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# PROPERTIES $A$ AND $B$ OF $n$TH ORDER LINEAR DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT

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Abstract. Sufficient conditions for the $n$th order linear differential equation

$$u^{(n)}(t)+p(t)u(\tau(t))=0, \quad n\geq 2,$$

to have Property $A$ or Property $B$ are established in both the delayed and the advanced cases. These conditions essentially improve many known results not only for differential equations with deviating arguments but for ordinary differential equations as well.

## 1. Introduction

Let $n \geq 2$ be a natural number, $R_+ = [0,+\infty[$, $p : R_+ \to R$ be a locally integrable function of constant sign and $\tau : R_+ \to R$ be a continuous function satisfying $\lim_{t\to+\infty} \tau(t) = +\infty$. Consider the equation

$$u^{(n)}(t)+p(t)u(\tau(t))=0. \tag{1.1}$$

Let $\tau_*(t) = \min\{t,\tau(t)\}$ for $t \geq 0$, $t_0 \geq 0$, and $\tau_0 = \min\{\tau_*(t) : t \geq t_0\}$. A continuous function $u : [\tau_0,+\infty[\to R$ is said to be *a proper solution* of equation (1.1) if it is absolutely continuous on $[t_0,+\infty[$ along with its derivatives up to the $(n-1)$th order inclusively on any finite subsegment of $[t_0,+\infty[$, satisfies (1.1) almost everywhere on $[t_0,+\infty[$, and $\sup\big\{|u(s)| : s \geq t\big\} > 0$ for $t \geq t_0$. A proper solution of (1.1) is called oscillatory if it has a sequence of zeros tending to $+\infty$. Otherwise it is called nonoscillatory.

**Definition 1.1.** We say that equation (1.1) has *Property A* if any of its proper solutions is oscillatory when $n$ is even and either is oscillatory or satisfies

$$\big|u^{(i)}(t)\big| \downarrow 0 \quad \text{as} \quad t \uparrow +\infty \quad (i=0,\dots,n-1) \tag{1.2}$$

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when $n$ is odd.

**Definition 1.2.** We say that equation (1.1) has *Property B* if any of its proper solutions either is oscillatory or satisfies either (1.2) or

$$\left|u^{(i)}(t)\right| \uparrow +\infty \quad \text{as} \quad t \uparrow +\infty \quad (i = 0, \dots, n-1) \tag{1.3}$$

when $n$ is even, and either is oscillatory or satisfies (1.3) when $n$ is odd.

The investigation of oscillatory properties of higher order linear ordinary differential equations began as far back as the end of the last century in the work of A. Kneser [1]. Later W. Fite [2], J. Mikusinski [3], G. Anan'eva and V. Balaganskiĭ [4] contributed to the subject. A new impetus to investigations in this direction was given by the works of V. Kondrat'ev [5] and I. Kiguradze [6]. Further investigations were carried out by I. Kiguradze [7, 8] and T. Chanturia [9, 10]. Their results concerning Properties $A$ and $B$ of equation (1.1) in the case $\tau(t) \equiv t$ are collected in Section 1 of the monograph [11]. For higher order differential equations with deviating arguments Properties $A$ and $B$ were studied by R. Koplatadze and T. Chanturia [12] and R. Koplatadze [13–16].

In the present paper new sufficient conditions are established for equation (1.1) to have Properties $A$ and $B$. They complement some results given in Section 6 of [16] and are new even in the case of ordinary differential equations.

In the recent paper [17] I. Kiguradze and I. P. Stavroulakis obtained sufficient conditions concerning the existence of oscillatory solutions of advanced differential equations in the case where the equations have Property $A$. The results of the present paper can be applied in this case.

## 2. Statement of the Main Results

In this section we formulate the sufficient conditions mentioned at the end of Introduction. As a preliminary, note that, as it follows from Lemma 4.1 of [15], the condition

$$\int^{+\infty} [\tau_*(t)]^{n-1} |p(t)| dt = +\infty, \quad \tau_*(t) = \min\{t, \tau(t)\}, \tag{2.1}$$

is necessary for both Properties $A$ and $B$ of (1.1). In the case where $\tau(t) \le t$ for $t \ge 0$, condition (2.1) turns into

$$\int^{+\infty} [\tau(t)]^{n-1} |p(t)| dt = +\infty \tag{2.2}$$

and is implied by the hypotheses of the corresponding theorems. As to the case where $\tau(t) \geq t$ for $t \geq 0$, the equivalent condition

$$\int^{+\infty} t^{n-1}|p(t)|dt = +\infty \tag{2.3}$$

does not follow from the hypotheses of the corresponding theorems and is included therein.

Theorems 2.1–2.3 below treat the case of Property $A$, while Theorems 2.4–2.7 concern Property $B$.

**Theorem 2.1.** *Let $\tau$ be nondecreasing,*

$$\tau(t) \leq t, \quad p(t) \geq 0 \quad \textit{for} \quad t \geq 0 \tag{2.4}$$

*and*

$$\limsup_{t\to+\infty} \left\{ \tau(t) \int_t^{+\infty} [\tau(s)]^{n-2} p(s)ds + \int_{\tau(t)}^{t} [\tau(s)]^{n-1} p(s)ds + \right.$$
$$\left. +[\tau(t)]^{-1} \int_0^{\tau(t)} s[\tau(s)]^{n-1} p(s)ds \right\} > (n-1)!. \tag{2.5}$$

*Then equation* (1.1) *has Property $A$.*

**Theorem 2.2.** *Let $\tau$ be nondecreasing,* (2.3) *be fulfilled, $n$ be even, $\tau(t) \geq t$, $p(t) \geq 0$ for $t \geq 0$, and*

$$\limsup_{t\to+\infty} \left\{ \tau(t) \int_{\tau(t)}^{+\infty} s^{n-2} p(s)ds + \int_t^{\tau(t)} s^{n-1} p(s)ds + \right.$$
$$\left. +[\tau(t)]^{-1} \int_0^{t} s^{n-1} \tau(s) p(s)ds \right\} > (n-1)!. \tag{2.6}$$

*Then equation* (1.1) *has Property $A$.*

**Theorem 2.3.** *Let $\tau$ be nondecreasing,* (2.3) *be fulfilled, $n$ be odd, $\tau(t) \geq t$, $p(t) \geq 0$ for $t \geq 0$,*

$$\limsup_{t\to+\infty} \Big\{ \tau(t) \int_{\tau(t)}^{+\infty} [\tau(s)] s^{n-3} p(s) ds + \int_t^{\tau(t)} s^{n-2} \tau(s) p(s) ds + \\ + [\tau(t)]^{-1} \int_0^t s^{n-2} [\tau(s)]^2 p(s) ds \Big\} > 2(n-2)!, \tag{2.7}$$

*and*

$$\limsup_{t\to+\infty} \Big\{ \tau(t) \int_{\tau(t)}^{+\infty} [\tau(s)]^{n-2} p(s) ds + \int_t^{\tau(t)} s[\tau(s)]^{n-2} |p(s)| ds + \\ + [\tau(t)]^{-1} \int_0^t s[\tau(s)]^{n-1} p(s) ds \Big\} > (n-1)!. \tag{2.8}$$

*Then equation* (1.1) *has Property A.*

**Theorem 2.4.** *Let $\tau$ be nondecreasing, $n$ be even, $\tau(t) \leq t$, $p(t) \leq 0$ for $t \geq 0$, and*

$$\limsup_{t\to+\infty} \Big\{ \tau(t) \int_t^{+\infty} [\tau(s)] s^{n-3} |p(s)| ds + \int_{\tau(t)}^t s[\tau(s)]^{n-2} |p(s)| ds + \\ + [\tau(t)]^{-1} \int_0^{\tau(t)} s^2 [\tau(s)]^{n-2} |p(s)| ds \Big\} > 2(n-2)!. \tag{2.9}$$

*Then equation* (1.1) *has Property B.*

**Theorem 2.5.** *Let $\tau$ be nondecreasing, $n$ be odd, $\tau(t) \leq t$, $p(t) \leq 0$ for $t \geq 0$,*

$$\limsup_{t\to+\infty} \Big\{ \tau(t) \int_t^{+\infty} s^{n-2} |p(s)| ds + \int_{\tau(t)}^t s^{n-2} \tau(s) |p(s)| ds + \\ + [\tau(t)]^{-1} \int_0^{\tau(t)} s^{n-1} \tau(s) |p(s)| ds \Big\} > (n-1)!, \tag{2.10}$$

*and*

$$\limsup_{t\to+\infty}\left\{\tau(t)\int_t^{+\infty}[\tau(s)]^{n-3}s|p(s)|ds+\int_{\tau(t)}^{t}s[\tau(s)]^{n-2}|p(s)|ds+\right.$$
$$\left.+[\tau(t)]^{-1}\int_0^{\tau(t)}s^2[\tau(s)]^{n-2}|p(s)|ds\right\}>2(n-2)!. \tag{2.11}$$

*Then equation* (1.1) *has Property B.*

**Theorem 2.6.** *Let $\tau$ be nondecreasing,* (2.3) *be fulfilled, $n$ be even, $\tau(t)\ge t$, $p(t)\le 0$ for $t\ge 0$, and*

$$\limsup_{t\to+\infty}\left\{\tau(t)\int_{\tau(t)}^{+\infty}s^{n-3}\tau(s)|p(s)|ds+\int_t^{\tau(t)}s^{n-2}\tau(s)|p(s)|ds+\right.$$
$$\left.+[\tau(t)]^{-1}\int_0^{t}s^{n-2}[\tau(s)]^2|p(s)|ds\right\}>2(n-2)!. \tag{2.12}$$

*Then equation* (1.1) *has Property B.*

**Theorem 2.7.** *Let $\tau$ be nondecreasing,* (2.3) *be fulfilled, $n$ be odd, $\tau(t)\ge t$, $p(t)\le 0$ for $t\ge 0$, and*

$$\limsup_{t\to+\infty}\left\{\tau(t)\int_{\tau(t)}^{+\infty}s^{n-2}|p(s)|ds+\int_t^{\tau(t)}s^{n-2}\tau(s)|p(s)|ds+\right.$$
$$\left.+[\tau(t)]^{-1}\int_0^{t}s^{n-1}\tau(s)|p(s)|ds\right\}>(n-1)!. \tag{2.13}$$

*Then equation* (1.1) *has Property B.*

A series of corollaries more convenient for applications can be deduced from the formulated theorems. We will give here only those of Theorem 2.1. Analogous corollaries of Theorems 2.2–2.7 can be stated without any difficulty.

**Corollary 2.1.** *Let $\tau$ be nondecreasing,* (2.4) *be fulfilled and either*

$$\limsup_{t\to+\infty}\tau(t)\int_t^{+\infty}[\tau(s)]^{n-2}p(s)ds+\liminf_{t\to+\infty}t^{-1}\int_0^{t}s[\tau(s)]^{n-1}p(s)ds>(n-1)!$$

*or*

$$\liminf_{t\to+\infty}\tau(t)\int_t^{+\infty}[\tau(s)]^{n-2}p(s)ds+\limsup_{t\to+\infty}t^{-1}\int_0^t s[\tau(s)]^{n-1}p(s)ds>(n-1)!.$$

*Then equation* (1.1) *has Property A.*

**Corollary 2.2.** *Let $\tau$ be nondecreasing,* (2.4) *be fulfilled and either*

$$\limsup_{t\to+\infty}\tau(t)\int_t^{+\infty}[\tau(s)]^{n-2}p(s)ds>(n-1)!$$

*or*

$$\limsup_{t\to+\infty}t^{-1}\int_0^t s[\tau(s)]^{n-1}p(s)ds>(n-1)!.$$

*Then equation* (1.1) *has Property A.*

In the case of ordinary differential equations ($\tau(t)\equiv t$) all the above theorems give the same results for both Properties $A$ and $B$. They read as follows.

**Theorem 2.8.** *Let $\tau(t)\equiv t$, $p(t)\geq 0$ for $t\geq 0$, and*

$$\limsup_{t\to+\infty}\left\{t\int_t^{+\infty}s^{n-2}|p(s)|ds+t^{-1}\int_0^t s^n|p(s)|ds\right\}>(n-1)!. \tag{2.14}$$

*Then equation* (1.1) *has Property A.*

**Theorem 2.9.** *Let $\tau(t)\equiv t$, $p(t)\leq 0$ for $t\geq 0$. Let, moreover,*

$$\limsup_{t\to+\infty}\left\{t\int_t^{+\infty}s^{n-2}|p(s)|ds+t^{-1}\int_0^t s^n|p(s)|ds\right\}>2(n-2)! \tag{2.15}$$

*hold if $n$ is even, and* (2.14) *hold if $n$ is odd. Then equation* (1.1) *has Property B.*

Theorems 2.8 and 2.9 essentially improve T. Chanturia's tests [11] for Property $A$ and $B$ which read as follows.

If $\tau(t)\equiv t$, $p(t)\geq 0$ for $t\geq 0$, and

$$\limsup_{t\to+\infty}t\int_t^{+\infty}s^{n-2}|p(s)|ds>(n-1)!, \tag{2.16}$$

then (1.1) has Property $A$; if $\tau(t) \equiv t$, $p(t) \le 0$ for $t \ge 0$, (2.16) holds when $n$ is odd and

$$\limsup_{t\to+\infty} t \int_t^{+\infty} s^{n-2}|p(s)|ds > 2(n-2)!$$

when $n$ is even, then (1.1) has Property $B$.

T. Chanturia's tests imply the following criterion of I. Kiguradze [6, 7]: if $\tau(t) \equiv t$ and for some $\varepsilon > 0$

$$\int^{+\infty} t^{n-1-\varepsilon}|p(t)|dt = +\infty, \tag{2.17}$$

then equation (1.1) has Property $A$ for $p(t) \ge 0$, and Property $B$ for $p(t) \le 0$. It should be noted that the following corollary of Theorems 2.8, 2.9 also implies this criterion. To be more precise, under (2.17) the left-hand sides of (2.18) and (2.19) equal $+\infty$.

**Corollary 2.3.** *Let $\tau(t) \equiv t$, $p(t) \ge 0$ $(p(t) \le 0)$ and*

$$\limsup_{t\to+\infty} t^{-1} \int_0^t s^n|p(s)|ds > (n-1)! \tag{2.18}$$

*((2.16) holds if $n$ is odd and*

$$\limsup_{t\to+\infty} t^{-1} \int_0^t s^n|p(s)|ds > 2(n-2)! \tag{2.19}$$

*if $n$ is even). Then equation (1.1) has Property $A$ ($B$).*

## 3. Auxiliary Statements

In this section we give some auxiliary statements which we will need in the sequel. Lemmas 3.1 and 3.2 below describe some properties of nonoscillatory solutions of (1.1). Lemma 3.1 is due to I. Kiguradze and its proof can be found in [11, Lemma 1.1].

By $\widetilde{C}_{loc}^{n-1}([t_0,+\infty[)$ we denote the set of all functions $u : [t_0,+\infty[\to R$ which are absolutely continuous along with their derivatives up to the $(n-1)$th order inclusively on any finite subsegment of $[t_0,+\infty[$.

**Lemma 3.1.** *Let $u \in \widetilde{C}_{loc}^{n-1}([t_0,+\infty[)$ be positive and satisfy $u^{(n)}(t) \le 0$ $(u^{(n)}(t) \ge 0)$ for $t \ge t_0$. Then there exist $t_1 \in [t_0,+\infty[$ and $l \in \{0,\dots,n\}$ such that $l+n$ is odd (even) and*

$$\begin{aligned} &u^{(i)}(t) > 0 \quad \text{for} \quad t \ge t_1 \quad (i = 0,\dots,l-1), \\ &(-1)^{i+l}u^{(i)}(t) \ge 0 \quad \text{for} \quad t \ge t_1 \quad (i = l,\dots,n). \end{aligned} \tag{3.1$_l$}$$

**Lemma 3.2.** *Let $u \in \widetilde{C}_{loc}^{n-1}([t_0,+\infty[)$ and $(3.1_l)$ be fulfilled for some $l \in \{1,\dots,n-1\}$. Then*

$$\int^{+\infty} t^{n-l-1}|u^{(n)}(t)|dt < +\infty. \tag{3.2}$$

*If, moreover,*

$$\int^{+\infty} t^{n-l}|u^{(n)}(t)|dt = +\infty, \tag{3.3}$$

*then there is $t_* \ge t_0$ such that for $t \ge t_*$*

$$\frac{u(t)}{t^l} \downarrow, \qquad \frac{u(t)}{t^{l-1}} \uparrow \tag{3.4}$$

*and*

$$u(t) \ge \frac{t^l}{l!(n-l)!}\int_t^{+\infty} s^{n-l-1}|u^{(n)}(s)|ds + \frac{t^{l-1}}{l!(n-l)!}\int_{t_*}^{t} s^{n-l}|u^{(n)}(s)|ds. \tag{3.5}$$

*Proof.* Condition (3.2) follows from the identity

$$\sum_{j=i}^{k-1} \frac{(-1)^j t^{j-i} u^{(j)}(t)}{(j-i)!} = \sum_{j=i}^{k-1} \frac{(-1)^j t_0^{j-i} u^{(j)}(t_0)}{(j-i)!} + \\ + \frac{(-1)^{k-1}}{(k-i-1)!}\int_{t_0}^{t} s^{k-i+1}u^{(k)}(s)ds \tag{3.6$_{ik}$}$$

if we put $i = l$, $k = n$ in it and take $(3.1_l)$ into account. The same identity also implies the inequality

$$\sum_{j=l}^{n-1} \frac{t^{j-l}|u^{(j)}(t)|}{(j-l)!} \ge \frac{1}{(n-l-1)!}\int_t^{+\infty} s^{n-l-1}|u^{(n)}(s)|ds \ \text{ for } \ t \ge t_1. \tag{3.7}$$

Let now (3.3) be fulfilled. Using ($3.1_l$), from ($3.6_{l-1,n}$) we have

$$u^{(l-1)}(t) - tu^{(l)}(t) = \sum_{j=l+1}^{n-1} \frac{t^{j-l+1}|u^{(j)}(t)|}{(j-l+1)!} + \\ + \sum_{j=l-1}^{n-1} \frac{(-1)^{j+l-1} t_0^{j-l+1} u^{(j)}(t_0)}{(j-l+1)!} + \frac{1}{(n-l)!} \int_{t_0}^{t} s^{n-l} |u^{(n)}(s)| ds \\ \text{for} \quad t \geq t_1 \tag{3.8}$$

which by (3.3) gives

$$\lim_{t\to+\infty} \big(u^{(l-1)}(t) - tu^{(l)}(t)\big) = +\infty \tag{3.9}$$

and

$$u^{(l-1)}(t) \geq \sum_{j=l}^{n-1} \frac{t^{j-l+1}|u^{(j)}(t)|}{(j-l+1)!} \quad \text{for large} \quad t. \tag{3.10}$$

For any $t \geq t_1$ and $i \in \{1, \dots, l\}$ put

$$\rho_i(t) = iu^{(l-i)}(t) - tu^{(l-i+1)}(t) = -t^{i+1}\big(t^{-i}u^{(l-i)}(t)\big)', \tag{3.11}$$

$$r_i(t) = tu^{(l-i+1)}(t) - (i-1)u^{(l-i)}(t) = t^i\big(t^{1-i}u^{(l-i)}(t)\big)'. \tag{3.12}$$

From (3.9) by De l'Hospital's rule we have

$$\lim_{t\to+\infty} t^{1-i}u^{(l-i)}(t) = +\infty \quad (i = 1, \dots, l)$$

so that in view of (3.12) there are $\alpha_l \geq \dots \geq \alpha_1 > t_1$ such that $r_i(\alpha_i) > 0$ $(i = 1, \dots, l)$. Since $r_1(t) = tu^{(l)}(t) > 0$ and $r'_{i+1}(t) = r_i(t)$ for $t \geq t_0$ $(i = 1, \dots, l-1)$, we have $r_i(t) > 0$ for $t \geq \alpha_i$. Analogously, since by (3.9) $\rho_1(t) \to +\infty$ as $t \to +\infty$ and $\rho'_{i+1}(t) = \rho(t)$, we have $\rho_i(t) \to +\infty$ as $t \to +\infty$ $(i = 1, \dots, l)$. In view of (3.11), (3.12) this proves (3.4). On the other hand, by (3.11) we have

$$iu^{(l-i)}(t) \geq tu^{(l-i+1)}(t) \quad \text{for large} \quad t \quad (i = 1, \dots, l),$$

which implies

$$u(t) \geq \frac{t^{l-1}}{l!} u^{(l-1)}(t) \quad \text{for large} \quad t. \tag{3.13}$$

It remains to prove (3.5). Choose $t_2 > t_1$ sufficiently large for (3.10) to hold with $t = t_2$. Using (3.8) with $t_0$ replaced by $t_2$, by (3.10) and (3.7) we get

$$u^{(l-1)}(t) \geq \sum_{j=l}^{n-1} \frac{t^{j-l+1}|u^{(j)}(t)|}{(j-l+1)!} + \frac{1}{(n-l)!} \int_{t_2}^{t} s^{n-l}|u^{(n)}(s)|ds \geq$$

$$\geq \frac{t}{n-l} \sum_{j=l}^{n-1} \frac{t^{j-l}|u^{(j)}(t)|}{(j-l)!} + \frac{1}{(n-l)!} \int_{t_2}^{t} s^{n-l}|u^{(n)}(s)|ds \geq$$

$$\geq \frac{t}{(n-l)!} \int_{t}^{+\infty} s^{n-l-1}|u^{(n)}(s)|ds + \frac{1}{(n-l)!} \int_{t_2}^{t} s^{n-l}|u^{(n)}(s)|ds.$$

Combining the latter inequality with (3.13), we obtain (3.5). □

## 4. Proof of the Main Results

**Proposition 4.1.** *Let $\tau$ be nondecreasing, $\tau(t) \leq t$, $p(t) \geq 0$ ($p(t) \leq 0$) for $t \geq 0$, $l \in \{1, \dots, n-1\}$, $l+n$ be odd (even) and*

$$\limsup_{t \to +\infty} \left\{ \tau(t) \int_{t}^{+\infty} [\tau(s)]^{l-1} s^{n-l-1} |p(s)| ds + \int_{\tau(t)}^{t} s^{n-l-1} [\tau(s)]^{l} |p(s)| ds + [\tau(t)]^{-1} \int_{0}^{\tau(t)} s^{n-l} [\tau(s)]^{l} |p(s)| ds \right\} > l!(n-l)!. \tag{4.1}$$

*Then equation (1.1) has no solution satisfying ($3.1_l$).*

*Proof.* Suppose, to the contrary, that (1.1) has a solution $u : [t_0, +\infty[\to R$ satisfying ($3.1_l$). Show that (3.3) is fulfilled. Indeed, if it is not the case, then since $u(t) \geq ct^{l-1}$ for $t \geq t_1$ with some $t_1 \geq t_0$ and $c_0 > 0$, by (1.1) we have

$$\int_{t_1}^{+\infty} s^{n-l} [\tau(s)]^{l-1} |p(s)| ds < +\infty. \tag{4.2}$$

Therefore, since $\tau$ is nondecreasing and $\tau(t) \le t$, we get

$$\tau(t) \int_t^{+\infty} [\tau(s)]^{l-1} s^{n-l-1} |p(s)| ds \le \int_t^{+\infty} [\tau(s)][\tau(s)]^{l-1} s^{n-l-1} |p(s)| ds \le$$

$$\le \int_t^{+\infty} s^{n-l} [\tau(s)]^{l-1} |p(s)| ds \to 0 \quad \text{as} \quad t \to +\infty. \tag{4.3}$$

Analogously,

$$\int_{\tau(t)}^{t} s^{n-l-1} [\tau(s)]^{l} |p(s)| ds \le \int_{\tau(t)}^{+\infty} s^{n-l} [\tau(s)]^{l-1} |p(s)| ds \to 0 \quad \text{as} \quad t \to +\infty. \tag{4.4}$$

On the other hand, for any sufficiently large $t_2 \ge t_1$ we have

$$[\tau(t)]^{-1} \int_0^{\tau(t)} s^{n-l} [\tau(s)]^{l} |p(s)| ds \le [\tau(t)]^{-1} \int_0^{t_2} s^{n-l} [\tau(s)]^{l} ds +$$

$$+ \int_{t_2}^{\tau(t)} s^{n-l} [\tau(s)]^{l-1} |p(s)| ds \quad \text{for} \quad t \ge t_2.$$

Passing here to the upper limit as $t \to +\infty$ and then to the limit as $t_2 \to +\infty$, we obtain

$$[\tau(t)]^{-1} \int_0^{\tau(t)} s^{n-2} [\tau(s)]^{l} |p(s)| ds \to 0 \quad \text{as} \quad t \to +\infty. \tag{4.5}$$

Conditions (4.3), (4.4), and (4.5) contradict (4.1). Thus (3.3) is proved.

Now we can use Lemma 3.2. Taking into account (3.4), (3.5) and the fact that $\tau(t) \le t$ and $\tau$ is nondecreasing, we get for $t \ge t_*$ with sufficiently large $t_*$

$$u(\tau(t)) \ge \frac{\tau^l(t)}{l!(n-l)!} \int_{\tau(t)}^{+\infty} s^{n-l-1} |u^{(n)}(s)| ds + \frac{\tau^{l-1}(t)}{l!(n-l)!} \int_{t_*}^{\tau(t)} s^{n-l} |u^{(n)}(s)| ds =$$

$$= \frac{\tau^l(t)}{l!(n-l)!} \int_t^{+\infty} s^{n-l-1} |p(s)| u(\tau(s)) ds + \frac{\tau^l(t)}{l!(n-l)!} \int_{\tau(t)}^{t} s^{n-l-1} |p(s)| u(\tau(s)) ds +$$

$$+\frac{\tau^{l-1}(t)}{l!(n-l)!}\int_{t_*}^{\tau(t)} s^{n-l}|p(s)|u(\tau(s))ds \geq \frac{\tau(t)u(\tau(t))}{l!(n-l)!}\int_{t}^{+\infty} s^{n-l-1}\tau^{l-1}(s)|p(s)|ds+$$

$$+\frac{u(\tau(t))}{l!(n-l)!}\int_{\tau(t)}^{t} s^{n-l-1}\tau^{l}(s)|p(s)|ds+\frac{[\tau(t)]^{-1}u(\tau(t))}{l!(n-l)!}\int_{t_*}^{\tau(t)} s^{n-l}\tau^{l}(s)|p(s)|ds.$$

The last inequality contradicts (4.1). □

**Proposition 4.2.** *Let $\tau$ be nondecreasing, $\tau(t) \geq t$, $p(t) \geq 0$ $(p(t) \leq 0)$ for $t \geq 0$, $l \in \{1,\dots,n-1\}$, $l+n$ be odd (even) and*

$$\limsup_{t\to+\infty}\Big\{\tau(t)\int_{\tau(t)}^{+\infty}[\tau(s)]^{l-1}s^{n-l-1}|p(s)|ds+\int_{t}^{\tau(t)} s^{n-l}[\tau(s)]^{l-1}|p(s)|ds+$$

$$+[\tau(t)]^{-1}\int_{0}^{t} s^{n-l}[\tau(s)]^{l}|p(s)|ds\Big\} > l!(n-l)! \tag{4.6}$$

*Let, moreover, condition* (2.3) *be fulfilled. Then equation* (1.1) *has no solution satisfying* $(3.1_l)$.

The proof is analogous to that of Proposition 4.1. The difference is that now condition (3.3) follows from (2.3).

*Proof of Theorem* 2.1. Let $u : [t_0,+\infty[\to]0,+\infty[$ be a proper nonoscillatory solution of (1.1). According to Lemma 1.1, there exists $l \in \{0,\dots,n-1\}$ such that $l+n$ is odd and $(3.1_l)$ is fulfilled. It can be easily seen that (2.4) implies (4.1) for any $l \in \{1,\dots,n-1\}$. Hence by Proposition 4.1, $l \notin \{1,\dots,n-1\}$. Thus it remains to prove the case $l = 0$, which is possible only when $n$ is odd. Using arguments similar to those of deriving (3.3) from (4.1) in Proposition 4.1, one can show that (2.2) is fulfilled which, in turn, easily implies that $u(t) \to 0$ as $t \to +\infty$. But this means that (1.2) holds and thus equation (1.1) has Property $A$. □

Theorems 2.2 and 2.3 can be proved analogously. Instead of Proposition 4.1 one has to use Proposition 4.2. For any $l \in \{1,\dots,n-1\}$ inequality (4.6) follows from (2.6) in the case of Theorem 2.2, and from (2.7)–(2.8) in the case of Theorem 2.3.

*Proof of Theorem* 2.4. First of all note that, as above, (2.9) implies (2.2). Let $u : [t_0,+\infty[\to]0,+\infty[$ be a nonoscillatory proper solution of (1.1). According to Lemma 3.1, there exists $l \in \{0,\dots,n\}$ such that $l+n$ is even and $(3.1_l)$ is fulfilled. As above, (2.9) implies (4.1) for any $l \in \{1,\dots,n-1\}$. Hence by Proposition 4.1 $l \notin \{1,\dots,n-1\}$. If $l = n$, then condition (2.2) implies (1.3), and if $l = 0$ which is possible only if $n$ is even, then the same condition (2.2) yields (1.2). This means that equation (1.1) has Property $B$. □

The proofs of Theorems 2.5–2.7 are analogous. One has to use the appropriate one of Propositions 4.1, 4.2 and note that (2.10)–(2.11) imply (4.1) for any $l \in \{1, \dots, n-1\}$, while (2.12) and (2.13) imply (4.6) for any $l \in \{1, \dots, n-1\}$.

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