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SUFFICIENT CONDITIONS FOR THE OSCILLATION OF BOUNDED SOLUTIONS OF A CLASS OF IMPULSIVE DIFFERENTIAL EQUATIONS OF SECOND ORDER WITH A CONSTANT DELAY

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Abstract. Sufficient conditions are found for oscillation of bounded solutions of a class of impulsive differential equations of second order with a constant delay. Some asymptotic properties are studied for the bounded solutions.

1. Introduction

The last twenty years have seen a significant increase in the number of papers devoted to the oscillation theory of differential equations with a deviating argument. The main part of these investigations is given in the monographs [1], [2], [3].

On the other hand, the last decade has been marked by a growing interest in impulsive differential equations due to their various applications in science and technology. In the monographs [4] and [5] numerous aspects of their qualitative theory are studied. However, the oscillation theory of impulsive differential equations has not yet been worked out.

In the present paper we obtain sufficient conditions for the oscillation of bounded solutions of a class of impulsive differential equations of second order with a constant delay and fixed moments of the impulse effect.

2. Preliminary Notes

We consider the impulsive differential equations of second order

$$\begin{aligned} &\big(r(t)y'(t)\big)' - \sum_{i=1}^{n} p_i(t)y(t-h_i) = 0, \quad t \neq \tau_k, \quad k \in \mathbb{N}, \\ &\Delta y'(\tau_k) = y'(\tau_k+0) - y'(\tau_k-0) = \beta_k y(\tau_k), \\ &\Delta y(\tau_k) = y(\tau_k+0) - y(\tau_k-0) = 0, \end{aligned} \tag{1}$$

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under the initial conditions

$$y(t) = \varphi(t), \quad t \in [-h, 0], \quad h = \max\{h_i \,:\, i \in \mathbb{N}_n\},$$
$$y'(0) = \varphi'(0) = y'_0. \tag{2}$$

Here $\mathbb{N}_n = \{1, 2, \ldots, n\}$; $\{\tau_k\}_{k=1}^{\infty}$ is a monotone increasing unbounded sequence of positive numbers; $\{\beta_k\}_{k=1}^{\infty}$ is a sequence of positive numbers; h_i, $i \in \mathbb{N}_n$, are positive constants, $\overline{\mathbb{R}}_+ = [0, +\infty)$; $\mathbb{R}_+ = (0, +\infty)$; $y'(\tau_k - 0) = y'(\tau_k)$.

We denote by $PC(\overline{\mathbb{R}}_+, \mathbb{R})$ the set of all functions $u\colon\ \overline{\mathbb{R}}_+ \to \mathbb{R}$ which are continuous for $t \in \overline{\mathbb{R}}_+$, $t \neq \tau_k$ $(k \in \mathbb{N})$, continuous from the left for $t \in \mathbb{R}_+$ and having a discontinuity of first kind at the points $\tau_k \in \mathbb{R}_+$ $(k \in \mathbb{N})$.

Let us introduce the following conditions:

H1. $\varphi \in C^2([-h, 0], \mathbb{R})$.

H2. $p_i \in PC(\overline{\mathbb{R}}_+, \mathbb{R}_+)$, $i \in \mathbb{N}_n$.

H3. $r \in PC(\overline{\mathbb{R}}_+, \mathbb{R}_+)$, $r(\tau_k + 0) > 0$, $k \in \mathbb{N}$.

Definition 1. We shall call a solution of equation (1) with the initial conditions (2) any function $y\colon [-h, +\infty) \to \mathbb{R}$ for which the following conditions are fulfilled:

1. If $-h \le t \le 0$, $y(t) = \varphi(t)$.
2. If $0 < t \le \tau_1$, the solution $y(t)$ coincides with the solution of problem (1), (2) without impulse effect.
3. If $\tau_k < t \le \tau_{k+1}$, $k \in \mathbb{N}$, the solution of problem (1), (2) coincides with the solution of the integro-differential equation

$$r(t)y'(t) = r(\tau_k + 0)y'(\tau_k + 0) + \int_{\tau_k}^{t} \sum_{i=1}^{n} p_i(s)y(s - h_i)\, ds$$

with the initial conditions (2).

Definition 2. The solution $y(t)$ of problem (1), (2) is said to be *oscillatory* if for each $a > 0$ we have

$$\bigl\{t \,:\, y(t) > 0,\ t > a\bigr\} \neq \varnothing \quad \text{and} \quad \bigl\{t \,:\, y(t) < 0,\ t > a\bigr\} \neq \varnothing.$$

Otherwise, the solution $y(t)$ is called *nonoscillatory.*

3. Main Results

Theorem 1. *Let the following conditions hold:*

1. *Conditions* H1–H3 *are fulfilled.*
2. $\lim\limits_{t\to+\infty} R(t) = +\infty$, *where* $R(t) = \displaystyle\int_0^t \frac{ds}{r(s)}$.

3. $\int^{\infty} R(s) \sum_{i=1}^{n} p_i(s) ds = +\infty.$

Then all bounded solutions of equation (1) *either tend to zero as* $t \to +\infty$, *or oscillate.*

Proof. Let $y(t)$ be a positive bounded solution of equation (1) for $t \geq t_1 > 0$. It is clear that $y(t-h_i) > 0$ for $t \geq t_2 = t_1 + h$. This fact, (1) and condition H2 imply that the function $r(t)y'(t)$ increases in the set $M = [t_2, \tau_s) \cup \big[\bigcup_{i=s}^{\infty} (\tau_i, \tau_{i+1})\big]$, where $\tau_{s-1} < t_2 < \tau_s$. On the other hand, $r(\tau_k)\Delta y'(\tau_k) = \beta_k r(\tau_k) y(\tau_k) > 0$ for $\tau_k > t_2$ and therefore $r(t)y'(t)$ is an increasing function for $t \geq t_2$.

The following cases are possible:

Case 1. Let $r(t)y'(t) > 0$ for $t \geq t_2$. Since $r(t)y'(t)$ is an increasing function for $t \geq t_2$, there exist a constant $c > 0$ and a point $t_3 \geq t_2$ such that

$$y'(t) \geq \frac{c}{r(t)}, \qquad t \geq t_3. \tag{3}$$

We integrate (3) from t_3 to t $(t \geq t_3)$ and obtain

$$y(t) \geq y(t_3) + \int_{t_3}^{t} \frac{c}{r(s)}\, ds. \tag{4}$$

Now (4) and condition 2 of the theorem imply that $\lim_{t \to +\infty} y(t) = +\infty$, which contradicts the assumption that $y(t)$ is a bounded solution.

Case 2. Let $r(t)y'(t) < 0$ for $t \geq t_2$. Therefore $y'(t) < 0$, $t \geq t_2$. On the other hand, $y(t) > 0$ for $t \geq t_2$. Then it follows that there exists a finite limit $\lim_{t \to +\infty} y(t) \geq 0$. The assumption that $r(t)y'(t) < 0$ and the fact that $r(t)y'(t)$ is an increasing function for $t \geq t_2$ lead to the existence of a finite limit $\lim_{t \to +\infty} r(t)y'(t) \leq 0$.

Suppose that $\lim_{t \to +\infty} r(t)y'(t) = c_1 < 0$, i.e., there exists a point $\bar{t} \geq t_2$ such that for $t \geq \bar{t}$ we have

$$y'(t) \leq \frac{c_1}{2r(t)}. \tag{5}$$

Integrating (5) from $\bar{t}$ to t $(t > \bar{t}\,)$, we obtain

$$y(t) \leq y(\bar{t}\,) + \int_{\bar{t}}^{t} \frac{c_1}{2r(s)}\, ds. \tag{6}$$

Thus (6) and condition 2 of the theorem yield $\lim_{t\to+\infty} y(t) = -\infty$, which contradicts the assumption that $y(t)$ is a positive solution of equation (1).

Therefore

$$\lim_{t\to+\infty} r(t)y'(t) = 0. \tag{7}$$

We integrate (1) from t_2 to t and arrive at

$$r(t)y'(t) = r(t_2)y'(t_2) + \sum_{t_2\le\tau_i<t} \beta_i r(\tau_i)y(\tau_i) + \int_{t_2}^{t} \sum_{i=1}^{n} p_i(s)y(s-h_i)\,ds. \tag{8}$$

Passing to the limit in (8) as $t\to+\infty$ and bearing in mind (7), we obtain

$$r(t_2)y'(t_2) = -\sum_{t_2\le\tau_i<\infty} \beta_i r(\tau_i)y(\tau_i) - \int_{t_2}^{\infty} \sum_{i=1}^{n} p_i(s)y(s-h_i)\,ds. \tag{9}$$

We divide (8) by $r(t) > 0$, integrate the equality obtained from t_2 to t and obtain

$$\begin{aligned} y(t) = y(t_2) &+ r(t_2)y'(t_2)\big[R(t)-R(t_2)\big] + \\ &+ \int_{t_2}^{t} \big[R(t)-R(s)\big] \sum_{i=1}^{n} p_i(s)y(s-h_i)\,ds + \\ &+ \int_{t_2}^{t} \frac{1}{r(s)} \sum_{t_2\le\tau_i<s} \beta_i r(\tau_i)y(\tau_i)\,ds. \end{aligned} \tag{10}$$

It follows from (9) and (10) that

$$\begin{aligned} y(t) = y(t_2) &- \big[R(t)-R(t_2)\big]\Big\{ \sum_{i=1}^{n} \int_{t_2}^{\infty} p_i(s)y(s-h_i)\,ds + \\ &+ \sum_{t_2\le\tau_i<\infty} \beta_i r(\tau_i)y(\tau_i) \Big\} + \int_{t_2}^{t} \frac{1}{r(s)} \sum_{t_2\le\tau_i<s} \beta_i r(\tau_i)y(\tau_i)\,ds + \\ &+ \int_{t_2}^{t} \big[R(t)-R(s)\big] \sum_{i=1}^{n} p_i(s)y(s-h_i)\,ds. \end{aligned} \tag{11}$$

Hence

$$y(t) \le y(t_2) - \big[R(t) - R(t_2)\big]\Bigg\{ \sum_{t_2 \le \tau_i < \infty} \beta_i r(\tau_i) y(\tau_i) + $$

$$+ \int\limits_{t_2}^{\infty} \sum_{i=1}^{n} p_i(s) y(s - h_i)\, ds \Bigg\} \big[R(t) - R(t_2)\big] \sum_{t_2 \le \tau_i < \infty} \beta_i r(\tau_i) y(\tau_i) +$$

$$+ \int\limits_{t_2}^{t} \big[R(t) - R(s)\big] \sum_{i=1}^{n} p_i(s) y(s - h_i)\, ds.$$

The latter inequality implies the relation

$$y(t) \le y(t_2) + \int\limits_{t_2}^{t} \big[R(t_2) - R(t)\big] \sum_{i=1}^{n} p_i(s) y(s - h_i)\, ds -$$

$$- \big[R(t) - R(t_2)\big] \int\limits_{t}^{\infty} \sum_{i=1}^{n} p_i(s) y(s - h_i)\, ds +$$

$$+ \int\limits_{t_2}^{t} \big[R(t) - R(s)\big] \sum_{i=1}^{n} p_i(s) y(s - h_i)\, ds =$$

$$= y(t_2) + \int\limits_{t_2}^{t} \big[R(t_2) - R(s)\big] \sum_{i=1}^{n} p_i(s) y(s - h_i)\, ds -$$

$$- \big[R(t) - R(t_2)\big] \int\limits_{t}^{\infty} \sum_{i=1}^{n} p_i(s) y(s - h_i)\, ds,$$

i.e.,

$$y(t) \le y(t_2) + R(t_2) \int\limits_{t_2}^{t} \sum_{i=1}^{n} p_i(s) y(s - h_i)\, ds -$$

$$- \int\limits_{t_2}^{t} R(s) \sum_{i=1}^{n} p_i(s) y(s - h_i)\, ds. \tag{12}$$

It follows from (8) and (12) that

$$y(t) \le y(t_2) + R(t_2) r(t) y'(t) - R(t_2) r(t_2) y'(t_2) -$$

$$- R(t_2) \sum_{t_2 \le \tau_i < t} \beta_i r(\tau_i) y(\tau_i) - \int_{t_2}^{t} R(s) \sum_{i=1}^{n} p_i(s) y(s - h_i)\, ds.$$

Therefore

$$y(t) \le y(t_2) - R(t_2) r(t_2) y'(t_2) - \int_{t_2}^{t} R(s) y(s - \bar{h}) \sum_{i=1}^{n} p_i(s)\, ds, \qquad (13)$$

where $\bar{h} = \min\{h_i \,:\, i \in \mathbb{N}_n\}$.

Now, from $y(t) > 0$ for $t \ge t_2$ and from the fact that $y(t)$ is a decreasing function in $[t_2, +\infty)$ we have $\inf_{s \in [t_2, t]} y(s - \bar{h}) = y(t - \bar{h})$. Thus (13) yields the inequality

$$y(t) \le y(t_2) - R(t_2) r(t_2) y'(t_2) - y(t - \bar{h}) \int_{t_2}^{t} R(s) \sum_{i=1}^{n} p_i(s)\, ds.$$

If we suppose that $\lim_{t \to +\infty} y(t) = c > 0$, then the latter inequality gives $\lim_{t \to +\infty} y(t) = -\infty$ as $t \to +\infty$, which contradicts the fact that $y(t)$ is a bounded positive solution of equation (1). Therefore $\lim_{t \to +\infty} y(t) = 0$. □

Theorem 2. *Let the following conditions hold:*

1. *Conditions* H1–H3 *are fulfilled.*
2. $\displaystyle\int_0^{\infty} \frac{dt}{r(t)} = +\infty.$
3. $\displaystyle\limsup_{t \to +\infty} \frac{1}{r(t)} \int_{t-\bar{h}}^{t} (s - t + \bar{h}) \sum_{i=1}^{n} p_i(s)\, ds > 1,$

where $\bar{h} = \min\{h_i \,:\, i \in \mathbb{N}_n\}$.

Then all bounded nontrivial solutions of equation (1) *are oscillatory.*

Proof. Let $y(t)$ be a bounded nonoscillatory solution of equation (1). Without loss of generality we may assume $y(t) > 0$ for $t \ge t_0 \ge 0$. Then $y(t - h_i) > 0$ for $t \ge t_0 + h = t_1$. Analogously to the proof of Theorem 1 we arrive at $r(t) y'(t) \le 0$, $t \ge t_1$.

Integrate (1) from s to t $(t > s \ge t_1)$ and obtain

$$r(t) y'(t) = r(s) y'(s) + \sum_{s \le \tau_i < t} \beta_i r(\tau_i) y(\tau_i) + \int_{s}^{t} \sum_{i=1}^{n} p_i(\sigma) y(\sigma - h_i) d\sigma. \qquad (14)$$

Now we integrate (14) from $t-\bar{h}$ to t, $t \geq t_1+\bar{h}$, and derive

$$r(t)y'(t)\bar{h} = \int_{t-\bar{h}}^{t} r(s)\,dy(s) + \int_{t-\bar{h}}^{t} \sum_{s\leq\tau_i<t} \beta_i r(\tau_i)y(\tau_i)\,ds + \\ + \int_{t-\bar{h}}^{t} [\sigma - t + \bar{h}] \sum_{i=1}^{n} p_i(\sigma)y(\sigma - h_i)\,d\sigma. \tag{15}$$

From (15) we obtain

$$0 \geq r(t)y(t) - r(t-\bar{h})y(t-\bar{h}) - \int_{t-\bar{h}}^{t} y(s)\,dr(s) + \\ + \int_{t-\bar{h}}^{t} \sum_{s\leq\tau_i<t} \beta_i r(\tau_i)y(\tau_i)\,ds + \int_{t-\bar{h}}^{t} [\sigma - t + \bar{h}] \sum_{i=1}^{n} p_i(\sigma)y(\sigma - h_i)\,d\sigma \geq \\ \geq r(t)y(t) - r(t-\bar{h})y(t-\bar{h}) - y(t-\bar{h})\big[r(t) - r(t-\bar{h})\big] + \\ + \int_{t-\bar{h}}^{t} [\sigma - t + \bar{h}] \sum_{i=1}^{n} p_i(\sigma)y(\sigma - h_i)\,d\sigma = \\ = r(t)y(t) - r(t)y(t-\bar{h}) + \int_{t-\bar{h}}^{t} [\sigma - t + \bar{h}] \sum_{i=1}^{n} p_i(\sigma)y(\sigma - h_i)\,d\sigma. \tag{16}$$

Since $y(t)$ is a nonincreasing function in $[t_1, +\infty)$, we have $y(\sigma - h_i) \geq y(\sigma - \bar{h})$, $\sigma \in [t-\bar{h}, t]$ and $\inf_{t-\bar{h}\leq\sigma\leq t} y(\sigma-\bar{h}) = y(t-\bar{h})$.

Then (16) implies

$$0 \geq r(t)y(t) - r(t)y(t-\bar{h}) + y(t-\bar{h}) \int_{t-\bar{h}}^{t} [\sigma - t + \bar{h}] \sum_{i=1}^{n} p_i(\sigma)\,d\sigma. \tag{17}$$

Divide (17) by $r(t)y(t-\bar{h}) > 0$ and obtain

$$\frac{y(t)}{y(t-\bar{h})} + \left[\frac{1}{r(t)} \int_{t-\bar{h}}^{t} (\sigma - t + \bar{h}\,) \sum_{i=1}^{n} p_i(\sigma)\,d\sigma - 1\right] \leq 0.$$

The latter inequality contradicts condition 3 of Theorem 2. $\square$

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