convergence of the series* $\sum_{k=1}^{\infty} \alpha_k$.

Proof. If a series $\sum_{k=1}^{\infty} \alpha_k\zeta_k$ converges a.s. absolutely, then it is clear that the series $\sum_{k=1}^{\infty} \alpha_k\zeta_k I_{[\,|\zeta_k|<c]}$ converges a.s. absolutely for all $c>0$, where I_A is the indicator of the set A. Therefore, without loss of generality, we can assume that all random variables ζ_k, $k=1,2,\dots$, are a.s. uniformly bounded by some constant $M<\infty$, where $M \ge m_2$.

Let us denote $\xi = \sum_{k=1}^{n} |\alpha_k\zeta_k|$. It is clear that

$$E\xi = \sum_{k=1}^{n} |\alpha_k| E|\zeta_k| \ge \sum_{k=1}^{n} |\alpha_k| E|\zeta_k| I_{[\,|\zeta_k|\in(m_1,m_2)]} \ge m_1\delta \sum_{k=1}^{n} |\alpha_k|;$$

$$E\xi^2 = \left(\left(E\left(\sum_{k=1}^{n} |\alpha_k\zeta_k|\right)^2\right)^{1/2}\right)^2 \le \left(\sum_{k=1}^{n} |\alpha_k| \left(E|\zeta_k|^2\right)^{1/2}\right)^2 \le \left(M\sum_{k=1}^{n} |\alpha_k|\right)^2.$$

Now for the non-negative random variable ξ we can apply the Kahane inequality (see, for instance, [15], p. 240):

$$P[\xi \ge \lambda E\xi] \ge (1-\lambda)^2 \frac{(E\xi)^2}{E\xi^2} \quad \text{for every} \quad \lambda \in (0,1).$$

We have

$$P\left[\sum_{k=1}^{n} |\alpha_k\zeta_k| > \lambda m_1\delta \sum_{k=1}^{n} |\alpha_k|\right] \ge P\left[\sum_{k=1}^{n} |\alpha_k\zeta_k| > \lambda E \sum_{k=1}^{n} |\alpha_k\zeta_k|\right]$$

$$\ge (1-\lambda)^2 \left(\frac{m_1\delta \sum_{k=1}^{n} |\alpha_k|}{M \sum_{k=1}^{n} |\alpha_k|}\right)^2 > 0 \quad \text{for every } n = 1,2,\dots. \quad \square$$

Note that the converce assertion of Lemma 2.1 in general is not true.

Proof of Theorem 2.1. According to Lemma 2.1 the a.s. unconditional convergence of the series $\sum_{k=1}^{\infty} a_k\zeta_k$ implies $\sum_{k=1}^{\infty} |\langle x^*, a_k\rangle| < \infty$ for all $x^* \in X^*$. It remains to prove that

$$\lim_{n\to\infty} \sup_{||x^*||_{X^*}\le 1} \sum_{k=n}^{\infty} |\langle x^*, a_k\rangle| = 0\,.$$

Let us denote

$$\eta_n = \sup_{\|x^*\|_{X^*}\le 1} \sum_{k=n}^{\infty} |\langle x^*, a_k\zeta_k\rangle|, \quad n=1,2,\dots.$$

By the assumption of Theorem 2.1, η_n converges a.s. to 0 as $n \to \infty$. It is clear that $\eta_n \cdot I_{[\eta_n < c]}$ also converges a.s. to 0 as $n \to \infty$ for every $c > 0$. Choosing $c \geq m_2$ by the monotone convegece theorem we have

$$0 = \lim_{n\to\infty} E\eta_n I_{[|\eta_n|<c]} \geq \lim_{n\to\infty} \sup_{||x^*||_{X^*}\leq 1} \sum_{k=n}^{\infty} |\langle x^*, a_k\rangle| E|\zeta_k| I_{[|\zeta_k|\in(m_1,m_2)]}$$

$$\geq m_1\delta \sup_{||x^*||_{X^*}\leq 1} \sum_{k=n}^{\infty} |\langle x^*, a_k\rangle|. \qquad \square$$

It is clear that a sequence of real standard Gaussian random variables (g_k), which is not necessary a Gaussian sequence, satisfies the condition of Theorem 2.1 and therefore the following corollary generalizes the result of [4].

Corollary. *Let (g_k) be a sequence of real standard Gaussian random variables and $a_k \in X$, $k = 1, 2, \dots$. If a series $\sum_{k=1}^{\infty} a_k g_k$ converges a.s. unconditionally, then the series $\sum_{k=1}^{\infty} a_k$ converges unconditionally in the Banach space X.*

3. Connection with the Geometry of Banach Spaces

Taking into account Theorem 1.1 it is easy to prove the validity of the converse assertion to Theorem 2.1 for some sequences of random variables in the spaces not containing l_∞^n's uniformly, the proof of which is omitted.

Proposition 3.1. *Let X be a Banach space not containing l_∞^n's uniformly, (ζ_k) be a sequence of real random variables for which $\sup_k E|\zeta_k|^p \leq c_p < \infty$ for every $p > 0$, $a_k \in X$, $k = 1, 2, \dots$. If a series $\sum_{k=1}^{\infty} a_k$ converges a.s. unconditionally in X, then the series $\sum_{k=1}^{\infty} a_k\zeta_k$ converges a.s. unconditionally.*

In particular, it is clear that here as ζ_k we can take a standard Gaussian random variable g_k, $k = 1, 2, \dots$.

For the sequence of independent standard Gaussian random variables we have the following result.

Theorem 3.1 ([3], [5]). *Let X be a Banach space, (γ_k) be a sequence of real independent standard Gaussian random variables, and $a_k \in X$, $k = 1, 2, \dots$. The following assertions are equivalent:*

1°. X does not contain l_∞^n's uniformly.

2°. The unconditional convergence of a series $\sum_{k=1}^{\infty} a_k$ in X implies the a.s. unconditional convergence of the series $\sum_{k=1}^{\infty} a_k\gamma_k$.

In a Banach space containing l_∞^n uniformly, a random analog of Theorem 1.2 is true which was announced in [3].

Theorem 3.2. *Let X be a Banach space and (g_k) be a Gaussian sequence of standard Gaussian random variables. Then the following assertions are equivalent:*

1°. *X contains l_∞^n's uniformly.*

2°. *For every real Gaussian sequence $(\alpha_k g_k)$ converging a.s. to 0 there exists a sequence (a_k) of elements of X such that the series $\sum_{k=1}^{\infty} a_k g_k$ converges a.s. unconditionally in X and $\|a_k\| = |\alpha_k|$, $k = 1, 2, \dots$.*

Proof. $1^\circ \Rightarrow 2^\circ$. According to the assumption, $(\alpha_k g_k)$ belongs a.s. to c_0. It is well-known that the norm in c_0 of the Gaussian random element is integrable (see, for instance, [14], p. 329): $M = E\|(\alpha_k g_k)\|_{c_0} = E\sup_k |\alpha_k g_k| < \infty$. By the monotone convergence theorem we have $\lim_{n\to\infty} E\sup_{k\ge n} |\alpha_k g_k| = 0$.

Let us consider a series of positive numbers $\sum_{j=1}^{\infty} \varepsilon_j < \infty$, where $\varepsilon_1 > M$. It is clear that for each ε_j there exists an integer n_j such that $E\sup_{k\ge n_j} |\alpha_k g_k| < \varepsilon_j$, $j = 1, 2, \dots$. Obviously, $n_1 = 0$. Then

$$E\sum_{j=1}^{\infty} \max_{n_j+1\le k\le n_{j+1}} |\alpha_k g_k| \le \sum_{j=1}^{\infty} E \sup_{k\ge n_j+1} |\alpha_k g_k| < \sum_{j=1}^{\infty} \varepsilon_j < \infty\,.$$

Therefore there is a measurable set Ω_0 with probability 1 on which the series $\sum_{j=1}^{\infty} \max_{n_j+1\le k\le n_{j+1}} |\alpha_k g_k|$ is convergent.

Let $m_j = n_{j+1} - n_j$, $j = 1, 2, \dots$. Since X contains l_∞^n's uniformly, for any $\varepsilon > 0$ there is an m_j-dimensional subspace $X_{m_j} \subset X$ and an isomorphic operator $T_{m_j} : l_\infty^{m_j} \to X_{m_j}$ such that $\|T_{m_j}\|\|T_{m_j}^{-1}\| \le 1 + \varepsilon$ for all $j = 1, 2, \dots$.

Let us consider the series in X:

$$\sum_{k=1}^{\infty} \alpha_k B_k e_k \|B_k e_k\|^{-1} g_k\,, \tag{3.1}$$

where $B_k = T_{m_j}$, $n_j + 1 \le k < n_{j+1}$, while e_k, $n_j + 1 \le k < n_{j+1}$, are the unit vectors in $l_\infty^{m_j}$, $j = 1, 2, \dots$. Hence we can rewrite (3.1) as

$$\sum_{j=1}^{\infty} \sum_{k=n_j+1}^{n_{j+1}} \alpha_k \|T_{m_j} e_k\|^{-1} T_{m_j} e_k g_k\,. \tag{3.2}$$

Taking into account that $\|T_{m_j}^{-1}\|\|T_{m_j} e_k\| \ge 1$, by (3.2) we have for all $x^* \in X^*$ and $\omega \in \Omega_0$

$$\sum_{j=1}^{\infty} \sum_{k=n_j+1}^{n_{j+1}} |\alpha_k| \cdot \|T_{m_j} e_k\|^{-1} |g_k(\omega)| |\langle x^*, T_{m_j} e_k\rangle|$$
$$\le \sum_{j=1}^{\infty} \|T_{m_j}^{-1}\| \cdot |\langle T_{m_j}^* x_j^*, \sum_{k=n_j+1}^{n_{j+1}} \theta_{k_j} |\alpha_k g_k(\omega)| e_k\rangle|$$

$$\le \sum_{j=1}^{\infty} \|T_{m_j}^{-1}\| \cdot \|T_{m_j}^* x_j^*\| \cdot \| \sum_{k=n_j+1}^{n_{j+1}} \theta_{k_j} |\alpha_k g_k(\omega)| e_k\|$$
$$\le (1+\varepsilon)\|x^*\| \sum_{j=1}^{\infty} \max_{n_j+1\le k\le n_{j+1}} |\alpha_k g_k(\omega)| < \infty\,.$$

Here $\theta_{k_j} = \operatorname{sgn}\langle x^*, T_{m_j} e_k\rangle$ for all $k, j = 1, 2, \dots$.

Thus series (3.1) converges weak-absolutely on the set Ω_0.

Since $\max_{n_j+1\le k\le n_{j+1}} |\alpha_k g_k(\omega)|$ converges to 0 as $n \to \infty$ for all $\omega \in \Omega_0$, we can by a similar reasoning prove that

$$\lim_{n\to\infty} \sup_{\|x^*\|_{X^*}\le 1} \sum_{k=n}^{\infty} |\alpha_k \langle x^*, B_k e_k\rangle| \, \|B_k e_k\|^{-1} |g_k(\omega)| = 0$$

for every $\omega \in \Omega_0$.

$2^\circ \Rightarrow 1^\circ$. Conversely, suppose that X does not contain l_∞^n's uniformly. Then there exists a number $p = p(X) \ge 2$ such that each a.s. unconditionally convergent Gaussian series $\sum_{k=1}^{\infty} a_k g_k$ converges a.s. p-absolutely, too [5], i.e., $\sum_{k=1}^{\infty} \|a_k\|^p < \infty$. Choose $\alpha_k = (\ln(k+1))^{-1}$, $k = 1, 2, \dots$. The sequence $((\ln(k+1))^{-1} g_k)$ a.s. converges to 0 (see [15], p. 48). By 2°, there is a sequence (a_k) in X such that the series $\sum_{k=1}^{\infty} a_k g_k$ converges a.s. unconditionally and $\|a_k\| = (\ln(k+1))^{-1}$, $k = 1, 2, \dots$. But this is impossible since the series $\sum_{k=1}^{\infty} a_k g_k$ does not converge a.s. p-absolutely for every $p \ge 2$. □

Formally, Theorem 3.2 implies Theorem 1.2 if $g_k = g$ for every $k = 1, 2, \dots$.

Note that, unlike Theorem 3.1, the independence of g_k, $k = 1, 2, \dots$, is not required in Theorem 3.2.

In a Banach space contaning l_∞^n's uniformly there exists an unconditionally convergent series $\sum_{k=1}^{\infty} a_k$ such that the sequence $(\|a_k\|\gamma_k)$ does not converge a.s. to 0 (see [12], [13] and [15], p. 48), and consequently the series $\sum_{k=1}^{\infty} a_k\gamma_k$ does not converge a.s. (here the independence of γ_k, $k = 1, 2, \dots$, is supposed). The question about the a.s. convergence of a series $\sum_{k=1}^{\infty} a_k\gamma_k$ under the assumptions: $\sum_{k=1}^{\infty} a_k$ converges unconditionally and the sequence $(\|a_k\|\gamma_k)$ converges a.s. to 0, remains unsolved. Nevertheless the following result holds.

Theorem 3.3. *Let X be a Banach space and (γ_k) be a sequence of independent standard Gaussian random variables. The following assertions are equivalent:*

1°. *X contains l_∞^n's uniformly.*

2°. *There exists an unconditionally convergent series* $\sum_{k=1}^{\infty} a_k$ *in* X *such that* $\sum_{k=1}^{\infty} \exp\{-\frac{\varepsilon}{\|a_k\|^2}\} < \infty$ *for every* $\varepsilon > 0$*, but the series* $\sum_{k=1}^{\infty} a_k\gamma_k$ *does not converge a.s. unconditionally.*

Proof. 1° ⇒2°. Let (i_n) be an arbitrary nondecreasing sequence of integers tending to ∞ as $n \to \infty$. Then there exists an unconditionally convergent series $\sum_{n=1}^{\infty} b_n$ in X such that $\sum_{n=1}^{\infty} i_n b_n$ also converges unconditionally. The existence of the series is guaranteed by the assumption [12], [13]

$$i_n\|b_n\| \to 0 \quad \text{as} \quad n \to \infty\,. \tag{3.3}$$

Let us consider a series $\sum_{k=1}^{\infty} a_k$ in X, where $a_k = b_n$ for $m_n + 1 \le k \le m_{n+1}$, $m_n = \sum_{j=0}^{n} i_j$, $n = 1, 2, \dots$, $m_0 = i_0 = 0$. It is clear that the series $\sum_{k=1}^{\infty} a_k$ converges unconditionally in X. The sequence $(a_k\gamma_k)$ converges a.s. to 0 if and only if (see [15], p. 48)

$$\sum_{n=1}^{\infty} \exp\Big\{-\frac{\varepsilon}{\|a_n\|^2}\Big\} = \sum_{n=1}^{\infty} i_n \exp\Big\{-\frac{\varepsilon}{\|b_n\|^2}\Big\} < \infty \text{ for every } \varepsilon > 0\,. \tag{3.4}$$

If the series $\sum_{k=1}^{\infty} a_k\gamma_k$ converges a.s. unconditionally, then, in particular, so does the series $\sum_{n=1}^{\infty}\Big(\sum_{k=m_{n-1}+1}^{m_n} a_k\gamma_k\Big)$ whose general term therefore must be converging a.s. to 0 as $n \to \infty$. But the general term of the series has the form

$$b_n(\gamma_{m_{n-1}+1} + \cdots + \gamma_{m_n}), \quad n = 1, 2, \dots,$$

whose convergence a.s. to 0 as $n \to \infty$ is equivalent to the condition

$$\sum_{n=1}^{\infty} \exp\Big\{-\frac{\varepsilon}{i_n\|b_n\|^2}\Big\} < \infty \text{ for every } \varepsilon > 0\,. \tag{3.5}$$

Choosing $i_0 = 0$ and $i_n = [(\ln(n+1))^{1/2}] - 1$, where $[\alpha]$ is the integer part of the number α and $\|b_n\| = (\ln(n+1))^{-3/4}$, $n = 1, 2, \dots$, it is easy to check that assumptions (3.3) and (3.4) are fulfilled, but condition (3.5) does not take place.

2° ⇒1°. Conversely, suppose that X does not contain l_∞^n's uniformly. Then contrary to 2°, according to Theorem 3.1 the unconditional convergence of a series $\sum_{k=1}^{\infty} a_k$ in X implies the a.s. unconditional convergence of the series $\sum_{k=1}^{\infty} a_k\gamma_k$. □

Now we shall consider multiplicative properties of the a.s. unconditionally convergent series $\sum_{k=1}^{\infty} a_k\gamma_k$.

Proposition 3.2. *Let X be a Banach space, (α_k) be a sequence of real numbers, (γ_k) be a sequence of real standard Gaussian random variables. If*

$$\sum_{k=1}^{\infty} exp\{-\varepsilon\alpha_k^2\} < \infty \text{ for some } \varepsilon > 0\,,$$

then the unconditional convergence of a series $\sum_{k=1}^{\infty} \alpha_k a_k$ in X implies the a.s. unconditional convergence of the series $\sum_{k=1}^{\infty} a_k\gamma_k$.

Proof. Without loss of generality, let us assume that $\alpha_k \neq 0$, $k = 1, 2, \ldots$. Taking into account that, by hypothesis, the sequence $\left(\frac{\gamma_k}{\alpha_k}\right)$ is a.s. bounded (cf. [14], p. 327) and using the relation $a_k\gamma_k = \alpha_k a_k(\gamma_k/\alpha_k)$ we get the a.s. unconditionally convergence of the series $\sum_{k=1}^{\infty} a_k\gamma_k$. □

Analogously, we can easily prove

Proposition 3.3. *Let X, (α_k), (γ_k) and (a_k) satisfy the conditions of Proposition* 3.1. *If*

$$\sum_{k=1}^{\infty} exp\{-\varepsilon\alpha_k^2\} < \infty \text{ for every } \varepsilon > 0\,,$$

then the weak absolute convergence of a series $\sum_{k=1}^{\infty} \alpha_k a_k$ implies the a.s. unconditional convergence of the series $\sum_{k=1}^{\infty} a_k\gamma_k$.

The converse assertions of Propositions 3.1 and 3.2 are connected with the geometry of a Banach space X.

Theorem 3.4. *Let X be a Banach space, (γ_k) be a sequence of independent Gaussian standard random variables, (α_k) be a sequence of real numbers, $a_k \in X$, $k = 1, 2, \ldots$. The following assertions are equivalent.*

1°. *X contains l_∞^n's uniformly.*

2°. *If the unconditional convergence of a series $\sum_{k=1}^{\infty} \alpha_k a_k$ in X implies the a.s. unconditional convergence of the series $\sum_{k=1}^{\infty} a_k\gamma_k$, then $\sum_{k=1}^{\infty} exp\{-\varepsilon\alpha_k^2\} < \infty$ for some $\varepsilon > 0$.*

Proof. 1° ⇒2°. First note that if the unconditional convergence of a series $\sum_{k=1}^{\infty} \alpha_k a_k$ implies the a.s. unconditional convergence of the series $\sum_{k=1}^{\infty} a_k\gamma_k$, then $\lim_{k\to\infty} \alpha_k = \infty$. Indeed, of we assume the contrary that there exists a number $M > 0$ such that the inequality $|\alpha_k| < M$ holds for infinitely many indices k, i.e. there is a subsequence (α_{k_j}) such that $|\alpha_{k_j}| < M$ for all $j = 1, 2, \ldots$. Choosing now a sequence (a_k) so that the series $\sum_{k=1}^{\infty} \alpha_k a_k$ converges unconditionally and

$|\alpha_{k_j}|\|a_{k_j}\| = M(\ln(j+1))^{-1/2}$ $j = 1, 2, \dots$, then it is easily seen that the series $\sum\limits_{k=1}^{\infty} a_k\gamma_k$ does not converge a.s. unconditionally, hence $\lim\limits_{k\to\infty} \alpha_k = \infty$.

To prove 2° suppose, contrary to our claim, that from the unconditional convergence of a series $\sum\limits_{k=1}^{\infty} \alpha_k a_k$ follows the a.s. unconditional convergence of the series $\sum\limits_{k=1}^{\infty} a_k\gamma_k$, but $\sum\limits_{k=1}^{\infty} \exp\{-\varepsilon\alpha_k^2\} = \infty$ for every $\varepsilon > 0$. From this we deduce the existence of a sequence of positive numbers (β_k^2), for which $\lim\limits_{k\to\infty} \beta_k^2 = \infty$ and $\sum\limits_{k=1}^{\infty} \exp\{-\beta_k^2\alpha_k^2\} = \infty$.

Let $\delta_k = |\beta_k|^{-1}$, $k = 1, 2, \dots$. It is obvious that $\lim\limits_{k\to\infty} \delta_k = 0$. For this reason there exists in X an unconditionally convergent series $\sum\limits_{k=1}^{\infty} b_k$ such that $\|b_k\| = \delta_k$, $k = 1, 2, \dots$. Setting $a_k = \alpha_k^{-1} b_k$, $k = 1, 2, \dots$, it is clear that the series $\sum\limits_{k=1}^{\infty} \alpha_k a_k$ converges unconditionally. Then, by assumption, the series $\sum\limits_{k=1}^{\infty} a_k\gamma_k$ converges a.s. unconditionally, which is impossible since the general term of the series does not a.s. converge to 0 because

$$\sum_{k=1}^{\infty} \exp\{-\|a_k\|^{-2}\} = \sum_{k=1}^{\infty} \exp\{-\beta_k^2\alpha_k^2\} = \infty\,.$$

2° ⇒1°. Assume the contrary that X does not contain l_∞^n's uniformly. Then, by Theorem 3.1, the unconditional convergence of a series $\sum\limits_{k=1}^{\infty} a_k$ implies the a.s. unconditional convergence of the series $\sum\limits_{k=1}^{\infty} a_k\gamma_k$ and therefore the condition $\sum\limits_{k=1}^{\infty} \exp\{-\varepsilon\alpha_k^2\} < \infty$ for some $\varepsilon > 0$ is not fulfilled since in that case $\alpha_k = 1$ for every $k = 1, 2, \dots$. □

Theorem 3.5. *Let X, (γ_k), (α_k) and (a_k) satisfy the assumptions of Theorem* 3.4*. Then the following asseritions are equivalent:*

1°*. X contains a subspace isomorphic to c_0.*

2°*. If the weak absolute convergence of a series $\sum\limits_{k=1}^{\infty} \alpha_k a_k$ implies the a.s. unconditional convergence of the series $\sum\limits_{k=1}^{\infty} a_k\gamma_k$ in X, then*

$$\sum_{k=1}^{\infty} \exp\{-\varepsilon\alpha_k^2\} < \infty \textit{ for every } \varepsilon > 0\,.$$

Proof. 1° ⇒2°. On the contrary, suppose that the weak absolute convergence of a series $\sum\limits_{k=1}^{\infty} \alpha_k a_k$ implies the a.s. unconditional convergence of the series $\sum\limits_{k=1}^{\infty} a_k\gamma_k$, but $\sum\limits_{k=1}^{\infty} \exp\{-\varepsilon\alpha_k^2\} = \infty$ for some $\varepsilon > 0$. It is well-known that in Banach spaces containing a subspace isomorphic to c_0 there exists a weakly absolutely converging series $\sum_{k=1}^{\infty} b_k$ such that $\inf_k \|b_k\| \geq \delta > 0$ [11]. Let $a_k = \alpha_k^{-1} b_k$, $k = 1, 2, \dots$. It is clear that the series $\sum_{k=1}^{\infty} \alpha_k a_k$ converges weakly absolutely. Then, by 2°, the series $\sum_{k=1}^{\infty} a_k \gamma_k$ converges a.s. unconditionally. But this is impossible, since the general term of the series a.s. converges to 0 if and only if

$$\sum_{k=1}^{\infty} \exp\{-\beta \|a_k\|^{-2}\} = \sum_{k=1}^{\infty} \exp\{-\beta \alpha_k^2 \|b_k\|^{-2}\} < \infty$$

for every $\beta > 0$. Now setting $\beta = \varepsilon \delta^2$, the last series will, by assumption, be diverging and consequently the series $\sum_{k=1}^{\infty} a_k \gamma_k$ does not converge.

2° ⇒1°. To obtain a contradiction, suppose that X does not contain a subspace isomorphic to c_0. Then the weak absolute convergence of the series implies its unconditional convergence [11]. If, in addition, X does not contain l_∞^n's uniformly, then, by Theorem 3.1, the unconditional convergence of a series $\sum_{k=1}^{\infty} a_k$ implies the a.s. unconditional convergence of the series $\sum_{k=1}^{\infty} a_k \gamma_k$, but in this case $\sum_{k=1}^{\infty} \exp\{-\varepsilon \alpha_k^2\} = \infty$ for every $\varepsilon > 0$, since in that case $\alpha_k = 1$ for all $k = 1, 2, \dots$. This contradiction proves 1° when X does not contain l_∞^n's uniformly.

Now let X contain l_∞^n's uniformly. Then by Theorem 3.4 the unconditional convergence of a series $\sum_{k=1}^{\infty} (\ln(k+1))^{1/2} a_k$ implies the a.s. unconditional convergence of the series $\sum_{k=1}^{\infty} a_k \gamma_k$, while the series $\sum_{k=1}^{\infty} \exp\{-\varepsilon \ln(k+1)\}$ is diverging for $0 < \varepsilon \leq 1$. The obtained contradiction completes the proof of Theorem 3.5. □

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References

1. M. M. Day, Normed linear spaces. *Springer-Velag, Berlin-Heidelberg-New York,* 1973.

2. V. V. Kvaratskhelia, On the convergence of series of Gaussian random elements. (Russian) *Soobshch. Akad. Nauk Gruz. SSR* **76**(1974), No. 1, 41–44.

3. V. V. Kvaratskhelia, On unconditional convergence in Banach spaces. (Russian) *Soobshch. Akad. Nauk Gruz. SSR* **90**(1978), No. 3, 533–536.

4. V. V. Kvaratskhelia, Unconditional convergence of Gaussian series. (Russian) *Tr. Vychisl. Tsentra im. N.I. Muskhelishvili* **18**(1978), No. 2, 71–79.

5. V. V. Kvaratskhelia, On unconditional convergence of random series in Banach spaces. *Lect. Notes Math.* **828**(1980), 162–166.

6. V. V. Kvaratskhelia, Some unconditionally convergent random series in Banach spaces. *3rd International Vilnius Conference on Probability Theory and Mathematical Statisties, Vilnius, Abstracts of communications,* 1981, 182–183.

7. V. V. Kvaratskhelia, Unconditional convergence of random series in Some Banach spaces. 14^{th} *European Meeting of Statisticians, Wroclaw, Poland, Abstracts of communications (supplement),* 1981.

8. V. V. Kvaratskhelia, Unconditional convergence of random series in Banach spaces. 5^{th} *Japan-USSR Symposium on Probability Theory, Abstracts of communications,* 1986, 28.

9. A. Pietsch, Verallgemeinerte vollkomene Folgenräume, *Akademie-Verlag, Berlin,* 1962.

10. A. Pietsch, Nukleare lokalkonvexe Räume. *Akademie-Verlag, Berlin,* 1965.

11. C. Bessaga, A. Pelczyński, On bases and unconditional convergence of series in Banach spaces. *Studia Math.* **17**(1958), No. 2, 151–164.

12. B. Maurey, Théorèmes de factorisation pour les opérateurs linéaires à valeurs dans les espaces L^p. *Astérisque* **11**(1974), 1–163.

13. S. A. Rakov, Banach spaces in which a theorem of Orlicz is not true. (Russian) *Mat. Zametki* **14**(1973), No. 1, 101–106.

14. N. N. Vakhania, V. I. Tarieladze, S. A. Chobanyan, Probability distributions on Banach spaces. *Dordrecht, Reidel,* 1987.

15. N. N. Vakhania, Probability distributions on linear spaces. *North Holland, N. Y.,* 1981.

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