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ASYMPTOTIC DISTRIBUTION OF EIGENELEMENTS OF THE BASIC TWO-DIMENSIONAL BOUNDARY-CONTACT PROBLEMS OF OSCILLATION IN CLASSICAL AND COUPLE-STRESS THEORIES OF ELASTICITY

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Abstract. The basic boundary-contact problems of oscillation are considered for a two-dimensional piecewise-homogeneous isotropic elastic medium bounded by several closed curves. Asymptotic formulas for the distribution of eigenfunctions and eigenvalues of the considered problems are derived using the correlation method.

1. In this paper the following notation will be used: $\mathbb{R}^2$ is a two-dimensional Euclidean space; $x=(x_1,x_2)$, $y=(y_1,y_2)$ are points in $\mathbb{R}^2$; $|x-y|$ is the Euclidean distance between x and y; $D_0\subset\mathbb{R}^2$ is a finite domain bounded by the closed curves $S_0,S_1,\dots,S_m$ of the class $\Lambda_2(\alpha)$, $0<\alpha\leq 1$ (the curves have a Hölder-continuous curvature), S_0 is enveloping all other S_k, while the latters are not enveloping one another, $S_i\cap S_k=\varnothing$ for $i\neq k$, $i,k=\overline{0,m}$; the finite domain bounded by the curve S_k ($k=\overline{1,m}$) is denoted by D_k, $\overline{D}_0=D_0\cup(\bigcup\limits_{k=0}^{m}S_k)$, $\overline{D}_k=D_k\cup S_k$, $k=\overline{1,m}$.

If u and v are the n-component real-valued vectors $u=(u_1,u_2,\dots,u_n)$, $v=(v_1,v_2,\dots,v_n)$, then uv denotes the scalar product of these vectors: $uv=\sum\limits_{k=1}^{n}u_kv_k$; $|u|=(\sum\limits_{k=1}^{n}u_k^2)^{1/2}$. The matrix product is obtained by multiplying a row vector by a column vector; if $A=\|A_{ij}\|_{n\times n}$ is an $n\times n$-matrix, then $|A|^2=\sum\limits_{i,j=1}^{n}A_{ij}^2$. Any vector $u=(u_1,u_2,\dots,u_n)$ is considered as an $n\times 1$ one-column matrix: $u=\|u_i\|_{n\times 1}$; $A_k=\|A_{jk}\|_{j=1}^{n}$ is the k-th column vector of the matrix A.

The vector $u(x)=(u_1(x),u_2(x),\dots,u_n(x))$ is called regular in D_k if $u_i\in C^1(D_k)\cap C^2(\overline{D}_k)$, $i=\overline{1,n}$.

A system of homogeneous differential equations of oscillation of the classical

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plane theory of elasticity for a homogeneous isotropic medium has the form [1]

$$\begin{cases}\mu\Delta u_1+(\lambda+\mu)\Big(\dfrac{\partial^2 u_1}{\partial x_1^2}+\dfrac{\partial^2 u_2}{\partial x_1\partial x_2}\Big)+\rho\sigma^2 u_1=0,\\ \mu\Delta u_2+(\lambda+\mu)\Big(\dfrac{\partial^2 u_1}{\partial x_1\partial x_2}+\dfrac{\partial^2 u_2}{\partial x_2^2}\Big)+\rho\sigma^2 u_2=0,\end{cases}\tag{1}$$

while a system of homogeneous differential equations of oscillation of the couple-stress plane theory of elasticity for a homogeneous isotropic centrosymmetric medium is written as [1, 2]

$$\begin{cases}(\mu+\alpha)\Delta u_1+(\lambda+\mu-\alpha)\Big(\dfrac{\partial^2 u_1}{\partial x_1^2}+\dfrac{\partial^2 u_2}{\partial x_1\partial x_2}\Big)+2\alpha\,\dfrac{\partial\omega}{\partial x_2}+\rho\sigma^2 u_1=0,\\ (\mu+\alpha)\Delta u_2+(\lambda+\mu-\alpha)\Big(\dfrac{\partial^2 u_1}{\partial x_1\partial x_2}+\dfrac{\partial^2 u_2}{\partial x_2^2}\Big)-2\alpha\,\dfrac{\partial\omega}{\partial x_1}+\rho\sigma^2 u_2=0,\\ (\nu+\beta)\Delta\omega+2\alpha\Big(\dfrac{\partial u_2}{\partial x_1}-\dfrac{\partial u_1}{\partial x_2}\Big)-4\alpha\omega+I\sigma^2\omega=0,\end{cases}\tag{2}$$

where Δ is the two-dimensional Laplace operator, $u(x)=(u_1,u_2)$ the displacement vector, ω a component of the rotation vector, $\rho=\mathrm{const}>0$ the medium density, $I=\mathrm{const}>0$ inertia moment, σ oscillation frequency; λ, μ, α, ν, β are the elastic constants satisfying the conditions: $\mu>0$, $3\lambda+2\mu>0$, $\alpha>0$, $\nu>0$, $\beta>0$.

Systems (1) and (2) can be written in the vector-matrix terms

$$A(\partial x)u(x)+\overset{1}{r}\sigma^2 u=0,\tag{3}$$

$$M(\partial x)v(x)+\overset{2}{r}\sigma^2 v=0,\tag{4}$$

respectively, where $A(\partial x)$ and $M(\partial x)$ are respectively 2×2 and 3×3 matrix differential operators whose elements are easily defined by virtue of (1) and (2); $v(x)=(u(x),\omega(x))=(u_1(x),u_2(x),\omega(x))=(v_1,v_2,v_3)$; $\overset{1}{r}$ and $\overset{2}{r}$ are respectively the 2×2 and 3×3 diagonal matrices: $\overset{1}{r}=\|r_{ij}\|_{2\times 2}$, $\overset{2}{r}=\|r_{ij}\|_{3\times 3}$ for $r_{ij}=0$, $i\neq j$, $r_{11}=r_{22}=\rho$, $r_{33}=I$.

(3) and (4) can be rewritten as

$$\widetilde{A}(\partial x)\widetilde{u}(x)+\sigma^2\widetilde{u}(x)=0,\quad \widetilde{M}(\partial x)\widetilde{v}(x)+\sigma^2\widetilde{v}(x)=0,$$

where $\widetilde{A}=\overset{1}{\widetilde{r}}{}^{-1}A\overset{1}{\widetilde{r}}{}^{-1}$, $\widetilde{M}=\overset{2}{\widetilde{r}}{}^{-1}M\overset{2}{\widetilde{r}}{}^{-1}$, $\widetilde{u}=\overset{1}{\widetilde{r}}u$, $\widetilde{v}=\overset{2}{\widetilde{r}}v$, $\overset{1}{\widetilde{r}}=\|\sqrt{r_{ij}}\|_{2\times 2}$, $\overset{2}{\widetilde{r}}=\|\sqrt{r_{ij}}\|_{3\times 3}$.

The matrix differential operator $T(\partial x,n(x))=\|T_{ij}(\partial x,n(x))\|_{2\times 2}$, where

$$T_{ij}(\partial x,n(x))=\lambda n_i(x)\,\frac{\partial}{\partial x_j}+\mu n_j(x)\,\frac{\partial}{\partial x_i}+\mu\delta_{ij}\,\frac{\partial}{\partial n(x)}$$

and $n(x)$ is an arbitrary unit vector at the point x (if $x\in S_k$, $k=\overline{0,m}$, then $n(x)$ is the external, with respect to D_0, normal unit vector of the curve S_k at the point x), is called the stress operator in the classical theory of elasticity, and the operator $T^M(\partial x, n(x)) = \|T^M_{ij}(\partial x, n(x))\|_{3\times 3}$, where

$$T^M_{ij}(\partial x, n(x)) = \lambda n_i(x)\,\frac{\partial}{\partial x_j} + (\mu-\alpha)n_j(x)\,\frac{\partial}{\partial x_i} + (\mu+\alpha)\delta_{ij}\,\frac{\partial}{\partial n(x)}\,, \quad i,j=1,2,$$

$$T^M_{ij}(\partial x, n(x)) = 2\alpha\sum_{k=1}^{2}\varepsilon_{ijk}n_k(x), \quad j=3, \quad i=1,2,$$

$$T^M_{ij}(\partial x, n(x)) = 0, \quad i=3, \quad j=1,2; \quad T^M_{33}(\partial x, n(x)) = (\nu+\beta)\,\frac{\partial}{\partial n(x)}\,,$$

is called the stress operator in the couple-stress theory of elasticity. Here δ_{ij} is the Kronecker symbol and ε_{ijk} the Levy-Civita symbol.

We assume that the domains D_k ($k=\overline{1,m_0}$) are filled with homogeneous isotropic elastic media with constants λ_k, μ_k, α_k, ν_k, β_k, ρ_k, I_k while the rest of the domains D_k ($k=m_0+1,\dots,m$) are hollow inclusions. When in the operators A, M, T, T^M figure λ_k, μ_k, α_k, ν_k, β_k instead of λ, μ, α, ν, β, then we will write $\overset{k}{A}$, $\overset{k}{M}$, $\overset{k}{T}$, $\overset{k}{T}{}^M$ respectively.

We introduce the notation

$$u^+(z) = \lim_{D_0\ni x\to z\in S_k} u(x), \quad k=\overline{0,m}; \quad u^-(z) = \lim_{D_k\ni x\to z\in S_k} u(x), \quad k=\overline{0,m_0}.$$

The notation $(T(\partial z, n(z)))^{\pm}$ has a similar meaning.

2. A matrix of fundamental solutions of the homogeneous equation of oscillation (3) has the form

$$\Gamma(x-y,\sigma^2) = \Big\|\Gamma_{kj}(x-y,\sigma^2)\Big\|_{2\times 2}, \quad \Gamma_{kj}(x-y,\sigma^2) = \frac{\delta_{kj}}{2\pi\mu}\left[\frac{\pi i}{2}\,H_0^{(1)}(k_2 r)\right] - \frac{1}{2\pi\rho\sigma^2}\,\frac{\partial^2}{\partial x_k\partial x_j}\left[\frac{\pi i}{2}\,H_0^{(1)}(k_1 r) - \frac{\pi i}{2}\,H_0^{(1)}(k_2 r)\right], \quad k,j=1,2,$$

where i is the imaginary unit, $r=|x-y|$, $H_n^{(1)}$ is the n-th order Hankel function of first kind with a nonnegative integer n; k_1 and k_2 are the nonnegative numbers defined by the equalities $k_1^2=\rho\sigma^2/(\lambda+2\mu)$, $k_2^2=\rho\sigma^2/\mu$.

A matrix of fundamental solutions of equation (4) is of the form

$$\Gamma^M(x-y,\sigma^2) = \Big\|\Gamma^M_{kj}(x-y,\sigma^2)\Big\|_{3\times 3},$$

$$\Gamma^M_{kj}(x-y,\sigma^2) = \frac{\delta_{kj}}{2\pi(\mu+\alpha)}\sum_{l=2}^{3}\frac{(-1)^l(\sigma_2^2-k_l^2)}{k_3^2-k_2^2}\,\frac{\pi i}{2}\,H_0^{(1)}(k_l r) - \frac{\partial^2}{\partial x_k\partial x_j}\left[\frac{1}{2\pi\rho\sigma^2}\,\frac{\pi i}{2}\,H_0^{(1)}(k_1 r) + \frac{1}{2\pi(\mu+\alpha)}\sum_{l=2}^{3}\frac{(-1)^l(\sigma_2^2-k_l^2)}{(k_2^2-k_3^2)k_l^2}\,\frac{\pi i}{2}\,H_0^{(1)}(k_l r)\right], \quad k,j=1,2;$$

$$\Gamma_{kj}^M(x-y,\sigma^2)$$
$$=\frac{\alpha}{\pi(\mu+\alpha)(\nu+\beta)(k_3^2-k_2^2)}\sum_{p=1}^{2}\varepsilon_{kjp}\frac{\partial}{\partial x_p}\frac{\pi i}{2}\Big[H_0^{(1)}(k_2r)-H_0^{(1)}(k_3r)\Big],$$
$$k=3,\quad j=1,2\quad\text{or}\quad j=3,\quad k=1,2,$$
$$\Gamma_{33}^M(x-y,\sigma^2)=\frac{1}{2\pi(\nu+\beta)}\sum_{l=2}^{3}\frac{(-1)^l(\sigma_1^2-k_l^2)}{k_3^2-k_2^2}\frac{\pi i}{2}H_0^{(1)}(k_lr),$$

where

$$k_1^2=\frac{\rho\sigma^2}{\lambda+2\mu},\quad k_2^2+k_3^2=\frac{\rho\sigma^2}{\mu+\alpha}+\frac{I\sigma^2-4\alpha}{\nu+\beta}+\frac{4\alpha^2}{(\mu+\alpha)(\nu+\beta)},$$
$$k_2^2k_3^2=\frac{\rho\sigma^2(I\sigma^2-4\alpha)}{(\mu+\alpha)(\nu+\beta)},\quad \sigma_1^2=\frac{\rho\sigma^2}{\mu+\alpha},\quad \sigma_2^2=\frac{I\sigma^2-4\alpha}{\nu+\beta}.$$

In what follows we will be interested in the asymptotic behavior of fundamental solutions for $\sigma\to\infty$. Therefore we take for k_2^2, k_3^2, σ_2^2 the following asymptotic values:

$$k_2^2\approx\frac{\rho\sigma^2}{\mu+\alpha},\quad k_3^2\approx\frac{I\sigma^2}{\nu+\beta},\quad \sigma_2^2\approx\frac{I\sigma^2}{\nu+\beta}. \tag{5}$$

Due to (5) the matrix of fundamental solutions $\Gamma^M(x-y,\sigma^2)$ can be written in the form (as $\sigma\to\infty$)

$$\Gamma_{kj}^M(x-y,\sigma^2)\approx\frac{\delta_{kj}}{2\pi(\mu+\alpha)}\frac{\pi i}{2}H_0^{(1)}(k_2r)$$
$$-\frac{1}{2\pi\rho\sigma^2}\frac{\partial^2}{\partial x_k\partial x_j}\Big[\frac{\pi i}{2}H_0^{(1)}(k_1r)-\frac{\pi i}{2}H_0^{(1)}(k_2r)\Big],\quad k,j=1,2,$$
$$\Gamma_{kj}^M(x-y,\sigma^2)\approx\frac{\alpha}{\pi\sigma^2(I(\mu+\alpha)-\rho(\nu+\beta))}\sum_{\rho=1}^{2}\varepsilon_{kjp}\frac{\partial}{\partial x_p}\Big[\frac{\pi i}{2}H_0^{(1)}(k_2r)$$
$$-\frac{\pi i}{2}H_0^{(1)}(k_3r)\Big],\quad k=3,\quad j=1,2\quad\text{or}\quad j=3,\quad k=1,2,$$
$$\Gamma_{33}^M(x-y,\sigma^2)\approx\frac{1}{2\pi(\nu+\beta)}\frac{\pi i}{2}H_0^{(1)}(k_3r).$$

Let $\varkappa_0$ be an arbitrary real positive fixed number and $\varkappa>\varkappa_0$ an arbitrary number. For $\sigma=i\varkappa$ we have

$$\Gamma_{kj}(x-y,-\varkappa^2)=\frac{\delta_{kj}}{2\pi\mu}\Big[\frac{\pi i}{2}H_0^{(1)}(ic_2\varkappa r)\Big]$$
$$+\frac{1}{2\pi\rho\varkappa^2}\frac{\partial^2}{\partial x_k\partial x_j}\Big[\frac{\pi i}{2}H_0^{(1)}(ic_1\varkappa r)-\frac{\pi i}{2}H_0^{(1)}(ic_2\varkappa r)\Big],\quad k,j=1,2,$$

where

$$c_1^2 = \frac{\rho}{\lambda + 2\mu}\,, \quad c_2^2 = \frac{\rho}{\mu}\,,$$

$$\Gamma_{kj}^M(x-y, -\varkappa^2) = \frac{\delta_{kj}}{2\pi(\mu+\alpha)} \Big[\frac{\pi i}{2}\, H_0^{(1)}(ic_2\varkappa r)\Big]$$

$$+\frac{1}{2\pi\rho\varkappa^2}\, \frac{\partial^2}{\partial x_k \partial x_j} \Big[\frac{\pi i}{2}\, H_0^{(1)}(ic_1\varkappa r) - \frac{\pi i}{2}\, H_0^{(1)}(ic_2\varkappa r)\Big], \quad k,j = 1,2,$$

$$\Gamma_{kj}^M(x-y, -\varkappa^2)$$

$$= -\frac{\alpha}{\pi\varkappa^2(I(\mu+\alpha) - \rho(\nu+\beta))} \sum_{p=1}^{2} \varepsilon_{kjp}\, \frac{\partial}{\partial x_p} \Big[\frac{\pi i}{2}\, H_0^{(1)}(ic_2\varkappa r)$$

$$-\frac{\pi i}{2}\, H_0^{(1)}(ic_3\varkappa r)\Big], \quad k=3, \quad j=1,2 \quad \text{or} \quad j=3, \quad k=1,2,$$

$$\Gamma_{33}^M(x-y, -\varkappa^2) = \frac{1}{2\pi(\nu+\beta)} \Big[\frac{\pi i}{2}\, H_0^{(1)}(ic_3\varkappa r)\Big],$$

where $c_1^2 = \frac{\rho}{\lambda+2\mu}$, $c_2^2 = \frac{\rho}{\mu+\alpha}$, $c_3^2 = \frac{I}{\nu+\beta}$.

We will consider two possible cases:

1. $\varkappa r$ is bounded for $r \to 0$ and $\varkappa \to \infty$;
2. $\varkappa r$ is unbounded for $r \to 0$ and $\varkappa \to \infty$.

When $\varkappa r$ is bounded, taking into account repersentations of the Hankel function, we have

$$|\Gamma_{pq}(x-y, -\varkappa^2)| \leq \text{const}\, |\ln \varkappa r|, \quad \Big|\frac{\partial^n \Gamma_{pq}(x-y, -\varkappa^2)}{\partial x_1^i \partial x_2^j}\Big| \leq \frac{\text{const}}{r^n}, \tag{6}$$
$$i+j = n, \quad n = 1,2,\dots\,; \quad p,q = 1,2.$$

The same estimates are obtained for $\Gamma_{pq}^M(x-y, -\varkappa^2)$.

When $\varkappa r$ is unbounded, we use the asymptotic representation of the Hankel function for large $|z|$ [3]

$$H_n^{(1)}(z) = \sqrt{\frac{2}{\pi z}}\, e^{i(z - n\,\frac{\pi}{2} - \frac{\pi}{4})} \Big[1 + O\Big(\frac{1}{z}\Big)\Big],$$

and the recurrent formula [3] $2\,\frac{d}{dz}\, H_n^{(1)}(z) = H_{n-1}^{(1)}(z) - H_{n+1}^{(1)}(z)$, to obtain

$$\Big|\frac{\partial^n \Gamma_{pq}(x-y, -\varkappa^2)}{\partial x_1^i \partial x_2^j}\Big| \leq \frac{\text{const}\, \varkappa^n}{\sqrt{\varkappa r}}\, e^{-a\varkappa r}, \tag{7}$$
$$i+j = n, \quad n = 0,1,2,\dots\,; \quad p,q = 1,2,$$

where $a = c_1 - \delta > 0$, $\delta < \frac{1}{2}\, c_1$ is an arbitrary positive number;

$$\Big|\frac{\partial^n \Gamma_{pq}^M(x-y, -\varkappa^2)}{\partial x_1^i \partial x_2^j}\Big| \leq \frac{\text{const}\, \varkappa^n}{\sqrt{\varkappa r}}\, e^{-b\varkappa r},$$

where $b = c^* - \delta$, $\delta < \frac{1}{2}c^*$ is an arbitrary positive number, c^* is the smallest of the numbers c_1, c_2, c_3.

3. Let $x, y \in D_k$, $k = \overline{0, m_0}$ and ly be the distance from the point y to the boundary of the domain D_k. We denote $\rho_y(x) = \max\{r, ly\}$ and introduce the auxiliary matrix

$$\overset{\widehat{k}}{\Gamma}(x-y,-\varkappa^2) = \Big[1 - \Big(1 - \frac{r^m}{\rho_y^m(x)}\Big)^n\Big]\overset{k}{\Gamma}(x-y,-\varkappa^2). \tag{8}$$

Let us denote by $K(y, ly)$ the circle of radius ly and center at the point y, and by $C(y, ly)$ the boundary of this circle. We easily find that $(1-r^m/\rho_y^m(x))^n$ vanishes together with its derivatives up to $(n-1)$-th order inclusive when the point $x \in K(y, ly)$ tends to a point of the boundary $C(y, ly)$. For $x \in D_k \backslash \overline{K(y, ly)}$ we have $1-(1-r^m/\rho_y^m(x))^n = 1$ and

$$\lim_{K(y,ly)\ni x\to z\in C(y,ly)} \Big[1 - \Big(1 - \frac{r^m}{\rho_y^m(x)}\Big)^n\Big] = 1.$$

Therefore $\overset{\widehat{k}}{\Gamma}(x-y,-\varkappa^2) = \overset{k}{\Gamma}(x-y,-\varkappa^2)$ for $x \in D_k \backslash K(y, ly)$, while, in crossing the boundary $C(y, ly)$, the function $\overset{\widehat{k}}{\Gamma}$ and its derivatives up to $(n-1)$-th order inclusive remain continuous.

We write the function $\overset{\widehat{k}}{\Gamma}$ in the form

$$\overset{\widehat{k}}{\Gamma}(x-y,-\varkappa^2) = \overset{k}{\Gamma}(x-y,-\varkappa^2)\Big(n\,\frac{r^m}{\rho_y^m(x)} + \dots\Big).$$

It is easy to verify that for $x = y$ the function $\overset{\widehat{k}}{\Gamma}$ and its derivatives up to $(m-2)$-th order inclusive are continuous. By virtue of (6), for $x \in K(y, ly)$, when $\varkappa r$ is bounded, we have the estimates

$$\Big|\overset{\widehat{k}}{\Gamma}_{pq}(x-y,-\varkappa^2)\Big| \le \frac{\text{const}\, r^{m-\alpha}}{\varkappa^\alpha l_y^m}, \quad \Big|\frac{\partial^s \overset{\widehat{k}}{\Gamma}_{pq}(x-y,-\varkappa^2)}{\partial x_1^i \partial x_2^j}\Big| \le \frac{\text{const}\, r^{m-s}}{l_y^m}, \tag{9}$$
$$i+j=s; \quad m \ge s+1, \quad p,q = 1,2,$$

where α is an arbitrary number, $0 < \alpha < 1$.

When $\varkappa r$ is unbounded, by virtue of (7) we have the estimates

$$\Big|\frac{\partial^s \overset{\widehat{k}}{\Gamma}_{pq}(x-y,-\varkappa^2)}{\partial x_1^i \partial x_2^j}\Big| \le \frac{\text{const}\,\varkappa^s e^{-a\varkappa r}}{\sqrt{\varkappa}\, l_y^m}\, r^{m-\frac{1}{2}}. \tag{10}$$

Analogous estimates hold for $\overset{\widehat{k}}{\Gamma}{}^M(x-y,-\varkappa^2)$.

4. Let us calculate the limit

$$\lim_{x\to y}\Big[\overset{k}{\Gamma}(x-y,-\varkappa^2)-\overset{k}{\Gamma}(x-y,-\varkappa_0^2)\Big],\quad x,\,y\in D_k,\quad k=\overline{0,m_0}.$$

Using the expansion

$$H_0^{(1)}(z)=I_0(z)+\frac{2i}{\pi}\,I_0(z)\ln\frac{z}{2}-\frac{2i}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{(k!)^2}\Big(\frac{z}{2}\Big)^{2k}\Big(1+\frac{1}{2}+\cdots+\frac{1}{k}-\gamma\Big),$$

where γ is the Euler's constant,

$$I_0(z)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(k!)^2}\Big(\frac{z}{2}\Big)^{2k}=1-\frac{z^2}{2^2(1!)^2}+\frac{z^4}{2^4(2!)^2}-\frac{z^6}{2^6(3!)^2}+\cdots,$$

$$I_0(ic_1\varkappa r)-I_0(ic_2\varkappa r)=\frac{\varkappa^2(c_1^2-c_2^2)}{2^2(1!)^2}\,r^2+\frac{\varkappa^4(c_1^4-c_2^4)}{2^4(2!)^2}\,r^4-\cdots$$

and the following evident relations

$$\frac{\partial^2 r^2}{\partial x_k\partial x_j}=2\delta_{kj}\,,\quad \ln\frac{ic_1\varkappa r}{2}=\ln\frac{ic_1}{2}+\ln\varkappa+\ln r,$$

for p, $q=1,2$ we have

$$\lim_{x\to y}\Big[\overset{k}{\Gamma}_{pq}(x-y,-\varkappa^2)-\overset{k}{\Gamma}_{pq}(x-y,-\varkappa_0^2)\Big]=\frac{1}{2\pi\mu_k}\,\delta_{pq}\,\ln\frac{\varkappa_0}{\varkappa}$$
$$+\frac{1}{4\pi}\,\delta_{pq}\Big(\frac{1}{\lambda_k+2\mu_k}-\frac{1}{\mu_k}\Big)\ln\frac{\varkappa_0}{\varkappa}=\frac{1}{4\pi}\Big(\frac{1}{\lambda_k+2\mu_k}+\frac{1}{\mu_k}\Big)\delta_{pq}\ln\frac{\varkappa_0}{\varkappa}\,. \tag{11}$$

Quite similarly, we obtain

$$\left.\begin{aligned}&\lim_{x\to y}\Big[\overset{k}{\Gamma}{}^M_{pq}(x-y,-\varkappa^2)-\overset{k}{\Gamma}{}^M_{pq}(x-y,-\varkappa_0^2)\\&=\frac{1}{4\pi}\Big(\frac{1}{\lambda_k+2\mu_k}+\frac{1}{\mu_k+\alpha_k}\Big)\delta_{pq}\ln\frac{\varkappa_0}{\varkappa}\,,\quad p,\,q=1,2,\\&\lim_{x\to y}\Big[\overset{k}{\Gamma}{}^M_{33}(x-y,-\varkappa^2)-\overset{k}{\Gamma}{}^M_{33}(x-y,-\varkappa_0^2)\Big]=\frac{1}{2\pi}\,\frac{1}{\nu_k+\beta_k}\,\ln\frac{\varkappa_0}{\varkappa}\,.\end{aligned}\right\} \tag{12}$$

5. Further we will investigate the first problem. The other problems are considered similarly.

The 2×2 matrix $G(x,y,-\varkappa_0^2)=\overset{k}{G}(x,y,-\varkappa_0^2)$, $x\in D_k$, $y\in D=\bigcup\limits_{k=0}^{m_0}D_k$, $x\neq y$, $k=\overline{0,m_0}$, denotes the Green tensor of the first basic boundary value problem of the operator $\overset{k}{A}(\partial x)-\varkappa_0^2E$ (E is the 2×2 unit matrix).

According to [4], $G(x,y,-\varkappa_0^2)$ possesses a symmetry property of the form

$$G(x,y,-\varkappa_0^2)=G^{\top}(y,x,-\varkappa_0^2), \tag{13}$$

where the symbol $\top$ denotes the matrix transposition operation. Moreover, we obtain the estimates [5]

$$\left.\begin{aligned} &\forall (x,y)\in D_k\times D_k:\ \ G_{pq}(x,y,-\varkappa_0^2)=O\Big(|\ln|x-y||\Big),\\ &\frac{\partial}{\partial x_j}G_{pq}(x,y,-\varkappa_0^2)=O(|x-y|^{-1}),\quad p,q,j=1,2,\ \ k=\overline{0,m_0}. \end{aligned}\right\}\tag{14}$$

In a similar way we can define the Green tensor of the first basic boundary-contact problem of the operator $\overset{k}{M}(\partial x)-\varkappa_0^2E$ (E is the 3×3 unit matrix) $G^M(x,y,-\varkappa_0^2)$ which is of size 3×3. This tensor has property (13) and estimates (14) hold for it.

6. Let $u(x)=\overset{k}{u}(x)$ and $v(x)=\overset{k}{v}(x)$, $x\in D_k$ be arbitrary regular vectors of the class $C^1(\overline{D}_k)\cap C^2(D_k)$. Then we have the Green formula [4]

$$\begin{aligned} &\sum_{k=0}^{m_0}\int\limits_{D_k}\Big[\overset{k}{v}\overset{k}{A}\overset{k}{u}+\overset{k}{E}(\overset{k}{v},\overset{k}{u})\Big]\,dx\\ &=\int\limits_{S}\overset{0}{v}{}^{+}(\overset{0}{T}\overset{0}{u})^{+}\,dS+\sum_{k=1}^{m_0}\int\limits_{S_k}\Big[\overset{0}{v}{}^{+}(\overset{0}{T}\overset{0}{u})^{+}-\overset{k}{v}{}^{-}(\overset{k}{T}\overset{k}{u})^{-}\Big]\,dS, \end{aligned}\tag{15}$$

where $S=S_0\cup(\bigcup\limits_{k=m_0+1}^{m}S_k)$ and

$$\overset{k}{E}(\overset{k}{v},\overset{k}{u})=\sum_{p,q=1}^{2}\Big(\mu_k\frac{\partial\overset{k}{v}_p}{\partial x_q}\frac{\partial\overset{k}{u}_p}{\partial x_q}+\lambda_k\frac{\partial\overset{k}{v}_p}{\partial x_q}\frac{\partial\overset{k}{u}_q}{\partial x_q}+\mu_k\frac{\partial\overset{k}{v}_p}{\partial x_q}\frac{\partial\overset{k}{u}_q}{\partial x_p}\Big).\tag{16}$$

We can rewrite (16) as

$$\begin{aligned} \overset{k}{E}(\overset{k}{v},\overset{k}{u})&=\frac{3\lambda_k+2\mu_k}{3}\Big(\frac{\partial\overset{k}{v}_1}{\partial x_1}+\frac{\partial\overset{k}{v}_2}{\partial x_2}\Big)\Big(\frac{\partial\overset{k}{u}_1}{\partial x_1}+\frac{\partial\overset{k}{u}_2}{\partial x_2}\Big)\\ &+\frac{\mu_k}{2}\sum_{p\neq q}\Big(\frac{\partial\overset{k}{v}_p}{\partial x_q}+\frac{\partial\overset{k}{v}_q}{\partial x_p}\Big)\Big(\frac{\partial\overset{k}{u}_p}{\partial x_q}+\frac{\partial\overset{k}{u}_q}{\partial x_p}\Big)\\ &+\frac{\mu_k}{3}\sum_{p,\,q}\Big(\frac{\partial\overset{k}{v}_p}{\partial x_p}-\frac{\partial\overset{k}{v}_q}{\partial x_q}\Big)\Big(\frac{\partial\overset{k}{u}_p}{\partial x_p}-\frac{\partial\overset{k}{u}_q}{\partial x_q}\Big). \end{aligned}\tag{17}$$

It follows from (17) that $\overset{k}{E}(\overset{k}{v},\overset{k}{u})=\overset{k}{E}(\overset{k}{u},\overset{k}{v})$ and $\overset{k}{E}(\overset{k}{v},\overset{k}{u})\geq 0$.

For the regular vector $u(x)$ in D_k, $k=\overline{0,m_0}$, we have the integral representation [4]

$$\begin{aligned} \forall y\in D_k:\ u_j(y)=&-\sum_{k=0}^{m_0}\int\limits_{D_k}\overset{k}{\Gamma}_j(x-y,-\varkappa_0^2)\Big[\overset{k}{A}(\partial x)\overset{k}{u}(x)-\varkappa^2\overset{k}{u}(x)\Big]\,dx\\ &+\int\limits_{S}\Big[\overset{0}{\Gamma}_j(z-y,-\varkappa^2)\Big(\overset{0}{T}(\partial z,n(z))\overset{0}{u}(z)\Big)^{+} \end{aligned}$$

$$- \overset{0}{u}{}^{+}(z)\overset{0}{T}(\partial z, n(z))\overset{0}{\Gamma}_j(z-y,-\varkappa^2)\Big]\, d_zS$$
$$+ \sum_{k=1}^{m_0}\int_{S_k}\Big[\overset{0}{\Gamma}_j(z-y,-\varkappa^2)\Big(\overset{0}{T}(\partial z, n(z))\overset{0}{u}(z)\Big)^{+}$$
$$- \overset{k}{\Gamma}_j(z-y,-\varkappa^2)\Big(\overset{k}{T}(\partial z, n(z))\overset{k}{u}(z)\Big)^{-}\Big]\, d_zS$$
$$- \sum_{k=1}^{m_0}\int_{S_k}\Big[\overset{0}{u}{}^{+}(z)\overset{0}{T}(\partial z, n(z))\overset{0}{\Gamma}_j(z-y,-\varkappa^2)$$
$$- \overset{k}{u}{}^{-}(z)\overset{k}{T}(\partial z, n(z))\overset{k}{\Gamma}_j(z-y,-\varkappa^2)\Big]\, d_zS, \quad j=1,2. \tag{18}$$

In the couple-stress theory of elasticity, formulas analogous to (15) and (18) are valid. To write these formuals, in (15) and (18) $\overset{k}{A}$ is to be replaced by $\overset{k}{M}$, $\overset{k}{T}$ by $\overset{k}{T}{}^M$, and $\overset{k}{E}$ by $\overset{k}{E}{}^M$ where

$$\overset{k}{E}{}^M(\overset{k}{v},\overset{k}{u})$$
$$= \sum_{p,\,q=1}^{2}\Big[(\mu_k+\alpha_k)\,\frac{\partial \overset{k}{v}_p}{\partial x_q}\,\frac{\partial \overset{k}{u}_p}{\partial x_q} + \lambda_k\,\frac{\partial \overset{k}{v}_p}{\partial x_p}\,\frac{\partial \overset{k}{u}_q}{\partial x_q} + (\mu_k-\alpha_k)\,\frac{\partial \overset{k}{v}_p}{\partial x_q}\,\frac{\partial \overset{k}{u}_q}{\partial x_p}\Big]$$
$$-2\alpha_k\Big(\frac{\partial \overset{k}{v}_2}{\partial x_1} - \frac{\partial \overset{k}{v}_1}{\partial x_2}\Big)\overset{k}{u}_3 + 2\alpha_k\Big(\frac{\partial \overset{k}{u}_1}{\partial x_2} - \frac{\partial \overset{k}{u}_2}{\partial x_1}\Big)\overset{k}{v}_3$$
$$+(\nu_k+\beta_k)\left(\frac{\partial \overset{k}{v}_3}{\partial x_1}\,\frac{\partial \overset{k}{u}_3}{\partial x_1} + \frac{\partial \overset{k}{v}_3}{\partial x_2}\,\frac{\partial \overset{k}{u}_3}{\partial x_2}\right) + 4\alpha_k\overset{k}{v}_3\overset{k}{u}_3,$$

$u(x)=\overset{k}{u}(x)$ and $v(x)=\overset{k}{v}(x)$ are arbitrary three-component regular vectors in D_k ($k=\overline{0,m_0}$). Note that $\overset{k}{E}{}^M(\overset{k}{v},\overset{k}{u}) = \overset{k}{E}{}^M(\overset{k}{u},\overset{k}{v})$ and $\overset{k}{E}{}^M(\overset{k}{v},\overset{k}{v}) \geq 0$.

7. To establish the asymptotic behavior of eigenfunctions and eigenvalues, it is necessary to estimate the regular parts of the Green tensors as $\varkappa \to \infty$. For this we consider the functional

$$L[u] = \sum_{k=0}^{m_0}\int_{D_k}\Big[\overset{k}{E}(\overset{k}{u},\overset{k}{u} + \varkappa^2\overset{k}{u}{}^2\Big]\, dx - 2\sum_{k=1}^{m_0}\int_{S_k}\Big[\overset{0}{u}{}^{+}(z)\overset{0}{T}(\partial z, n(z))\overset{0}{\Gamma}_j(z-y,-\varkappa^2)$$
$$-\overset{k}{u}{}^{-}(z)\overset{k}{T}(\partial z, n(z))\overset{k}{\Gamma}_j(z-y,-\varkappa^2)\Big]\, d_zS, \tag{19}$$

where $j=1,2$ is a fixed number, y an arbitrary fixed point in D_k, $k=\overline{0,m_0}$. Functional (18) is defined in the class of regular vector functions in D_k ($k=\overline{0,m_0}$) satisfying the conditions:

1) $\forall z \in S_k:\ \overset{0}{u}{}^{+}(z) - \overset{k}{u}{}^{-}(z) = \overset{0}{\Gamma}_j(z-y,-\varkappa^2) - \overset{k}{\Gamma}_j(z-y,-\varkappa^2)$,

$$\Big(\overset{0}{T}\overset{0}{u}(z)\Big)^{+} - \Big(\overset{k}{T}\overset{k}{u}(z)\Big)^{-}$$
$$= \overset{0}{T}(\partial z, n(z))\overset{0}{\Gamma}_j(z-y,-\varkappa^2) - \overset{k}{T}(\partial z, n(z))\overset{k}{\Gamma}_j(z-y,-\varkappa^2), \quad k = \overline{1,m_0};$$

2) $\forall z \in S_k:\ \overset{0}{u}{}^{+}(z) = \overset{0}{\Gamma}_j(z-y,-\varkappa^2)$, $k = 0, m_0+1, \dots, m$.

Quite similarly, we introduce the functional $L^M[u]$ for the couple-stress theory. For this $u(x)$ is assumned to be a three-component vector and in (19) $\overset{k}{E}$ is replaced by $\overset{k}{E}{}^M$, $\overset{k}{T}$ by $\overset{k}{T}{}^M$ and $\overset{k}{\Gamma}_j$ by $\overset{k}{\Gamma}{}^M_j$.

Theorem 1 (see [6]). *The functional L takes the minimal value at $u = g_j(x,y,-\varkappa^2)$.*

The similar assertion is valid for the functional L^M.

Theorem 2. *The estimate*

$$\Big|g_{jj}(y,y,-\varkappa^2) - g_{jj}(y,y,-\varkappa_0^2)\Big| \le \frac{\text{const}}{l_y^{1/2+\delta}}, \quad y \in D, \ \ \delta > 0, \tag{20}$$

holds for the function $g_{jj}(y,y,-\varkappa^2)$.

Proof. We write formula (18) for $u_j(x) = g_{jj}(x,y,-\varkappa^2)$ and $\Gamma_j(x-y,-\varkappa^2) = G_j(x,y,-\varkappa^2)$. Then, taking into account the boundary and contact conditions for g and G, we obtain

$$\forall x \in D_k:\ \ g_{jj}(x,y,-\varkappa^2)$$
$$= -\int\limits_S \Big[\overset{0}{\Gamma}_j(z-y,-\varkappa^2)\overset{0}{T}(\partial z,n(z))\overset{0}{G}_j(z,x,-\varkappa^2)\Big]\,d_zS$$
$$+\sum_{k=1}^{m_0}\int\limits_{S_k}\Big[\overset{0}{G}{}^{+}_j(z,x,-\varkappa^2)\overset{0}{T}(\partial z,n(z))\overset{0}{\Gamma}_j(z-y,-\varkappa^2)$$
$$-\overset{k}{G}{}^{-}_j(z,x,-\varkappa^2)\overset{k}{T}(\partial z,n(z))\overset{k}{\Gamma}_j(z-y,-\varkappa^2)\Big]\,d_zS$$
$$-\sum_{k=1}^{m_0}\int\limits_{S_k}\Big[\overset{0}{\Gamma}_j(z-y,-\varkappa^2)\Big(\overset{0}{T}(\partial z,n(z))\overset{0}{G}_j(z,x,-\varkappa^2)\Big)^{+}$$
$$-\overset{k}{\Gamma}_j(z-y,-\varkappa^2)\Big(\overset{k}{T}(\partial z,n(z))\overset{k}{G}_j(z,x,-\varkappa^2)\Big)\Big]\,d_zS. \tag{21}$$

Using (15) for $u = v$, we can rewrite (19) as

$$L[u] = -\sum_{k=0}^{m_0}\int\limits_{D_k}\overset{k}{u}(x)\Big[\overset{k}{A}(\partial x)\overset{k}{u}(x) - \varkappa^2\overset{k}{u}(x)\Big]\,dx + \int\limits_S \overset{0}{u}{}^{+}(z)\Big(\overset{0}{T}\overset{0}{u}(z)\Big)^{+}\,d_zS$$

$$+\sum_{k=1}^{m_0}\int_{S_k}\Big[\overset{0}{u}{}^{+}(z)\Big(\overset{0}{T}\overset{0}{u}(z)\Big)^{+}-\overset{k}{u}{}^{-}(z)\Big(\overset{k}{T}\overset{k}{u}(z)\Big)^{-}\Big]\,d_zS$$

$$-2\sum_{k=1}^{m_0}\int_{S_k}\Big[\overset{0}{u}{}^{+}(z)\overset{0}{T}(\partial z,n(z))\overset{0}{\Gamma}_j(z-y,-\varkappa^2)$$

$$-\overset{k}{u}{}^{-}(z)\overset{k}{T}(\partial z,n(z))\overset{k}{\Gamma}_j(z-y,-\varkappa^2)\Big]\,d_zS, \qquad (22)$$

which, for $u(x)=g_j(x,y,-\varkappa^2)=\Gamma_j(x-y,-\varkappa^2)-G_j(x,y,-\varkappa^2)$, implies

$$L[g_j]=\int_S\overset{0}{\Gamma}_j(z-y,-\varkappa^2)\overset{0}{T}(\partial z,n(z))\overset{0}{\Gamma}_j(z-y,-\varkappa^2)\,d_zS$$

$$-\int_S\overset{0}{\Gamma}_j(z-y,-\varkappa^2)\Big(\overset{0}{T}(\partial z,n(z))\overset{0}{G}_j(z-y,-\varkappa^2)\Big)^{+}\,d_zS$$

$$-\sum_{k=1}^{m_0}\int_{S_k}\Big[\overset{0}{\Gamma}_j(z-y,-\varkappa^2)\overset{0}{T}(\partial z,n(z))\overset{0}{\Gamma}_j(z-y,-\varkappa^2)$$

$$-\overset{k}{\Gamma}_j(z-y,-\varkappa^2)\overset{k}{T}(\partial z,n(z))\overset{k}{\Gamma}_j(z-y,-\varkappa^2)\Big]\,d_zS$$

$$+\sum_{k=1}^{m_0}\int_{S_k}\Big[\overset{0}{G}{}_j^{+}(z,y,-\varkappa^2)\overset{0}{T}(\partial z,n(z))\overset{0}{\Gamma}_j(z-y,-\varkappa^2)$$

$$-\overset{k}{G}{}_j^{-}(z,y,-\varkappa^2)\overset{k}{T}(\partial z,n(z))\overset{k}{\Gamma}_j(z-y,-\varkappa^2)\Big]\,d_zS$$

$$-\sum_{k=1}^{m_0}\int_{S_k}\Big[\overset{0}{\Gamma}_j(z-y,-\varkappa^2)\Big(\overset{0}{T}(\partial z,n(z))\overset{0}{G}_j(z,y,-\varkappa^2)\Big)^{+}$$

$$-\overset{k}{\Gamma}_j(z-y,-\varkappa^2)\Big(\overset{k}{T}(\partial z,n(z))\overset{k}{G}_j(z,y,-\varkappa^2)\Big)^{-}\Big]\,d_zS. \qquad (23)$$

By virtue of (23) formula (21) gives

$$g_{jj}(y,y,-\varkappa^2)$$

$$=L[g_j]-\int_S\overset{0}{\Gamma}_j(z-y,-\varkappa^2)\overset{0}{T}(\partial z,n(z))\overset{0}{\Gamma}_j(z-y,-\varkappa^2)\,d_zS$$

$$+\sum_{k=1}^{m_0}\int_{S_k}\Big[\overset{0}{\Gamma}_j(z-y,-\varkappa^2)\overset{0}{T}(\partial z,n(z))\overset{0}{\Gamma}_j(z-y,-\varkappa^2)$$

$$-\overset{k}{\Gamma}_j(z-y,-\varkappa^2)\overset{k}{T}(\partial z,n(z))\overset{k}{\Gamma}_j(z-y,-\varkappa^2)\Big]\,d_zS. \qquad (24)$$

The vector $\overset{\widehat{k}}{\Gamma}_j(x-y,-\varkappa^2)$ defined by (8) belongs to the definition domain of the functional L and, since $g_j(x,y,-\varkappa^2)$ imparts the minimal value to the functional L, it is obvious that

$$L[g_j] \le L[\widehat{\Gamma}_j].$$

Now from (24) we have

$$\begin{aligned} & g_{jj}(y,y,-\varkappa^2) \\ & \le L[\widehat{\Gamma}_j] - \int\limits_S \overset{0}{\Gamma}_j \overset{0}{T}\overset{0}{\Gamma}_j \, dS + \sum_{k=1}^{m_0} \int\limits_{S_k} \Big[\overset{0}{\Gamma}_j \overset{0}{T}\overset{0}{\Gamma}_j - \overset{k}{\Gamma}_j \overset{k}{T}\overset{k}{\Gamma}_j \Big] \, dS, \quad y \in D_k. \end{aligned} \tag{25}$$

By virtue of the properties of $\widehat{\Gamma}$ formula (22) implies

$$\begin{aligned} L[\widehat{\Gamma}_j] = & - \int\limits_{k(y,ly)} \widehat{\Gamma}_j \Big(A\widehat{\Gamma}_j - \varkappa^2 \widehat{\Gamma}_j \Big) \, dx + \int\limits_S \overset{0}{\Gamma}_j \overset{0}{T}\Gamma_j \, dS \\ & - \sum_{k=1}^{m_0} \int\limits_{S_k} \Big[\overset{0}{\Gamma}_j \overset{0}{T}\overset{0}{\Gamma}_j - \overset{k}{\Gamma}_j \overset{k}{T}\overset{k}{\Gamma}_j \Big] \, dS. \end{aligned} \tag{26}$$

Using (26), from (25) we obtain

$$g_{jj}(y,y,-\varkappa^2) \le - \int\limits_{k(y,ly)} \widehat{\Gamma}_j \Big(A\widehat{\Gamma}_j - \varkappa^2 \widehat{\Gamma}_j \Big) \, dx, \quad y \in D_k, \quad k = \overline{0, m_0}. \tag{27}$$

When $\varkappa r$ is bounded, taking into account estimates (13) for $m = 5$, we have

$$\Big| \widehat{\Gamma}_{mj}(x-y, -\varkappa^2) \Big| \le \frac{\text{const}}{l_y^\alpha} \, ,$$

$$\Big| \varkappa^2 \widehat{\Gamma}_{mj}(x-y, -\varkappa^2) \Big| \le \varkappa^2 \, \frac{\text{const} \, r^{5-\alpha}}{\varkappa^\alpha l_y^5} = \frac{\text{const} (\varkappa r)^{2-\alpha} r^3}{l_y^5} \le \frac{\text{const}}{l_y^2} \, ,$$

$$\Big| A\widehat{\Gamma}_j(x-y, -\varkappa^2) \Big| \le \frac{\text{const} \, r^3}{l_y^5} \le \frac{\text{const}}{l_y^2} \, .$$

Hence (27) implies

$$g_{jj}(y,y,-\varkappa^2) \le \frac{\text{const}}{l_y^{2+\alpha}} \, \pi l_y^2 = \frac{\text{const}}{l_y^\alpha} \, . \tag{28}$$

When $\varkappa r$ is unbounded, taking into account estimates (10) for $m = 5$, we have

$$\Big| \widehat{\Gamma}_{mj}(x-y, -\varkappa^2) \Big| \le \frac{\text{const}}{l_y^{1/2}} \, ,$$

$$\begin{aligned} \Big| \varkappa^2 \widehat{\Gamma}_{mj}(x-y, -\varkappa^2) \Big| & \le \varkappa^2 \, \frac{\text{const} \, e^{-a\varkappa r}}{\sqrt{\varkappa} \, l_y^5} \, r^{5-\frac{1}{2}} \\ & = \frac{\text{const} \, r^3}{l_y^5} \, (\varkappa r)^{3/2} e^{-a\varkappa r} \le \frac{\text{const}}{l_y^2} \, , \end{aligned}$$

$$\left|A\widehat{\Gamma}_j(x-y,-\varkappa^2)\right| \le \frac{\text{const}\,\varkappa^2 r^{-a\varkappa r}}{\sqrt{\varkappa} l_y^5}\, r^{5-\frac{1}{2}} \le \frac{\text{const}}{l_y^2}\,.$$

Hence (27) implies

$$g_{jj}(y,y,-\varkappa^2) \le \frac{\text{const}}{l_y^{5/2}}\,\pi l_y^2 = \frac{\text{const}}{l_y^{1/2}}\,.$$

We assume that $\alpha = \frac{1}{2}$ in (28).

Let us estimate $g_{jj}(y,y,-\varkappa^2)$ from below. For this we introduce the following notation:

$$M[u] = \sum_{k=0}^{m_0} \int\limits_{D_k} \Big[\overset{k}{E}(\overset{k}{u},\overset{k}{u}) + \varkappa^2 \overset{k}{u}{}^2\Big]\,dx,$$

$$M_0[u] = \sum_{k=0}^{m_0} \int\limits_{D_k} \Big[\overset{k}{E}(\overset{k}{u},\overset{k}{u}) + \varkappa_0^2 \overset{k}{u}{}^2\Big]\,dx,$$

$$N[u] = \sum_{k=1}^{m_0} \int\limits_{S_k} \Big[\overset{0}{u}{}^+(z)\overset{0}{T}(\partial z, n(z))\overset{0}{\Gamma}_j(z-y,-\varkappa^2)$$
$$- \overset{k}{u}{}^-(z)\overset{k}{T}(\partial z, n(z))\overset{k}{\Gamma}_j(z-y,-\varkappa^2)\Big]\,d_zS.$$

Then $L[u] = M[u] - 2N[u]$ and, since $\varkappa_0^2 \le \varkappa^2$, we have $M_0[u] \le M[u]$. Now

$$L\Big[g_j(x,y,-\varkappa^2)\Big] = \min L[u] = \min\Big(M[u]-2N[u]\Big) \ge\ge \min\Big(M_0[u]-2N[u]\Big).$$

Let the vector function $\varphi(x,y)$ impart a minimal value to the functional $P[u] = M_0[u] - 2N[u]$. Then $\varphi(x,y)$ belongs to the definition domain of the functional L and is a regular in the domain D_k ($k = \overline{0,m_0}$) solution of the equation

$$\forall x \in D_k: \quad A(\partial x)\varphi(x,y) - \varkappa_0^2\varphi(x,y) = 0, \quad k = \overline{0,m_0}.$$

After writing formula (18) for $\varphi(x,y)$, where $\Gamma = G$, we obtain

$$\forall (x,y) \in D_k \times D_k: \ \varphi(x,y) = -\int\limits_S \overset{0}{\Gamma}_j(z-x,-\varkappa^2)\overset{0}{T}\overset{0}{G}_j(z,y,-\varkappa_0^2)\,d_zS$$
$$+\sum_{k=1}^{m_0}\int\limits_{S_k}\Big[\overset{0}{G}_j(z,x,-\varkappa^2)\overset{0}{T}\overset{0}{\Gamma}_j(z-y,-\varkappa^2)$$
$$-\overset{k}{G}{}_j^-(z,x,-\varkappa_0^2)\overset{k}{T}\overset{k}{\Gamma}_j(z-y,-\varkappa^2)\Big]\,d_zS$$
$$-\sum_{k=1}^{m_0}\int\limits_{S_k}\Big[\overset{0}{\Gamma}_j(z,x,-\varkappa^2)\Big(\overset{0}{T}\overset{0}{G}_j(z,y,-\varkappa_0^2)\Big)$$
$$-\overset{k}{\Gamma}_j(z-x,-\varkappa^2)\Big(\overset{k}{T}\overset{k}{G}_j(z,y,-\varkappa_0^2)\Big)^-\Big]\,d_zS,$$

which, by virtue of (6), (14) and the theorem on kernel composition [5], implies

$$\forall(x,y)\in\overline{D}_k\times D_k:\quad |\varphi(x,y)|\le\frac{\text{const}}{|x-y|^{1/2}}\,,\quad \Big|\frac{\partial}{\partial x_i}\,\varphi(x,y)\Big|\le\frac{\text{const}}{|x-y|}\,,\quad k=\overline{0,m_0},\quad i=1,2. \tag{29}$$

Since

$$M_0[u]=-\sum_{k=0}^{m_0}\int\limits_{D_k}\overset{k}{u}\Big(\overset{k}{A}\overset{k}{u}-\varkappa_0^2\overset{k}{u}\Big)\,dx+\int\limits_S\overset{0}{u}{}^+(\overset{0}{T}\overset{0}{u})^+\,dS$$
$$+\sum_{k=1}^{m_0}\int\limits_{S_k}\Big[\overset{0}{u}{}^+(\overset{0}{T}\overset{0}{u})^+-\overset{k}{u}{}^-(\overset{k}{T}\overset{k}{u})^-\Big]\,dS,$$

we have

$$L\Big[g_j(x,y,-\varkappa^2)\Big]\ge\int\limits_S\overset{0}{\Gamma}_j(z-y,-\varkappa^2)\Big(\overset{0}{T}\overset{0}{\varphi}(z,y)\Big)^+\,d_zS$$
$$+\sum_{k=1}^{m_0}\int\limits_{S_k}\Big[\overset{0}{\Gamma}_j(z-y,-\varkappa^2)\Big(\overset{0}{T}\overset{0}{\varphi}(z,y)\Big)^+-\overset{k}{\Gamma}_j(z-y,-\varkappa^2)\Big(\overset{k}{T}\overset{k}{\varphi}(z,y)\Big)^-\Big]\,d_zS$$
$$-2\sum_{k=1}^{m_0}\int\limits_{S_k}\overset{0}{\Gamma}_j(z-y,-\varkappa^2)\overset{0}{T}\overset{0}{\Gamma}_j(z-y,-\varkappa^2)$$
$$-\overset{k}{\Gamma}_j(z-y,-\varkappa^2)\overset{k}{T}\overset{k}{\Gamma}_j(z-y,-\varkappa^2)\Big]\,d_zS. \tag{30}$$

By (29) we obtain

$$\forall(z,y)\in S_k\times D_k:\quad\Big|\frac{\partial}{\partial x_i}\,\varphi(x,y)\Big|\le\frac{\text{const}}{|z-y|}=\frac{\text{const}}{|z-y|^{\delta}|z-y)^{1-\delta}}$$
$$\le\frac{\text{const}}{l_y^{\delta}|z-y|^{1-\delta}}\,,\quad \delta>0.$$

Now (30) gives

$$L[g_j]\ge-\frac{\text{const}}{l^{1/2+\delta}}\,,\quad \delta>0.$$

By representation (24) the latter inequality readily yields the estimate

$$\forall y\in D:\quad g_{jj}(y,y,-\varkappa^2)\ge-\frac{\text{const}}{l_y^{1/2+\delta}}\,. \tag{31}$$

Formulas (28) and (31) imply (20). □

By a similar technique we establish estimate (20) for $g_{jj}^M(y,y,-\varkappa^2)$.

8. Let us consider the first boundary-contact problem with eigenvalues: Find, in D_k ($k=\overline{0,m_0}$), a regular vector $w(x)=\overset{k}{w}(x)=(\overset{k}{w}_1,\overset{k}{w}_2)$ which is a nontrivial solution of the equations

$$\forall x\in D_k:\quad \overset{k}{\widetilde{A}}(\partial x)\overset{k}{w}(x)+\gamma\overset{k}{w}(x)=0,\quad k=\overline{0,m_0},$$

and satisfies the contact conditions

$$\forall z\in S_k:\quad \overset{0}{w}{}^{+}(z)=\overset{k}{w}{}^{-}(z),\quad \Big(\overset{0}{T}\overset{0}{w}(z)\Big)^{+}=\Big(\overset{k}{T}\overset{k}{w}(z)\Big)^{-},\quad k=\overline{1,m_0},$$

and the boundary condition

$$\forall z\in S_k:\quad \overset{0}{w}{}^{+}(z)=0,\quad k=0,m_0+1,\dots,m.$$

We denote this problem by $\overset{c}{I}_\gamma$. If in problem $\overset{c}{I}_\gamma$ we replace $\widetilde{A}$ by $\widetilde{M}$, T by T^M and assume $w(x)$ to be a three-component vector, then the resulting problem with eigenvalues is denoted by $\overset{c}{I}{}^M_\gamma$.

It can be shown by the known technique [4] that problem $\overset{c}{I}_\gamma$ is equivalent to a system of integral equations

$$w(x)=(\gamma+\varkappa_0^2)\int\limits_D \widetilde{G}(x,y,-\varkappa_0^2)w(y)\,dy, \tag{32}$$

where $\widetilde{G}=\overset{1}{\widetilde{r}}G\overset{1}{\widetilde{r}}$, and problem $\overset{c}{I}{}^M_\gamma$ is equivalent to a system of integral equations

$$w(x)=(\gamma+\varkappa_0^2)\int\limits_D \widetilde{G}{}^M(x,y,-\varkappa_0^2)w(y)\,dy, \tag{33}$$

where $\widetilde{G}{}^M=\overset{2}{\widetilde{r}}G^M\overset{2}{\widetilde{r}}$. By virtue of (13) and (14) equations (32) and (33) are integral equations with a symmetric kernel of the class $L_2(D)$. Hence it follows that there exists a countable system of eigenvalues $(\gamma_n+\varkappa_0^2)_{n=1}^\infty$ and the corresponding (orthonormal in D) system of eigenvectors $(w^{(n)}(x))_{n=1}^\infty=(\overset{k}{w}{}^{(n)}(x))_{n=1}^\infty$, $x\in D_k$, $k=\overline{0,m_0}$, of equation (32). This in turn implies that $(\gamma_n)_{n=1}^\infty$ and $(w^{(n)}(x))_{n=1}^\infty$ are the eigenvalues and eigenvectors of problem $\overset{c}{I}_\gamma$. It has been established [1] that all $\gamma_n>0$. Moreover, in [7] it is proved that the system $(w^{(n)}(x))_{n=1}^\infty$ forms a complete system in $L_2(D)$. By the properties of a volume potential we conclude that the eigenvectors are regular. What has been said above can be repeated for the eigenvalues and eigenvectors of problem $\overset{c}{I}{}^M_\gamma$.

9. In deriving asymptotic formulas, the Tauber type theorem due to Ikehara [8], [9] plays a decisive role. Let us formulate this theorem for series [9].

Theorem 3. *Let λ_n be an increasing sequence of real numbers,*

$$F(z)=\sum_{n=1}^\infty\frac{a_n}{\lambda_n^z},\quad \operatorname{Re}(z)>1,$$

where $a_n \geq 0$, $n = 1, 2, \dots$, and assume that $F(z)$ can be analytically continued onto the straight line $\mathrm{Re}(z) = 1$ *and has no singularities at the points of this straight line except for the point $z = 1$ at which it has a first order pole with the principal part $\frac{A}{z-1}$. Then*

$$\sum_{k=1}^{n} a_k \sim A\lambda_n.$$

By expanding the kernel in eigenfunctions we obtain

$$\widetilde{G}(x, y, -\varkappa^2) - \widetilde{G}(x, y, -\varkappa_0^2) = (\varkappa_0^2 - \varkappa^2) \sum_{n=1}^{\infty} \frac{w^{(n)}(x) \times w^{(n)}(y)}{(\gamma_n + \varkappa^2)(\gamma_n + \varkappa_0^2)}\,, \tag{34}$$

where $x,\, y \in D_k$, $k = \overline{0, m_0}$, and the symbol $\times$ denotes the matrix product of a column vector by a row vector (the dyad product)

$$w^{(n)}(x) \times w^{(n)}(y) = \left\| w_i^{(n)}(x) w_k^{(n)}(x) \right\|_{i,k=1,2}.$$

After passing in equality (34) to the limit as $x \to y$, we obtain

$$\begin{aligned}
&(\varkappa_0^2 - \varkappa^2) \sum_{n=1}^{\infty} \frac{[w_j^{(n)}(y)]^2}{(\gamma_n + \varkappa^2)(\gamma_n + \varkappa_0^2)} \\
&= \lim_{x \to y} \Big[\overset{1}{\tilde{r}} \Gamma_{jj}(x - y, -\varkappa^2) \overset{1}{\tilde{r}} - \overset{1}{\tilde{r}} \Gamma_{jj}(x - y, -\varkappa_0^2) \overset{1}{\tilde{r}}\Big] \\
&\quad - \lim_{x \to y} \Big[\overset{1}{\tilde{r}} g_{jj}(x, y, -\varkappa^2) \overset{1}{\tilde{r}} - \overset{1}{\tilde{r}} g_{jj}(x, y, -\varkappa_0^2) \overset{1}{\tilde{r}}\Big], \\
&\qquad x,\, y \in D_k, \quad k = \overline{0, m_0}, \quad j = 1, 2.
\end{aligned} \tag{35}$$

Denote

$$\begin{aligned}
h_j(y, \varkappa) &= \overset{1}{\tilde{r}} g_{jj}(y, y, -\varkappa_0^2) \overset{1}{\tilde{r}} - \overset{1}{\tilde{r}} g_{jj}(y, y, -\varkappa^2) \overset{1}{\tilde{r}} \\
&= \rho\big[g_{jj}(y, y, -\varkappa_0^2) - g_{jj}(y, y, -\varkappa^2)\big].
\end{aligned}$$

By (20) $h_j(\varkappa) = O(1)$ as $\varkappa \to \infty$. Using (11), we have

$$\begin{aligned}
&\lim_{x \to y} \Big[\overset{1}{\tilde{r}} \Gamma_{jj}(x - y, -\varkappa^2) \overset{1}{\tilde{r}} - \overset{1}{\tilde{r}} \Gamma_{jj}(x - y, -\varkappa_0^2) \overset{1}{\tilde{r}}\Big] \\
&= \lim_{x \to y} \rho_k \Big[\overset{k}{\Gamma}_{jj}(x - y, -\varkappa^2) - \overset{k}{\Gamma}_{jj}(x - y, -\varkappa_0^2)\Big] = -A_n \ln \frac{\varkappa}{\varkappa_0}\,,
\end{aligned} \tag{36}$$

where

$$A_k = \frac{\rho_k}{4\pi} \left(\frac{1}{\lambda_k + 2\mu_k} + \frac{1}{\mu_k} \right), \quad k = \overline{0, m_0}.$$

Let $\lambda = \varkappa_0^2 - \varkappa^2$, $\widetilde{\gamma}_n = \gamma_n + \varkappa_0^2$. Let us choose $\varkappa_0$ such that $\gamma_1 + \varkappa_0^2 = \widetilde{\gamma}_1 > 0$. Now by virtue of (36) it follows from (35) that

$$R(y, \lambda) = \lambda \sum_{n=1}^{\infty} \frac{[w_j^{(n)}(y)]^2}{\widetilde{\gamma}_n (\widetilde{\gamma}_n - \lambda)}\,, \tag{37}$$

where

$$R(y,\lambda)=-A_k\ln\frac{\varkappa}{\varkappa_0}+h_j(y,\lambda)=-A_k\ln\sqrt{1-\frac{\lambda}{\varkappa_0^2}}+h_j(y,\lambda)$$
$$=-A_k\ln\sqrt{-\lambda}-A_k\ln\sqrt{\frac{1}{\varkappa_0^2}-\frac{1}{\lambda}}+h_j(y,\lambda)$$
$$=-A_k\sqrt{-\lambda}+O(1)\quad\text{for}\quad\lambda\to-\infty,\quad y\in D_k,\quad k=\overline{0,m_0},\quad j=1,2.$$

Divide both parts of (37) by $2\pi i\lambda^z$ and integrate from $\varepsilon-i\infty$ to $\varepsilon+i\infty$, where $0<\varepsilon<\widetilde{\gamma}_1$ (z is a complex-valued parameter whose real part is sufficiently large). Applying the basic residue theorem, we obtain

$$\frac{1}{2\pi i}\int\limits_{\varepsilon-i\infty}^{\varepsilon+i\infty}\frac{\lambda\,d\lambda}{\widetilde{\gamma}_n(\widetilde{\gamma}_n-\lambda)\lambda^z}=\frac{1}{\widetilde{\gamma}_n^z}\,. \tag{38}$$

Due to (38) we can rewrite (37) as

$$\sum_{n=1}^{\infty}\frac{[w_j^{(n)}(y)]^2}{\widetilde{\gamma}_n^z}=\frac{1}{2\pi i}\int\limits_{\varepsilon-i\infty}^{\varepsilon+i\infty}\frac{R(y,\lambda)}{\lambda^z}\,d\lambda. \tag{39}$$

We perform the Carleman transformation [10] of the integral in the right-hand part of (39). The function $R(y,\lambda)\lambda^{-z}$ is analytic on the entire plane except for the points $0,\widetilde{\gamma}_1,\widetilde{\gamma}_2,\dots$ and, by the Cauchy theorem, the integral of this function taken over the closed contour not containing these points internally, is equal to zero. By the Cauchy theorem the integral in the right-hand part of (39) can be written as

$$\int\limits_{\varepsilon-i\infty}^{\varepsilon+i\infty}\frac{R(y,\lambda)}{\lambda^z}\,d\lambda=\int\limits_{L_1}\frac{R(y,\lambda)}{\lambda^z}\,d\lambda+\int\limits_{C}\frac{R(y,\lambda)}{\lambda^z}\,d\lambda+\int\limits_{L_2}\frac{R(y,\lambda)}{\lambda^z}\,d\lambda, \tag{40}$$

where $L_1=(-\infty-0i,-\varepsilon-0i)$, $L_2(-\varepsilon+0i,-\infty+0i)$, C is the circumference $|\lambda|=\varepsilon$. Clearly, $\lambda=|\lambda|e^{-i\pi}$ on L_1, $\lambda=|\lambda|e^{i\pi}$ on L_2, and $\lambda=\varepsilon e^{i\theta}$ on C. We have

$$\int\limits_{C}\frac{R(y,\lambda)}{\lambda^z}\,d\lambda=i\varepsilon^{1-z}\int\limits_{-\pi}^{\pi}R(y,\varepsilon e^{i\theta})e^{i(1-z)\theta}\,d\theta,$$

$$\int\limits_{L_1}\frac{R(y,\lambda)}{\lambda^z}\,d\lambda=e^{i\pi z}\int\limits_{\varepsilon}^{\infty}\frac{R(y,-\lambda)}{\lambda^z}\,d\lambda,\quad\int\limits_{L_2}\frac{R(y,\lambda)}{\lambda^z}\,d\lambda=-e^{-i\pi z}\int\limits_{\varepsilon}^{\infty}\frac{R(y,-\lambda)}{\lambda^z}\,d\lambda.$$

Hence by virtue of (39) and (40)

$$\sum_{n=1}^{\infty}\frac{[w_j^{(n)}(y)]^2}{\widetilde{\gamma}_n^z}$$

$$= \frac{\varepsilon^{1-z}}{2\pi} \int\limits_{-\pi}^{\pi} R(y, \varepsilon e^{i\theta}) e^{i(1-z)\theta}\, d\theta + \frac{\sin \pi z}{\pi} \int\limits_{\varepsilon}^{\infty} \frac{R(y,-\lambda)}{\lambda^z}\, d\lambda. \tag{41}$$

Integrating by parts we obtain

$$\int\limits_{\varepsilon}^{\infty} \frac{\ln\sqrt{\lambda}}{\lambda^z}\, d\lambda = \frac{\varepsilon^{1-z}\ln\varepsilon}{2(z-1)} + \frac{\varepsilon^{1-z}}{2(z-1)^2}\,. \tag{42}$$

By virtue of (42) we can rewrite (41) as

$$\sum_{n=1}^{\infty} \frac{[w_j^{(n)}(y)]^2}{\widetilde{\gamma}_n^{\,z}} = \frac{\varepsilon^{1-z}}{2\pi} \int\limits_{-\pi}^{\pi} R(y, \varepsilon e^{i\theta}) e^{i(1-z)\theta}\, d\theta$$

$$-\frac{\sin \pi(z-1)}{\pi} \Big[-\frac{A_k \ln\varepsilon\, \varepsilon^{1-z}}{2(z-1)} - \frac{A_k \varepsilon^{1-z}}{2(z-1)^2} + \int\limits_{\varepsilon}^{\infty} \frac{O(1)}{\lambda^z}\, d\lambda \Big]. \tag{43}$$

Due to Ikehara's Theorem 3, (43) gives

$$\sum_{p=1}^{n} \Big[w_j^{(p)}(y) \Big]^2 \sim \frac{A_k}{2}\, \gamma_n, \quad j = 1, 2,$$

or, finally,

$$\sum_{p=1}^{n} \Big[w^{(p)}(y) \Big]^2 \sim \frac{\rho_k}{4\pi} \Big(\frac{1}{\lambda_k + 2\mu_k} + \frac{1}{\mu_k} \Big) \gamma_n, \quad y \in D_k, \quad k = \overline{0, m_0}. \tag{44}$$

On repeating the above reasoning, we obtain asymptotic formulas for eigenvectors of problem $\overset{c}{I}{}^M_\gamma$. Note only that in this case the eigenvectors $(w^{(n)}(x))_{n=1}^{\infty}$ are three-component and, as follows from (12),

$$\lim_{x\to y} \Big[\overset{2}{\widetilde{r}}_{jj} \overset{k}{\Gamma}{}^M_{jj}(x-y, -\varkappa^2) \overset{2}{\widetilde{r}}_{jj} - \overset{2}{\widetilde{r}}_{jj} \overset{k}{\Gamma}{}^M_{jj}(x-y, -\varkappa_0^2) \overset{2}{\widetilde{r}}_{jj} \Big]$$

$$= \frac{\rho_k}{4\pi} \Big(\frac{1}{\lambda_k + 2\mu_k} + \frac{1}{\mu_k + \alpha_k} \Big) \ln \frac{\varkappa_0}{\varkappa}\,, \quad j = 1, 2,$$

$$\lim_{x\to y} \Big[\overset{2}{\widetilde{r}}_{33} \overset{k}{\Gamma}{}^M_{33}(x-y, -\varkappa^2) \overset{2}{\widetilde{r}}_{33} - \overset{2}{\widetilde{r}}_{33} \overset{k}{\Gamma}{}^M_{33}(x-y, -\varkappa_0^2) \overset{2}{\widetilde{r}}_{33} \Big] = \frac{I_k}{2\pi} \, \frac{1}{\nu_k + \beta_k} \, \ln \frac{\varkappa_0}{\varkappa}\,.$$

Therefore in the couple-stress elasticity

$$\sum_{p=1}^{n} \Big[w_j^{(p)}(y) \Big]^2 \sim \frac{\rho_k}{8\pi} \Big(\frac{1}{\lambda_k + 2\mu_k} + \frac{1}{\mu_k + \alpha_k} \Big) \gamma_n, \quad j = 1, 2, \tag{45}$$

$$\sum_{p=1}^{n} \Big[w_3^{(p)}(y) \Big]^2 \sim \frac{I_k}{4\pi} \, \frac{1}{\nu_k + \beta_k}\, \gamma_n, \quad y \in D_k, \quad k = \overline{0, m_0}. \tag{46}$$

10. From (35) we obtain

$$\frac{-R(y,\varkappa)}{\varkappa^2-\varkappa_0^2}=\sum_{n=1}^{\infty}\frac{[w^{(n)}(y)]^2}{(\gamma_n+\varkappa_0^2)(\gamma_n+\varkappa_0^2)}\le\sum_{n=1}^{\infty}\frac{[w^{(n)}(y)]^2}{(\gamma_n+\varkappa_0^2)^2}\,. \tag{47}$$

Applying the Bessel inequality, we have

$$\sum_{n=1}^{\infty}\frac{[w^{(n)}(x)]^2}{(\gamma_n+\varkappa_0^2)^2}\le\int\limits_D\left|\widetilde{G}(x,y,-\varkappa_0^2)\right|^2dy,\quad x\in D_k,$$

which, taking into account estimates (14), implies the existence and uniform boundedness of the sum of the series

$$\sum_{n=1}^{\infty}\frac{[w^{(n)}(x)]^2}{(\gamma_n+\varkappa_0^2)^2}$$

in $\overline{D}_k$. Now from (47) it follows that

$$\forall y\in\overline{D}_k,\quad k=\overline{0,m_0}:\quad |R(y,\varkappa)|\le\operatorname{const}(\varkappa^2-\varkappa_0^2). \tag{48}$$

If we integrate (47) in the domain D, then, recalling that the vectors $(w^{(n)}(x))_{n=1}^{\infty}$ are orthonormal in D, we obtain

$$-\int\limits_D R(y,\varkappa)\,dy=(\varkappa^2-\varkappa_0^2)\sum_{n=1}^{\infty}\frac{1}{(\gamma_n+\varkappa^2)(\gamma_n+\varkappa_0^2)}\,,$$

$$\begin{aligned}-\int\limits_D R(y,\varkappa)\,dy&=\int\limits_D 2A_k\ln\frac{\varkappa}{\varkappa_0}\,dy-\int\limits_D h_j(y,\varkappa)\,dy\\&=2\ln\frac{\varkappa}{\varkappa_0}\sum_{k=0}^{m_0}A_k\operatorname{mes}D_k-\int\limits_D h_j(y,\varkappa)\,dy.\end{aligned} \tag{49}$$

Denote by $(D_k)_\eta$ that part of D_k $(k=\overline{0,m_0})$ where the distance from the points to the boundary of D_k is less than η; $D_\eta=\bigcup\limits_{k=0}^{m_0}(D_k)_\eta$. Now,

$$\int\limits_D h_j(y,\varkappa)\,dy=\int\limits_{D\setminus D_\eta}h_j(y,\varkappa)\,dy+\int\limits_{D_\eta}R(y,\varkappa)\,dy+\int\limits_{D_\eta}2A_k\ln\frac{\varkappa}{\varkappa_0}\,dy, \tag{50}$$

$$\int\limits_{D_\eta}2A_k\ln\frac{\varkappa}{\varkappa_0}\,dy=2\ln\frac{\varkappa}{\varkappa_0}\sum_{k=0}^{m_0}A_k\operatorname{mes}(D_k)_\eta=2\ln\frac{\varkappa}{\varkappa_0}\sum_{k=0}^{m_0}A_kO(\eta), \tag{51}$$

$$\int\limits_{D_\eta}R(y,\varkappa)\,dy\le\operatorname{const}(\varkappa^2-\varkappa_0^2)\operatorname{mes}D_\eta=(\varkappa^2-\varkappa_0^2)O(\eta), \tag{52}$$

$$\int\limits_{D\setminus D_\eta}h_j(y,\varkappa)\,dy=O(\eta^{\frac{1}{2}-\delta}). \tag{53}$$

The validity of (51) is obvious. (52) holds by virtue of (48), while (53) by virtue of (20). Let $\lambda = \varkappa_0^2 - \varkappa^2$, $\eta = \frac{1}{\varkappa^2 - \varkappa_0^2} = -\frac{1}{\lambda}$. Then (51), (52), (53) imply

$$\int\limits_{D_\eta} 2A_k \ln \frac{\varkappa}{\varkappa_0}\, dy = \sum_{k=0}^{m_0} A_k 2\Big(\ln\sqrt{-\lambda} + \ln\sqrt{\frac{1}{\varkappa_0^2} - \frac{1}{\lambda}}\Big) O\Big(\frac{1}{\lambda}\Big), \tag{54}$$

$$\int\limits_{D_\eta} R(y,\varkappa)\, dy = \lambda O\Big(\frac{1}{\lambda}\Big) = O(1), \tag{55}$$

$$\int\limits_{D\setminus D_\eta} h_j(y,\varkappa)\, dy = O\Big(\frac{1}{\lambda^{\frac{1}{2}-\delta}}\Big). \tag{56}$$

On account of (54), (55), (56), from (50) we obtain

$$\int\limits_{D} h_j(y,\varkappa)\, dy = 2\ln\sqrt{-\lambda}\sum_{k=0}^{m_0} A_k O\Big(\frac{1}{\lambda}\Big)$$
$$+2\ln\sqrt{\frac{1}{\varkappa_0^2} - \frac{1}{\lambda}}\sum_{k=0}^{m_0} A_k O\Big(\frac{1}{\lambda}\Big) + O(1) + O\Big(\frac{1}{\lambda^{\frac{1}{2}-\delta}}\Big). \tag{57}$$

By virtue of (57), formula (49) gives

$$-\int\limits_{D} R(y,\varkappa)\, dy = 2\ln\sqrt{-\lambda}\sum_{k=0}^{m_0} A_k \operatorname{mes} D_k$$
$$+2\ln\sqrt{\frac{1}{\varkappa_0^2} - \frac{1}{\lambda}}\sum_{k=0}^{m_0} A_k \operatorname{mes} D_k - 2\ln\sqrt{-\lambda}\sum_{k=0}^{m_0} A_k O\Big(\frac{1}{\lambda}\Big)$$
$$-2\ln\sqrt{\frac{1}{\varkappa_0^2} - \frac{1}{\lambda}}\sum_{k=0}^{m_0} A_k O\Big(\frac{1}{\lambda}\Big) + O(1) + O\Big(\frac{1}{\lambda^{\frac{1}{2}-\delta}}\Big) = R^*(\lambda).$$

Hence (47) implies

$$R^*(\lambda) = \lambda\sum_{n=1}^{\infty} \frac{1}{\widetilde{\gamma}_n(\widetilde{\gamma}_n - \lambda)}\,. \tag{58}$$

Divide both parts of (58) by $2\pi i\lambda^z$ and integrate from $\varepsilon - i\infty$ to $\varepsilon + i\infty$, where $0 < \varepsilon < \widetilde{\gamma}_1$. Now in the same manner as above we have

$$\sum_{n=1}^{\infty} \frac{1}{\widetilde{\gamma}_n^z} = -\frac{\varepsilon^{1-z}}{2\pi}\int\limits_{-\pi}^{\pi} R^*(\varepsilon e^{i\theta}) e^{i(1-z)\theta}\, d\theta$$
$$-\frac{\sin\pi(z-1)}{\pi}\Bigg[-\frac{\sum\limits_{k=0}^{m_0} A_k \operatorname{mes} D_k \ln\varepsilon\, \varepsilon^{1-z}}{z-1} - \frac{\sum\limits_{k=0}^{m_0} A_k \operatorname{mes} D_k \varepsilon^{1-z}}{(z-1)^2}$$
$$+\int\limits_{\varepsilon}^{\infty} \frac{O(1)}{\lambda^z}\, d\lambda + \text{const}\,\frac{\ln\varepsilon\,\varepsilon^{-z}}{z} + \text{const}\,\frac{e^{-z}}{z^2} + \text{const}\,\frac{\varepsilon^{\frac{1}{2}+\delta-z}}{\frac{1}{2}+\delta-z}\Bigg]. \tag{59}$$

Applying Ikehara's theorem, from (59) we obtain

$$\lim_{n\to\infty}\frac{n}{\gamma_n}=\sum_{k=0}^{m_0}A_k\operatorname{mes}D_k$$

or, finally,

$$\lim_{n\to\infty}\frac{n}{\gamma_n}=\frac{1}{4\pi}\sum_{k=0}^{m_0}\rho_k\Big(\frac{1}{\lambda_k+2\mu_k}+\frac{1}{\mu_k}\Big)\operatorname{mes}D_k. \tag{60}$$

If $(\gamma_n)_{n=1}^{\infty}$ are the eigenvalues of problem $\overset{c}{I}{}_{\gamma}^{M}$, then, as above, we obtain

$$\lim_{n\to\infty}\frac{n}{\gamma_n}=\frac{1}{4\pi}\sum_{k=0}^{m_0}\Big[\rho_k\Big(\frac{1}{\lambda_k+2\mu_k}+\frac{1}{\mu_k+\alpha_k}\Big)+\frac{I_k}{\nu_k+\beta_k}\Big]\operatorname{mes}D_k. \tag{61}$$

To conclude, the results of this paper can be formulated as

Theorem 4. *The asymptotic distribution of eigenelements of the basic two-dimensional boundary-contact problems of oscillation is given by formulas* (44) *and* (60) *in the classical theory of elasticity, and by formulas* (45), (46) *and* (61) *in the couple-stress theory of elasticity.*

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