GEORGIAN MATHEMATICAL JOURNAL: Vol. 5, No. 4, 1998, 341-360

ON SOME BOUNDARY VALUE PROBLEMS FOR AN ULTRAHYPERBOLIC EQUATION

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Abstract. The correct formulation of a characteristic problem and a Darboux type problem in the special weighted functional spaces for an ultrahyperbolic equation is investigated.

In the space of variables x_1, x_2, y_1 and y_2 we consider the ultrahyperbolic equation

$$u_{y_1y_1} + u_{y_2y_2} - u_{x_1x_1} - u_{x_2x_2} = F. \tag{1}$$

Denote by $D : -y_1 < x_1 < y_1$, $0 < y_1 < +\infty$, a dihedral angle bounded by the characteristic surfaces $S_1 : x_1 - y_1 = 0$, $0 \le y_1 < +\infty$, and $S_2 : x_1 + y_1 = 0$, $0 \le y_1 < +\infty$, of equation (1).

We shall consider a characteristic problem formulated as follows: in the domain D find a solution $u(x_1, x_2, y_1, y_2)$ of equation (1) by the boundary conditions

$$u\,|_{S_i} = f_i, \quad i = 1, 2, \tag{2}$$

where f_i, $i = 1, 2$, are given real functions on S_i and $(f_1 - f_2)\,|_{S_1 \cap S_2} = 0$.

Characteristic problems formulated similarly were considered in [1–3].

Let $G = \{(x_1, \xi_1, y_1, \xi_2) \in R^4 : -y_1 < x_1 < y_1,\ 0 < y_1 < +\infty;\ -\infty < \xi_i < +\infty,\ i = 1, 2\}$.

Denote by $\Phi^k(\overline{D})$, $k \ge 2$, the space of functions $u(x_1, x_2, y_1, y_2)$ of the class $C^k(\overline{D})$ whose partial Fourier transforms $\widehat{u}(x_1, \xi_1, y_1, \xi_2)$ with respect to the variables x_2 and y_2 are continuous functions in $\overline{G}$ together with partial derivatives with respect to the variables x_1 and y_1 up to kth order inclusive and satisfy the following estimates: for any natural N there exist positive

1991 *Mathematics Subject Classification.* 35L20.

Key words and phrases. Ultrahyperbolic equation, characteristic problem, Darboux type problem.

integers $C_N = C_N(x_1, y_1)$ and $K_N = K_N(x_1, y_1)$ not depending on ξ_1, ξ_2 such that the inequalities

$$\left|\partial^{i_1+i_2}/\partial x_1^{i_1}\partial y_1^{i_2}\ \widehat{u}(x_1,\xi_1,y_1,\xi_2)\right| \le C_N e^{-N(|\xi_1|+|\xi_2|)},\ \ 0 \le i_1+i_2 \le k, \quad (3)$$

are fulfilled for $-y_1 \le x_1 \le y_1$, $0 \le y_1 < +\infty$ and $|\xi_1| + |\xi_2| > K_N$. Note that

$$C_N^0(x_1,y_1) = \sup_{(x_1^0,y_1^0)\in I(x_1,y_1)} C_N(x_1^0,y_1^0) < +\infty,$$
$$K_N^0(x_1,y_1) = \sup_{(x_1^0,y_1^0)\in I(x_1,y_1)} K_N(x_1^0,y_1^0) < +\infty,$$

where $I(x_1,y_1) = \{(x_1^0,y_1^0) \in R^2 : 0 \le y_1^0 - x_1^0 \le y_1 - x_1,\ 0 \le y_1^0 + x_1^0 \le y_1 + x_1\}$ is the closed rectangle.

The spaces $\Phi^k(S_i)$, $i = 1, 2$, are introduced similarly. Note that the trace $u\,|_{S_i}$ of the function u from the space $\Phi^k(\overline{D})$ belongs to the space $\Phi^k(S_i)$.

When considering a solution of problem (1), (2) in the space $\Phi^k(\overline{D})$, $k \ge 2$, it will be assumed that $F \in \Phi^{k-1}(\overline{D})$, $f_i \in \Phi^k(S_i)$, $i = 1, 2$.

If u is a solution of problem (1), (2) of the class $\Phi^k(\overline{D})$, then after the Fourier transformation with respect to the variables x_2 and y_2, equation (1) and conditions (2) can be rewritten as

$$v_{y_1y_1} - v_{x_1x_1} + (\xi_1^2 - \xi_2^2)v = F_0, \quad (4)$$
$$v\,|_{l_j} = g_j, \quad j = 1, 2, \quad (5)$$

where v, F_0, g_1, g_2 are respectively the Fourier transforms of the functions u, F, f_1, f_2 with respect to the variables x_2 and y_2, i.e.,

$$v(x_1,y_1,\xi_1,\xi_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x_1,x_2,y_1,y_2)e^{-ix_2\xi_1-iy_2\xi_2}dx_2\,dy_2,$$

$$F_0(x_1,y_1,\xi_1,\xi_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x_1,x_2,y_1,y_2)e^{-ix_2\xi_1-iy_2\xi_2}dx_2\,dy_2,$$

$$g_j(x_1,y_1,\xi_1,\xi_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_j(x_1,x_2,y_1,y_2)e^{-ix_2\xi_1-iy_2\xi_2}dx_2\,dy_2,$$
$$(x_1y_1) \in l_j, \quad j = 1, 2,$$

and $l_1 : x_1 - y_1 = 0$, $0 \le y_1 < +\infty$, $l_2 : x_1 + y_1 = 0$, $0 \le y_1 < +\infty$, are the beams lying in the plane of variables x_1, y_1 and outgoing from the origin $O(0, 0)$.

Remark. Thus by the Fourier transformation with respect to the variables x_2 and y_2 the spatial problem (1), (2) is reduced to the Goursat plane problem (4), (5) with the parameters ξ_1 and ξ_2 lying in the domain D_0 : $-y_1 < x_1 < y_1$, $0 < y_1 < +\infty$ of the plane of variables x_1, y_1. It is easy to see that in the class of functions defined by inequalities (3), this reduction is equivalent, i.e., problem (1), (2) is equivalent to problem (4), (5).

Using for the functions v, F_0, g_i the previous notation, in terms of the new variables

$$x = \frac{1}{2}(y_1 + x_1), \quad y = \frac{1}{2}(y_1 - x_1), \tag{6}$$

problem (4), (5) takes the form

$$\frac{\partial^2 v}{\partial x \partial y} + (\xi_1^2 - \xi_2^2)v = F_0, \tag{7}$$

$$v\,|_{\gamma_j} = g_j, \quad j = 1, 2. \tag{8}$$

Here the solution $v = v(x, y, \xi_1, \xi_2)$ of equation (7) is considered in the domain $\Omega_0 : 0 < x < +\infty$, $0 < y < +\infty$ of the plane of variables x, y, which is the image of the domain D_0 for the linear transform (6), and $\gamma_1 : y = 0$, $0 \le x < +\infty$, and $\gamma_2 : x = 0$, $0 \le y < +\infty$, are the images l_1 and l_2 for the same transform (6).

With our assumptions for the functions F_0, g_1, g_2, problem (7), (8) has a unique solution v of the class $C^2(\overline{\Omega}_0)$ which has the form [4]

$$\begin{aligned} v(x, y, \xi_1, \xi_2) &= R(x, 0; x, y)g_1(x, \xi_1, \xi_2) + R(0, y; x, y)g_2(y, \xi_1, \xi_2) - \\ &-R(0, 0; x, y)g_1(0, \xi_1, \xi_2) - \int_0^x \frac{\partial R(\xi, 0; x, y)}{\partial \xi} g_1(\xi, \xi_1, \xi_2)d\xi - \\ &-\int_0^y \frac{\partial R(0, \eta; x, y)}{\partial \eta} g_2(\eta, \xi_1, \xi_2)d\eta + \int_0^x d\xi \int_0^y R(\xi, \eta; x, y)F_0(\xi, \eta, \xi_1, \xi_2)d\eta, \end{aligned} \tag{9}$$

where $g_1(x, \xi_1, \xi_2) = v(x, 0, \xi_1, \xi_2)$, $g_2(y, \xi_1, \xi_2) = v(0, y, \xi_1, \xi_2)$ are the Goursat data for v, and $R(\xi, \eta; x, y)$ is the Riemann function for equation (7). For simplicity, we omit the dependence of R on the parameters ξ_1 and ξ_2.

It is well known that the Riemann function $R(\xi, \eta; x, y)$ for equation (7) can be expressed in terms of the Bessel function $\mathcal{J}_0$ of zero order as follows:

$$\begin{aligned} R(\xi, \eta; x, y) &= \mathcal{J}_0\left(\sqrt{4(\xi_1^2 - \xi_2^2)(x - \xi)(y - \eta)}\right) = \\ &= \sum_{k=0}^{\infty} (\xi_2^2 - \xi_1^2)^k \frac{(x - \xi)^k (y - \eta)^k}{(k!)^2}. \end{aligned} \tag{10}$$

The representation [6] $\mathcal{J}_0(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(iz\sin\theta)d\theta$ readily implies

$$|R(\xi,\eta;x,y)| \leq \begin{cases} 1 \quad \text{for} \quad |\xi_1| \geq |\xi_2|, \\ \exp\sqrt{4|\xi_1^2-\xi_2^2|(x-\xi)(y-\eta)} \quad \text{for} \quad |\xi_1| < |\xi_2|. \end{cases} \tag{11}$$

Using the equalities

$$\mathcal{J}_0'(z) = -\frac{z}{2\pi}\int_{-\pi}^{\pi} \cos^2\theta \exp(iz\sin\theta)d\theta,$$

$$\frac{d\mathcal{J}_0(2\lambda\sqrt{\nu x})}{dx} = -\frac{\lambda^2\nu}{\pi}\int_{-\pi}^{\pi} \cos^2\theta\exp(i2\lambda\sqrt{\nu x}\sin\theta)d\theta, \quad \nu > 0,$$

we obtain

$$\Big|\frac{d\mathcal{J}_0(2\lambda\sqrt{\nu x})}{dx}\Big| \leq \begin{cases} 2\lambda^2\nu \quad \text{for} \quad \lambda = 0, \\ 2|\lambda|^2\nu\exp 2|\lambda|\sqrt{\nu x} \quad \text{for} \quad \operatorname{Re}\lambda = 0. \end{cases}$$

Hence for derivatives of R we easily derive the estimates

$$\Big|\frac{\partial R(\xi,0;x,y)}{\partial \xi}\Big| \leq \begin{cases} 2(\xi_1^2-\xi_2^2)\sqrt{y} \quad \text{for} \quad |\xi_1| \geq |\xi_2|, \\ 2(\xi_1^2-\xi_2^2)\sqrt{y}\exp\sqrt{4(\xi_1^2-\xi_2^2)(x-\xi)y} \\ \qquad\qquad \text{for} \quad |\xi_1| < |\xi_2|, \end{cases} \tag{12}$$

$$\Big|\frac{\partial R(0,\eta;x,y)}{\partial \eta}\Big| \leq \begin{cases} 2(\xi_1^2-\xi_2^2)\sqrt{x} \quad \text{for} \quad |\xi_1| \geq |\xi_2|, \\ 2(\xi_1^2-\xi_2^2)\sqrt{x}\exp\sqrt{4(\xi_1^2-\xi_2^2)x(y-\eta)} \\ \qquad\qquad \text{for} \quad |\xi_1| < |\xi_2|. \end{cases} \tag{13}$$

Next, we assume without loss of generality that for the functions F_0, g_1, g_2 estimates (3) hold with respect to ξ_1, ξ_2, with the same integers C_N, C_N^0, K_N, and K_N^0. Then, using (11)–(13), for the solution $v(x,y,\xi_1,\xi_2)$ of problem (7), (8) written in form (9) we obtain for $|\xi_1|+|\xi_2| > K_N^0$ the following estimates:

$$\begin{aligned} |v(x,y,\xi_1,\xi_2)| \leq |g_1(x,\xi_1,\xi_2)| + |g_2(y,\xi_1,\xi_2)| + |g_1(0,\xi_1,\xi_2) \times \\ \times \exp\sqrt{4|\xi_1^2-\xi_2^2|xy} + 2|\xi_2^2-\xi_1^2|\sqrt{y}\exp\sqrt{4|\xi_2^2-\xi_1^2|xy} \times \\ \times \int_0^x |g_1(\xi,\xi_1,\xi_2)|d\xi + 2|\xi_2^2-\xi_1^2|\sqrt{x}\exp\sqrt{4|\xi_2^2-\xi_1^2|xy} \times \\ \times \int_0^y |g_2(\eta,\xi_1,\xi_2)|d\eta + \exp\sqrt{4|\xi_2^2-\xi_1^2|xy}\int_0^x d\xi \int_0^y |F_0(\xi,\eta,\xi_1,\xi_2)|d\eta \leq \end{aligned}$$

$$\leq 2C_N^0 \exp(-N(|\xi_1|+|\xi_2|) + C_N^0 \exp\sqrt{4|\xi_1^2-\xi_2^2|xy}\times$$
$$\times \exp[-N(|\xi_1|+|\xi_2|)] + 2|\xi_2^2-\xi_1^2|\sqrt{y}x \exp\sqrt{4|\xi_2^2-\xi_1^2|xy}C_N^0\times$$
$$\times \exp[-N(|\xi_1|+|\xi_2|)] + 2|\xi_2^2-\xi_1^2|\sqrt{x}y \exp\sqrt{4|\xi_2^2-\xi_1^2|xy}C_N^0\times$$
$$\times \exp[-N(|\xi_1|+|\xi_2|)] + xy \exp\sqrt{4|\xi_2^2-\xi_1^2|xy}C_N^0\times$$
$$\times \exp[-N(|\xi_1|+|\xi_2|)] \leq \big[3+2(\sqrt{x}y+\sqrt{y}x)|\xi_2^2-\xi_1^2|+xy\big]\times$$
$$\times C_N^0 \exp 2\sqrt{xy}(|\xi_1|+|\xi_2|)\exp[-N(|\xi_1|+|\xi_2|)]. \quad (14)$$

Now choose a least natural number $N_0 = N_0(x,y)$ such that

$$N_0 > N + 2\sqrt{xy} + 1. \quad (15)$$

Taking into account that

$$A_N(x,y) = \sup_{(\xi_1,\xi_2)\in R^2} C_{N_0}^0\big[3+2(\sqrt{x}y+\sqrt{y}x)|\xi_2^2-\xi_1^2|+xy\big]\times$$
$$\times \exp\big[-(|\xi_1|+|\xi_2|)\big] < +\infty,$$

from (14) and (15), for $|\xi_1|+|\xi_2| > K_{N_0}^0$, we obtain

$$|v(x,y,\xi_1,\xi_2)| \leq \big[3+2(\sqrt{x}y+\sqrt{y}x)|\xi_2^2-\xi_1^2|+xy\big]C_{N_0}^0 \exp 2\sqrt{xy}\times$$
$$\times(|\xi_1|+|\xi_2|)\exp[-N(|\xi_1|+|\xi_2|)] \leq C_{N_0}^0\big[3+2(\sqrt{x}y+\sqrt{y}x)|\xi_2^2-\xi_1^2|+$$
$$+xy\big]\exp 2\sqrt{xy}(|\xi_1|+|\xi_2|)\exp[-(|\xi_1|+|\xi_2|)]\exp\big[-2\sqrt{xy}(|\xi_1|+|\xi_2|)\big]\times$$
$$\times \exp\big[-(N_0-N-2\sqrt{xy}-1)(|\xi_1|+|\xi_2|)\big]\times$$
$$\times \exp[-N(|\xi_1|+|\xi_2|)] \leq A_N(x,y)\exp[-N(|\xi_1|+|\xi_2|)]. \quad (16)$$

The latter inequality implies estimate (3) for $i_1 = i_2 = 0$. The proof of this estimate for $i_1 + i_2 > 0$ is similar. Thus we have

Theorem 1. *For any $F \in \Phi^{k-1}(\overline{D})$, $f_i \in \Phi^k(S_i)$, $i = 1,2$, problem (1), (2) is uniquely solvable in the class $\Phi^k(\overline{D})$, $k \geq 2$.*

We denote by $D_1 : -k_2y_1 < x_1 < k_1y$, $0 < y_1 < +\infty$, $0 < k_i < 1$, $i = 1,2$, a dihedral angle bounded by the surfaces $S_1^0 : x_1 - k_1y_1 = 0$, $0 \leq y_1 < +\infty$ and $S_2^0 : x_1 + k_2y_1 = 0$, $0 \leq y_1 < +\infty$.

Let us consider a multidimensional variant of the second Darboux problem fomulated as follows: in the domain D_1 find a solution $u(x_1,y_1,x_2,y_2)$ of equation (1) by the boundary conditions

$$(M_1u_{x_1} + N_1u_{y_1} + \widetilde{M}_1u_{x_2} + \widetilde{N}_1u_{y_2} + \widetilde{S}_1u)\big|_{S_1^0} = f_1, \quad (17)$$

$$(M_2u_{x_1} + N_2u_{y_1} + \widetilde{M}_2u_{x_2} + \widetilde{N}_2u_{y_2} + \widetilde{S}_2u)\big|_{S_2^0} = f_2, \quad (18)$$

where M_i, N_i, $\widetilde{M}_i$, $\widetilde{N}_i$, $\widetilde{S}_i$, $i = 1, 2$, are given real functions on S_i^0; the coefficients M_i, N_i, $\widetilde{M}_i$, $\widetilde{N}_i$, $\widetilde{S}_i$, $i = 1, 2$, depend only on the variables x_1 and y_1.

Note that some multidimensional variants of the second Darboux problem for a wave equation are investigated in [7, 8].

Denote by $\overset{\circ}{\Phi}{}^k_\alpha(\overline{D}_1)$, $k \geq 2$, $\alpha \geq 0$, the space of functions $u(x_1, x_2, y_1, y_2)$ of the class $C^k(\overline{D}_1)$ for which

$$\partial^{i_1+i_2}/\partial x_1^{i_1}\partial y_1^{i_2} u(0, x_2, 0, y_2) = 0,$$
$$-\infty < x_2 < +\infty, \ \ -\infty < y_2 < +\infty \ \ 0 \leq i_1 + i_2 \leq k,$$

and whose partial Fourier transforms $\widehat{u}(x_1, \xi_1, y_1, \xi_2)$ with respect to the variables x_2 and y_2 are continuous functions in

$$\overline{G}_1 = \big\{(x_1, \xi_1, y_1, \xi_2) \in R^4 : -k_2 y_1 \leq x_1 \leq k_1 y_1,$$
$$0 \leq y_1 < +\infty; \ -\infty < \xi_i < +\infty, \ i = 1, 2\big\}$$

together with partial derivatives with respect to the variables x_1 and y_1 up to kth order inclusive and satisfy the following estimates: for any natural N there exist positive integers $C_N = C_N(x_1, y_1)$ and $K_N = K_N(x_1, y_1)$ not depending on ξ_1, ξ_2 such that for $-k_2 y_1 \leq x_1 \leq k_1 y_1$, $0 \leq y_1 < +\infty$, and $|\xi_1| + |\xi_2| > K_N$ we have the inequalities

$$\big|\partial^{i_1+i_2}/\partial x_1^{i_1}\partial y_1^{i_2} \widehat{u}(x_1, \xi_1, y_1, \xi_2)\big| \leq C_N y_1^{k+\alpha-i_1-i_2} e^{-N(|\xi_1|+|\xi_2|)}, \quad (19)$$
$$0 \leq i_1 + i_2 \leq k,$$

with

$$C_N^0(x_1, y_1) = \sup_{(x_1^0, y_1^0) \in I_1(x_1, y_1)} C_N(x_1^0, y_1^0) < +\infty,$$
$$K_N^0(x_1, y_1) = \sup_{(x_1^0, y_1^0) \in I_1(x_1, y_1)} K_N(x_1^0, y_1^0) < +\infty,$$

where $I_1(x_1, y_1) = \big\{(x_1^0, y_1^0) \in R^2 : -k_2 y_1^0 \leq x_1^0 \leq k_1 y_1^0, y_1^0 - x_1^0 \leq y_1 - x_1, y_1^0 + x_1^0 \leq y_1 + x_1, -k_2 y_1 \leq x_1 \leq k_1 y_1\big\}$ is the closed rectangle.

The spaces $\overset{\circ}{\Phi}{}^k_\alpha(S_i^0)$, $i = 1, 2$, are derived in a similar manner. Note that the trace $u|_{S_i^0}$ of the function u from the space $\overset{\circ}{\Phi}{}^k_\alpha(\overline{D}_1)$ belongs to the space $\overset{\circ}{\Phi}{}^k_\alpha(S_i^0)$.

Remark. When considering problem (1), (17), (18) in the class $\overset{\circ}{\Phi}{}^k_\alpha(\overline{D}_1)$, we require of the functions F, f_i, $i = 1, 2$, and the coefficients M_i, N_i, $\widetilde{M}_i$, $\widetilde{N}_i$, $\widetilde{S}_i$, $i = 1, 2$, in the boundary conditions (17), (18) that

$$F \in \overset{\circ}{\Phi}{}^{k-1}_\alpha(\overline{D}_1), \ f_i \in \overset{\circ}{\Phi}{}^{k-1}_\alpha(S_i^0), \ i = 1, 2;$$

$$M_i, N_i, \widetilde{M}_i, \widetilde{N}_i, \widetilde{S}_i \in C^{k-1}(S_i^0),\ i = 1, 2.$$

Using the notation $u_{y_1} = v_1$, $u_{x_1} = v_2$, $u_{y_2} = v_3$, $u_{x_2} = v_4$, we reduce equation (1) to the first-order system

$$u_{y_1} = v_1, \tag{20}$$

$$v_{1y_1} + v_{3y_2} - v_{2x_1} - v_{4x_2} = F, \tag{21}$$

$$v_{2y_1} - v_{1x_1} = 0, \tag{22}$$

$$v_{3y_1} - v_{1y_2} = 0, \tag{23}$$

$$v_{4y_1} - v_{1x_2} = 0, \tag{24}$$

and write the boundary conditions (17) and (18) in the form

$$\left(M_1 v_2 + N_1 v_1 + \widetilde{M}_1 v_4 + \widetilde{N}_1 v_3 + \widetilde{S}_1 u\right)\Big|_{S_1} = f_1, \tag{25}$$

$$\left(M_2 v_2 + N_2 v_1 + \widetilde{M}_2 v_4 + \widetilde{N}_2 v_3 + \widetilde{S}_2 u\right)\Big|_{S_2} = f_2. \tag{26}$$

Along with conditions (25), (26) we shall also consider the boundary conditions

$$(u_{x_1} - v_2)\Big|_{S_1^0 \cup S_2^0} = 0, \tag{27}$$

$$(u_{x_2} - v_4)\Big|_{S_1^0 \cup S_2^0} = 0, \tag{28}$$

$$(u_{y_2} - v_3)\Big|_{S_1^0 \cup S_2^0} = 0. \tag{29}$$

Clearly, if u is a regular solution of problem (1), (17), (18) of the class $\mathring{\Phi}_\alpha^k(\overline{D}_1)$, then the system of functions u, v_1, v_2, v_3, v_4 will be a regular solution of the boundary value problem (20)–(29), where $v_i \in \mathring{\Phi}_\alpha^{k-1}(\overline{D}_1)$, $i = 1, \dots, 4$. Conversely, let the system of functions u, v_1, v_2, v_3, v_4 of the class $\mathring{\Phi}_\alpha^{k-1}(\overline{D}_1)$ be a solution of problem (20)–(29). We shall prove that then $v_1 = u_y$, $v_2 = u_{x_1}$, $v_3 = u_{y_2}$, $v_4 = u_{x_2}$ and therefore the function u is a solution of problem (1), (17), (18) in the class $\mathring{\Phi}_\alpha^k(\overline{D}_1)$. Indeed, by equality (22) we have

$$(u_{x_1} - v_2)_{y_1} = (u_{y_1})_{x_1} - v_{2y_1} = v_{1x_1} - v_{2y_1} = 0,$$

which by virtue of the boundary condition (27) implies that $v_2 \equiv u_{x_1}$ in $\overline{D}_1$.

Further, the use of equality (23) gives

$$(u_{y_2} - v_3)_{y_1} = (u_{y_1})_{y_2} - v_{3y_1} = v_{1y_2} - v_{3y_2} = 0,$$

which by virtue of the boundary condition (29) implies that $v_3 \equiv u_{y_2}$ in $\overline{D}_1$.

Finally, on account of (24) and (28) we find by a similar reasoning that $v_4 \equiv u_{x_2}$ in $\overline{D}_1$.

Thus problem (1), (17), (18) in the class $\overset{\circ}{\Phi}{}^k_\alpha(\overline{D}_1)$ is equivalent to the problem of finding a system of functions u, v_1, v_2, v_3, v_4 of the class $\overset{\circ}{\Phi}{}^{k-1}_\alpha(\overline{D}_1)$ satisfying (20)–(29).

If u, v_1, v_2, v_3, v_4 are a solution of problem (20)–(29), then, after the Fourier transformation with respect to the variables x_2 and y_2, equations (20)–(24) and boundary conditions (25)–(29) take the form

$$\widehat{u}_{y_1} = \widehat{v}_1, \tag{30}$$

$$\widehat{v}_{1y_1} - \widehat{v}_{2x_1} + i\xi_2\widehat{v}_3 - i\xi_1\widehat{v}_4 = \widehat{F}, \tag{31}$$

$$\widehat{v}_{2y_1} - \widehat{v}_{1x_1} = 0, \tag{32}$$

$$\widehat{v}_{3y_1} - i\xi_2\widehat{v}_1 = 0, \tag{33}$$

$$\widehat{v}_{4y_1} - i\xi_1\widehat{v}_1 = 0, \tag{34}$$

$$\big(M_1\widehat{v}_2 + N_1\widehat{v}_1 + \widetilde{M}_1\widehat{v}_4 + \widetilde{N}_1\widehat{v}_3 + \widetilde{S}_1\widehat{u}\big)\big|_{l_1} = \widehat{f}_1, \tag{35}$$

$$\big(M_2\widehat{v}_2 + N_2\widehat{v}_1 + \widetilde{M}_2\widehat{v}_4 + \widetilde{N}_2\widehat{v}_3 + \widetilde{S}_2\widehat{u}\big)\big|_{l_2} = \widehat{f}_2, \tag{36}$$

$$(\widehat{u}_{x_1} - \widehat{v}_2)\big|_{l_1\cup l_2} = 0, \tag{37}$$

$$(\widehat{v}_4 - i\xi_1\widehat{u})\big|_{l_1\cup l_2} = 0, \tag{38}$$

$$(\widehat{v}_3 - i\xi_2\widehat{u})\big|_{l_1\cup l_2} = 0, \tag{39}$$

where $\widehat{u}$, $\widehat{v}_j$, $j = 1, \dots, 4$, $\widehat{F}$, $\widehat{f}_1$, $\widehat{f}_2$ are respectively the Fourier transforms of the functions u, v_j, $j = 1, \dots, 4$, F, f_1, f_2 with respect to the variables x_2 and y_2; $l_1 : x_1 - k_1y_1 = 0$, $0 \le y_1 < +\infty$, $l_2 : x_1 + k_2y_1 = 0$, $0 \le y_1 < +\infty$, are the beams lying in the plane of the variables x_1, y_1 and outgoing from the origin $O(0,0)$. Here $i = \sqrt{-1}$.

Thus, after the Fourier transformation with respect to the variables x_2 and y_2, the spatial problem (20)–(29) is reduced to the plane problem (30)–(39) with the parameters ξ_1 and ξ_2 lying in the domain $\Omega_1 : -k_2y_1 < x_1 < k_1y_1$, $0 < y_1 < +\infty$, of the plane of variables x_1, y_1. It is easy to verify that in the class $\overset{\circ}{\Phi}{}^k_\alpha(\overline{\Omega}_1)$ of functions defined by inequalities (19) this reduction is equivalent.

Assuming $v = (\widehat{v}_1, \widehat{v}_2)$ and $v_5 = (\widehat{v}_3, \widehat{v}_4)$, we rewrite system (31), (32) in the vector form

$$v_{y_1} - Av_{x_1} + A_1v_5 = F_1, \tag{40}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} i\xi_2 & -i\xi_1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} \widehat{F} \\ 0 \end{pmatrix}.$$

We easily see that with respect to v system (40) is strictly hyperbolic and its characteristics $L_1(x_1^0, y_1^0)$ and $L_2(x_1^0, y_1^0)$ passing through the point

(x_1^0, y_1^0) are the straight lines defined by the equations

$$L_1(x_1^0, y_1^0) : x_1 - y_1 = x_1^0 - y_1^0, \quad L_2(x_1^0, y_1^0) : x_1 + y_1 = x_1^0 + y_1^0.$$

If

$$B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{41}$$

then, as is easy to verify,

$$B^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad B^{-1}AB = \Lambda. \tag{42}$$

Therefore by replacing $v = Bw$ we can rewrite system (40) as

$$w_{y_1} - \Lambda w_{x_1} + B^{-1}A_1 v_5 = B^{-1}F_1. \tag{43}$$

By virtue of (41)–(43), after replacing $v = Bw$, problem (30)–(39) is rewritten as

$$\widehat{u}_{y_1} = w_1 - w_2, \tag{44}$$

$$w_{1y_1} - w_{1x_1} + \frac{1}{2}i\xi_2\widehat{v}_3 - \frac{1}{2}\xi_1\widehat{v}_4 = F_2, \tag{45}$$

$$w_{1y_1} + w_{1x_1} - \frac{1}{2}i\xi_2\widehat{v}_3 + \frac{1}{2}\xi_1\widehat{v}_4 = F_3, \tag{46}$$

$$\widehat{v}_{3y_1} - i\xi_2(w_1 - w_2) = 0, \tag{47}$$

$$\widehat{v}_{4y_1} - i\xi_1(w_1 - w_2) = 0, \tag{48}$$

$$[(M_1 + N_1)w_1 + (M_1 - N_1)w_2 + \widetilde{M}_1\widehat{v}_4 + \widetilde{N}_1\widehat{v}_3 + \widetilde{S}_1\widehat{u}]\big|_{l_1} = \widehat{f}_1, \tag{49}$$

$$[(M_2 + N_2)w_1 + (M_2 - N_2)w_2 + \widetilde{M}_2\widehat{v}_4 + \widetilde{N}_1\widehat{v}_3 + \widetilde{S}_2\widehat{u}]\big|_{l_2} = \widehat{f}_2, \tag{50}$$

$$(\widehat{u}_{x_1} - (w_1 + w_2))\big|_{l_1\cup l_2} = 0, \tag{51}$$

$$(\widehat{v}_4 - i\xi_1\widehat{u})\big|_{l_1\cup l_2} = 0, \tag{52}$$

$$(\widehat{v}_3 - i\xi_2\widehat{u})\big|_{l_1\cup l_2} = 0, \tag{53}$$

where $(F_2, F_3) = B^{-1}F_1$.

It is easy to see that the characteristic $L_1(x_1^0, y_1^0) : x_1 + y_1 = x_1^0 + y_1^0$ of equation (45) passing through the point $(x_1^0, y_1^0) \in \overline{\Omega}_1$ intersects the beam l_1 at a single point whose ordinate is denoted by $\omega_1(x_1^0, y_1^0)$. In a similar manner, the characteristic $L_2(x_1^0, y_1^0) : x_1 - y_1 = x_1^0 - y_1^0$, $(x_1^0, y_1^0) \in \overline{\Omega}_1$, of equation (46) intersects l_2 at a single point with ordinate $\omega_2(x_1^0, y_1^0)$. Clearly,

$$\omega_i(x_1^0, y_1^0) \in C^\infty(\overline{\Omega}_1), \quad \omega_i(x_1^0, y_1^0) \le y_1^0, \quad i = 1, 2. \tag{54}$$

It should also be noted that a segment of each characteristic $L_i(x_1^0, y_1^0)$, $i = 1, 2$, $(x_1^0, y_1^0) \in \overline{\Omega}_1$, drawn from the point (x_1^0, y_1^0) towards decreasing ordinate values to the intersection with one of the beams l_1 or l_2, admits a parametrization of the form

$$x_1 = z_i(x_1^0, y_1^0; t), \quad y_1 = t, \quad \omega_i(x_1^0, y_1^0) \le t \le y_1^0, \tag{55}$$

where $z_1(x_1^0, y_1^0; t) = x_1^0 + y_1^0 - t \in C^\infty[\omega_1(x_1^0, y_1^0), y_1^0]$, $z_2(x_1^0, y_1^0; t) = x_1^0 - y_1^0 + t \in C^\infty[\omega_2(x_1^0, y_1^0), y_1^0]$.

Remark. The functions $\widehat{u}$, w_1, w_2, $\widehat{v}_3$, $\widehat{v}_4$, F_2, F_3, $\widehat{f}_1$, $\widehat{f}_2$ depend not only on the independent variables x_1 and x_2, but also on the parameters ξ_1 and ξ_2. For simplicity, these parameters will be omitted below. For example, instead of $\widehat{u}(x_1, y_1, \xi_1, \xi_2)$ we shall write $\widehat{u}(x_1, y_1)$.

Using (44), (51) and the fact that $\widehat{u}(0,0) = 0$, for $u \in \overset{\circ}{\Phi}{}^k_\alpha(\overline{D}_1)$ we obtain

$$\begin{aligned} \widehat{u}(x_1, y_1) &= \int_0^{y_1} (k_1 u_{x_1} + u_{y_1})(k_1 t, t)dt = \int_0^{y_1} [(1+k_1)w_1 + \\ &\quad + (1-k_1)w_2](k_1 t, t)dt, \quad (x_1, y_1) \in l_1, \end{aligned} \tag{56}$$

$$\begin{aligned} \widehat{u}(x_1, y_1) &= \int_0^{y_1} (-k_2 u_{x_1} + u_{y_1})(-k_2 t, t)dt = \int_0^{y_1} [(1-k_2)w_1 + \\ &\quad + (1+k_2)w_2](-k_2 t, t)dt, \quad (x_1, y_1) \in l_2. \end{aligned} \tag{57}$$

For $(x_1, y_1) \in \overline{\Omega}_1$ and $x_1 > 0$, by integrating equations (44), (47), (48) with respect to the variable y_1 and taking into account the boundary conditions (52), (53) and equalities (56), (57) we obtain

$$\begin{aligned} \widehat{u}(x_1, y_1) = &\int_0^{k_1^{-1}x_1} [(1+k_1)w_1 + (1-k_1)w_2](k_1 t, t)dt + \\ &+ \int_{k_1^{-1}x_1}^{y_1} (w_1 - w_2)(x_1, t)dt, \end{aligned} \tag{58}$$

$$\begin{aligned} \widehat{v}_3(x_1, y_1) = i\xi_2 &\int_0^{k_1^{-1}x_1} [(1+k_1)w_1 + (1-k_1)w_2](k_1 t, t)dt + \\ &+ i\xi_2 \int_{k_1^{-1}x_1}^{y_1} (w_1 - w_2)(x_1, t)dt, \end{aligned} \tag{59}$$

$$\widehat{v}_4(x_1,y_1)=i\xi_1\int\limits_0^{k_1^{-1}x_1}[(1+k_1)w_1+(1-k_1)w_2](k_1t,t)dt+$$

$$+\,i\xi_1\int\limits_{k_1^{-1}x_1}^{y_1}(w_1-w_2)(x_1,t)dt, \tag{60}$$

and for $(x_1,y_1)\in\overline{\Omega}_1$ and $x_1\le 0$ we have

$$\widehat{u}(x_1,y_1)=\int\limits_0^{-k_1^{-1}x_1}[(1-k_2)w_1+(1+k_2)w_2](-k_2t,t)dt+$$

$$+\int\limits_{-k_2^{-1}x_1}^{y_1}(w_1-w_2)(x_1,t)dt, \tag{61}$$

$$\widehat{v}_3(x_1,y_1)=i\xi_2\int\limits_0^{-k_2^{-1}x_1}[(1-k_2)w_1+(1+k_2)w_2](-k_2t,t)dt+$$

$$+\,i\xi_2\int\limits_{-k_2^{-1}x_1}^{y_1}(w_1-w_2)(x_1,t)dt, \tag{62}$$

$$\widehat{v}_4(x_1,y_1)=i\xi_1\int\limits_0^{-k_2^{-1}x_1}[(1-k_2)w_1+(1+k_2)w_2](-k_2t,t)dt+$$

$$+\,i\xi_1\int\limits_{-k_2^{-1}x_1}^{y_1}(w_1-w_2)(x_1,t)dt. \tag{63}$$

Now, by integrating equations (45) and (46) along the respective characteristics from the point $(x_1,y_1)\in\overline{\Omega}_1$ to the intersection points of these characteristics with the beams l_1 and l_2 we obtain

$$w_1(x_1,y_1)=w_1(k_1\omega_1(x_1,y_1),\omega_1(x_1,y_1))+$$

$$+\frac{1}{2}i\int\limits_{\omega_1(x_1,y_1)}^{y_1}(\xi_1\widehat{v}_4-\xi_2\widehat{v}_3)(z_1(x_1,y_1;t),t)dt+\widetilde{F}_2(x_1,y_1), \tag{64}$$

$$w_2(x_1,y_1)=w_2(-k_2\omega_2(x_1,y_1),\omega_2(x_1,y_1))+$$

$$+\frac{1}{2}i\int\limits_{\omega_2(x_1,y_1)}^{y_1}(\xi_2\widehat{v}_3-\xi_1\widehat{v}_4)(z_2(x_1,y_1;t),t)dt+\widetilde{F}_3(x_1,y_1), \tag{65}$$

where

$$\widetilde{F}_2(x_1,y_1)=\int\limits_{\omega_1(x_1,y_1)}^{y_1}F_2(z_1(x_1,y_1;t),t)dt,$$

$$\widetilde{F}_3(x_1,y_1)=\int\limits_{\omega_2(x_1,y_1)}^{y_1}F_3(z_2(x_1,y_1;t),t)dt.$$

We set

$$\varphi=w_1|_{l_1},\quad \psi=w_2|_{l_2}.$$

On substituting the obtained expressions for $\widehat{u}$, w_1, w_2, $\widehat{v}_3$, $\widehat{v}_4$ from (58)–(65) into the boundary conditions (49), (50) and taking into account that

$$\omega_1(x_1,y_1)|_{l_1}=y_1,\quad \omega_1(x_1,y_1)=\frac{1-k_2}{1+k_1}y_1,$$

$$\omega_2(x_1,y_1)=\frac{1-k_1}{1+k_2}y_1,\quad \omega_2(x_1,y_1)|_{l_2}=y_1$$

we obtain

$$(M_1+N_1)\varphi(y_1)+(M_1-N_1)\psi(\tau_2y_1)+(M_1-N_1)\frac{1}{2}i\times$$
$$\times\int\limits_{\tau_2y_1}^{y_1}(\xi_2\widehat{v}_3-\xi_1\widehat{v}_4)(z_2(k_1y_1,y_1;t),t)dt+(\widetilde{M}_1i\xi_1+\widetilde{N}_1i\xi_2+\widetilde{S}_1)\times$$
$$\times\int\limits_0^{y_1}[(1+k_1)w_1+(1-k_1)w_2](k_1t,t)dt=\widehat{f}_1(y_1)-(M_1+N_1)(y_1)\times$$
$$\times\widetilde{F}_2(k_1y_1,y_1)-(M_1-N_1)(y_1)\widetilde{F}_3(k_1y_1,y_1), \tag{66}$$

$$(M_2+N_2)\varphi(\tau_1y_1)+(M_2-N_2)\psi(y_1)+(M_2+N_2)\frac{1}{2}i\times$$
$$\times\int\limits_{\tau_1y_1}^{y_1}(\xi_1\widehat{v}_4-\xi_2\widehat{v}_3)(z_1(-k_2y_1,y_1;t),t)dt+(\widetilde{M}_2i\xi_1+\widetilde{N}_2i\xi_2+\widetilde{S}_2)\times$$
$$\times\int\limits_0^{y_1}[(1-k_2)w_1+(1+k_2)w_2](-k_2t,t)dt=\widehat{f}_2(y_1)-$$
$$-(M_2+N_2)(y_1)\widetilde{F}_2(-k_2y_1,y_1)-(M_2-N_2)(y_1)\widetilde{F}_3(-k_2y_1,y_1). \tag{67}$$

Now, since $0 < k_i < 1$, $i = 1, 2$, we have

$$0 < \tau_1 = \frac{1-k_2}{1+k_1} < 1, \quad 0 < \tau_2 = \frac{1-k_1}{1+k_2} < 1, \quad 0 < \tau = \tau_1\tau_2 < 1. \quad (68)$$

It will assumed below that

$$(M_1 + N_1)\big|_{l_1} \neq 0, \quad (M_2 - N_2)\big|_{l_2} \neq 0.$$

Since by assumptions the functions M_j, N_j, $j = 1, 2$, do not depend on the variables x_2 and y_2, these conditions are evidently equivalent to the conditions

$$(M_1 + N_1)\big|_{S_1^0} \neq 0, \quad (69)$$

$$(M_2 - N_2)\big|_{S_2^0} \neq 0. \quad (70)$$

Solving system (66), (67) with respect to φ and ψ, we obtain

$$\varphi(y_1) - a(y_1)\varphi(\tau y_1) = (T_1(w))(y_1) + (T_2(v_5))(y_1) + g_1(y_1), \quad (71)$$

$$\psi(y_1) - b(y_1)\psi(\tau y_1) = (T_3(w))(y_1) + (T_5(v_5))(y_1) + g_2(y_1), \quad (72)$$

where

$$a(y_1) = \big[(M_1 - N_1)(M_1 + N_1)^{-1}\big](y_1)\big[(M_2 + N_2)(M_2 - N_2)^{-1}\big](\tau_2 y_1),$$
$$b(y_1) = \big[(M_2 + N_2)(M_2 - N_2)^{-1}\big](y_1)\big[(M_1 - N_1)(M_1 + N_1)^{-1}\big](\tau_1 y_1),$$

and T_j, $j = 1, \dots, 4$, are linear integral operators of the form

$$(T_1(w))(y_1) = (E_{11}\xi_1 + E_{12}\xi_2 + E_{13}) \int_0^{y_1} w(k_1 t, t)dt +$$
$$+ (E_{14}\xi_1 + E_{15}\xi_2 + E_{16}) \int_0^{\tau_2 y_1} w(-k_2 t, t)dt,$$

$$(T_2(v_5))(y_1) = (E_{21}\xi_1 + E_{22}\xi_2) \int_{\tau_2 y_1}^{y_1} v_5(z_2(k_1 y_1, y_1; t), t)dt +$$
$$+ (E_{23}\xi_1 + E_{24}\xi_2) \int_{\tau y_1}^{\tau_2 y_1} v_5(z_1(-k_2\tau_2 y_1, \tau_2 y_1; t), t)dt,$$

$$(T_3(w))(y_1) = (E_{31}\xi_1 + E_{32}\xi_2 + E_{33}) \int_0^{y_1} w(-k_2 t, t)dt +$$

$$+(E_{34}\xi_1+E_{35}\xi_2+E_{36})\int\limits_0^{\tau_1 y_1} w(k_1t,t)dt,$$

$$(T_4(v_5))(y_1)=(E_{41}\xi_1+E_{42}\xi_2)\int\limits_{\tau_1 y_1}^{y_1} v_5(z_1(-k_2y_1,y_1;t),t)dt+$$

$$+(E_{43}\xi_1+E_{44}\xi_2)\int\limits_{\tau y_1}^{\tau_1 y_1} v_5(z_2(k_1\tau_1y_1,\tau_1y_1;t),t)dt.$$

Here $v_5=(\widehat{v}_3,\widehat{v}_4)$, E_{ij} are the completely defined 1×2 matrices of the class C^{k-1}, and g_1, g_2 are functions expressed in terms of the well-known functions F and f_1, f_2.

As mentioned above, in equations (58)–(65), (71), (72) the unknown functions and the right-hand sides depend on the parameter $\xi=(\xi_1,\xi_2)$, which we omit for simplicity.

Remark. As follows from the above reasoning, if conditions (69), (70) are fulfilled, then in the class $\overset{\circ}{\Phi}{}^k_\alpha(\overline{D}_1)$ problem (1), (17), (18) is equivalent to the problem of finding a system of functions $\widehat{u}$, w_1, w_2, $\widehat{v}_3$, $\widehat{v}_4$, φ, and ψ from equations (58)–(65), (71), (72), where

$$\widehat{u},\ w_1,\ w_2,\ \widehat{v}_3,\ \widehat{v}_4\in\overset{\circ}{\Phi}{}^{k-1}_\alpha(\overline{\Omega}_1),\ \ \varphi\in\overset{\circ}{\Phi}{}^{k-1}_\alpha[0,+\infty),\ \ \psi\in\overset{\circ}{\Phi}{}^{k-1}_\alpha[0,+\infty)$$

and

$$\widetilde{F}_2,\ \widetilde{F}_3\in\overset{\circ}{\Phi}{}^{k-1}_\alpha(\overline{\Omega}_1),\ \ g_1\in\overset{\circ}{\Phi}{}^{k-1}_\alpha[0,+\infty),\ \ g_2\in\overset{\circ}{\Phi}{}^{k-1}_\alpha[0,+\infty).$$

Let $Q(x_1^0,y_1^0)\in\Omega_1$. Denote by P_1 and P_2 the points of intersection of the characteristics $L_1(x_1^0,y_1^0):x_1-y_1=x_1^0-y_1^0$ and $L_2(x_1^0,y_1^0):x_1+y_1=x_1^0+y_1^0$ from system (40) with the beams l_1 and l_2, respectively. Denote by $\Omega_{1Q}\subset\Omega_1$ a rectangle with vertices at the points $O(0,0)$, P_1, P_2, and Q.

We set

$$\sigma=a(0)=b(0)=\Big[\frac{(M_1-N_1)(M_2+N_2)}{(M_2-N_2)(M_1+N_1)}\Big](0),$$

where $a(y_1)$ and $b(y_1)$ are the coefficients in equations (71), (72).

Lemma 1. *Let conditions* (69), (70) *be fulfilled. Then for* $k+\alpha>\log|\sigma|/\log\tau+1$ *the boundary value problem* (44)–(53) *is uniquely solvable in the class* $\overset{\circ}{\Phi}{}^{k-1}_\alpha(\overline{\Omega}_1)$*; the domain of the dependence of the solution of this problem for the point* $Q(x_1^0,y_1^0)\in\Omega_1$ *is* $\overline{\Omega}_{1Q}$.

Remark. As mentioned above, the boundary value problem (44)–(53) is equivalent to the system of integro-functional equations (58)–(65), (71), (72). When problem (44)–(53) is considered in the domain Ω_{1Q}, it is sufficient to investigate equations (71) and (72) on the segments $[0, d_1]$ and $[0, d_2]$, where d_1 and d_2 are the ordinates of the points P_1 and P_2.

Let us consider the functional equations

$$(K_{1j}(\varphi))(y_1) = \varphi(y_1) - a_j(y_1)\varphi(\tau y_1) = \chi_1(y_1), \quad 0 \le y_1 \le d_1, \tag{73}$$
$$(K_{2j}(\psi))(y_1) = \psi(y_1) - b_j(y_1)\psi(\tau y_1) = \chi_2(y_2), \quad 0 \le y_1 \le d_2, \tag{74}$$

where $a_j(y_1) = \tau^j a(y_1)$, $b_j(y_1) = \tau^j b(y_1)$, $j = 0, 1, \ldots, k-1$. Note that if we differentiate j-times the expression $(K_{10}(\varphi))(y_1)$, which is the left-hand side of equation (71), with respect to y_1, then in the obtained expression the sum of the terms which contain the function $\varphi(y_1)$ with its derivative $\varphi^{(j)}(y_1)$ gives $(K_{1j}(\varphi^{(j)}))(y_1)$. A similar statement also holds for K_{2j}.

Let in equations (73), (74) the right-hand sides $\chi_p(y_1) \in \overset{\circ}{\Phi}_{k-1+\alpha-j}[0, d_p]$, $p = 1, 2$. It is assumed that $\overset{\circ}{\Phi}{}^k_\alpha[0, d_p] = \overset{\circ}{\Phi}_\alpha[0, d_p]$ for $k = 0$. In that case, by the definition of the space $\overset{\circ}{\Phi}_{k-1+\alpha-j}[0, d_p]$, for any natural number N there exist positive numbers $C_p = C_p(y, N, \chi_p)$, $B_p = B_p(y_1, N, \chi_p)$, not depending on $\xi = (\xi_1, \xi_2)$, such that for $0 \le y_1 \le d_p$ and $|\xi| = |\xi_1| + |\xi_2| > B_p$ the inequality

$$|\chi_p(y_1)| \le C_p y_1^{k-1+\alpha-j} e^{-N|\xi|}$$

is fulfilled and

$$C_p^0(y_1) = \sup_{0 \le y_1^0 \le y_1} C_p(y_1^0) < +\infty, \quad B_p^0(y_1) = \sup_{0 \le y_1^0 \le y_1} B_p(y_1^0) < +\infty.$$

Lemma 2. *Let conditions* (69), (70) *be fulfilled. Then for* $k - 1 + \alpha > -\frac{\log|\sigma|}{\log \tau}$ *equations* (73) *and* (74) *are uniquely solvable in the spaces* $\overset{\circ}{\Phi}_{k-1+\alpha-j}[0, d_1]$ *and* $\overset{\circ}{\Phi}_{k-1+\alpha-j}[0, d_2]$, *and for* $|\xi| > B_1$ *and* $|\xi| > B_2$ *the estimates*

$$\left|(K_{1j}^{-1}(\chi_1))(y_1)\right| = |\varphi(y_1)| \le \sigma_1 C_1 y_1^{k-1+\alpha-j} e^{-N|\xi|}, \tag{75}$$
$$\left|(K_{2j}^{-1}(\chi_2))(y_1)\right| = |\psi(y_1)| \le \sigma_2 C_2 y_1^{k-1+\alpha-j} e^{-N|\xi|} \tag{76}$$

hold, where the positive constants σ_1 *and* σ_2 *do not depend on* N, ξ, *and the functions* χ_1, χ_2.

The proof of this lemma repeats the reasoning from [9].

To prove Lemma 1, we shall solve the system of integro-functional equations (58)–(65), (71), (72) with respect to the unknowns

$$\widehat{u}, w_1, w_2, \widehat{v}_3, \widehat{v}_4 \in \overset{\circ}{\Phi}{}^{k-1}_\alpha(\overline{\Omega}_{1Q}), \quad \varphi \in \overset{\circ}{\Phi}{}^{k-1}_\alpha[0, d_1], \quad \psi \in \overset{\circ}{\Phi}{}^{k-1}_\alpha[0, d_2]$$

by the method of successive approximations.

We set

$$\widehat{u}_0(x_1,y_1)\equiv 0,\quad w_{1,0}(x_1,y_1)\equiv 0,\quad w_{2,0}(x_1,y_1)\equiv 0,$$
$$\widehat{v}_{3,0}(x_1,y_1)\equiv 0,\quad \widehat{v}_{4,0}(x_1,y_1)\equiv 0,\quad \varphi_0(y_1)\equiv 0,\quad \psi_0(y_1)\equiv 0,$$

$$\widehat{u}_n(x_1,y_1)=\int\limits_0^{\beta^{-1}x_1}[(1+\beta)w_{1,n-1}+(1-\beta)w_{2,n-1}](\beta t,t)dt+$$
$$+\int\limits_{\beta^{-1}x_1}^{y_1}(w_{1,n-1}-w_{2,n-1})(x_1,t)dt, \tag{77}$$

$$\widehat{v}_{3,n}(x_1,y_1)=i\xi_2\int\limits_0^{\beta^{-1}x_1}[(1+\beta)w_{1,n-1}+(1-\beta)w_{2,n-1}](\beta t,t)dt+$$
$$+i\xi_2\int\limits_{\beta^{-1}x_1}^{y_1}(w_{1,n-1}-w_{2,n-1})(x_1,t)dt, \tag{78}$$

$$\widehat{v}_{4,n}(x_1,y_1)=i\xi_1\int\limits_0^{\beta^{-1}x_1}[(1+\beta)w_{1,n-1}+(1-\beta)w_{2,n-1}](\beta t,t)dt+$$
$$+i\xi_1\int\limits_{\beta^{-1}x_1}^{y_1}(w_{1,n-1}-w_{2,n-1})(x_1,t)dt, \tag{79}$$

$$\beta=\begin{cases}k_1 & \text{for}\quad x_1>0,\\ -k_2 & \text{for}\quad x_1\le 0,\end{cases}\qquad \beta^{-1}x_1\le y_1\quad\text{for}\quad (x_1,y_1)\in\Omega_{1Q},$$

$$w_{1,n}(x_1,y_1)=\varphi_n(\omega_1(x_1,y_1))+$$
$$+\frac{1}{2}i\int\limits_{\omega_1(x_1,y_1)}^{y_1}(\xi_1\widehat{v}_{4,n-1}-\xi_2\widehat{v}_{3,n-1})(z_1(x_1,y_1;t),t)dt+\widetilde{F}_2(x_1,y_1), \tag{80}$$

$$w_{2,n}(x_1,y_1)=\psi_n(\omega_2(x_1,y_1))+$$
$$+\frac{1}{2}i\int\limits_{\omega_2(x_1,y_1)}^{y_1}(\xi_2\widehat{v}_{3,n-1}-\xi_1\widehat{v}_{4,n-1})(z_2(x_1,y_1;t),t)dt+\widetilde{F}_3(x_1,y_1). \tag{81}$$

To define the functions $\varphi_n(y_1)$ and $\psi_n(y_1)$, we shall use the equations

$$(K_{10}(\varphi_n))(y_1)=(T_1(w_{n-1}))(y_1)+(T_2(v_{5,n-1}))(y_1)+g_1(y_1), \tag{82}$$
$$(K_{20}(\psi_n))(y_1)=(T_3(w_{n-1}))(y_1)+(T_4(v_{5,n-1}))(y_1)+g_2(y_1), \tag{83}$$

where $v_5 = (\widehat{v}_3, \widehat{v}_4)$.

If the conditions of Lemma 2 are fulfilled, then using estimate (75), (76) for $j = 0$ and applying (54), (68), also the inequality $\beta^{-1}x_1 \le y_1$ for $(x_1, y_1) \in \Omega_{1Q}$, $Q(x_1^0, y_1^0) \in \Omega_1$, and taking into account the Volterra structure of the integral operators contained in equalities (77)–(78), for $|\xi| > \widetilde{\beta}$ we obtain by the method of mathematical induction the inequalities:

$$\big|\widehat{u}_{n+1}(x_1, y_1) - \widehat{u}_n(x_1, y_1)\big| \le M^* \frac{M_*^n}{n!}(1+|\xi|)^n e^{-N|\xi|} y_1^{n+k-1+\alpha}, \quad (84)$$

$$\big|w_{p,n+1}(x_1, y_1) - w_{p,n}(x_1, y_1)\big| \le M^* \frac{M_*^n}{n!}(1+|\xi|)^n e^{-N|\xi|} y_1^{n+k-1+\alpha}, \quad (85)$$
$$p = 1, 2,$$

$$\big|\widehat{v}_{q,n+1}(x_1, y_1) - \widehat{v}_{q,n}(x_1, y_1)\big| \le M^* \frac{M_*^n}{n!}(1+|\xi|)^n e^{-N|\xi|} y_1^{n+k-1+\alpha}, \quad (86)$$
$$q = 3, 4,$$

$$\big|\varphi_{n+1}(y_1) - \varphi_n(y_1)\big| \le M^* \frac{M_*^n}{n!}(1+|\xi|)^n e^{-N|\xi|} y_1^{n+k-1+\alpha}, \quad (87)$$

$$\big|\psi_{n+1}(y_1) - \psi_n(y_1)\big| \le M^* \frac{M_*^n}{n!}(1+|\xi|)^n e^{-N|\xi|} y_1^{n+k-1+\alpha}, \quad (88)$$

where the positive numbers $\widetilde{B}=\widetilde{B}(x_1^0, y_1^0, N, f_1, f_2, F)$, $M_*=M_*(x_1^0, y_1^0, \sigma_1, \sigma_2)$ and $M^* = M^*(x_1^0, y_1^0, N, f_1, f_2, F, \sigma_1, \sigma_2)$ do not depend on ξ; σ_1 and σ_2 are the constants from (75) and (76).

Remark. As follows from equalities (77)–(79), for $x_1 > 0$ and $x_1 \le 0$ the functions $\widehat{u}_n$, $\widehat{v}_{3,n}$ and $\widehat{v}_{4,n}$ are defined by different formulas. This does not cause discontinuities of the functions $\widehat{u}_n$, $\widehat{v}_{3,n}$ and $\widehat{v}_{4,n}$ and their partial derivatives with respect to x_1 and y_1 up to order $(k-1)$ inclusive along the axis $Oy_1 : x_1 = 0$, since it is assumed that the functions $\widetilde{F}_2$, $\widetilde{F}_3$, g_1, g_2 from (72)–(83) and their derivatives up to order $(k-1)$ inclusive are equal to zero at the point $O(0, 0)$.

After differentiating equalities (77)–(83) with respect to x_1 and y_1 and using the above estimates (84)–(88) and (75), (76), for $|\xi| > \widetilde{B}_{j_1+j_2}$ we obtain by the method of mathematical induction the inequalities

$$\big|[\partial^{j_1,j_2}(\widehat{u}_{n+1} - \widehat{u}_n)](x_1, y_1)\big| \le M^*_{j_1+j_2} \frac{M^n_{*j_1+j_2}}{n!}(1+|\xi|)^n e^{-N|\xi|} \times$$
$$\times y_1^{n+k+\alpha-j_1-j_2-1}, \quad (89)$$

$$\big|[\partial^{j_1,j_2}(w_{p,n+1} - w_{p,n})(x_1, y_1)\big| \le M^*_{j_1+j_2} \frac{M^n_{*j_1+j_2}}{n!}(1+|\xi|)^n e^{-N|\xi|} \times$$
$$\times y_1^{n+k+\alpha-j_1-j_2-1}, \quad p = 1, 2, \quad (90)$$

$$\big|[\partial^{j_1,j_2}(\widehat{v}_{q,n+1} - \widehat{v}_{q,n})(x_1, y_1)\big| \le M^*_{j_1+j_2} \frac{M^n_{*j_1+j_2}}{n!}(1+|\xi|)^n e^{-N|\xi|} \times$$

$$\times\, y_1^{n+k+\alpha-j_1-j_2-1}, \quad q=3,4, \tag{91}$$

$$\Big|\Big[\frac{d^{j_1+j_2}}{dy_1^{j_1+j_2}}(\varphi_{n+1}-\varphi_n)\Big](y_1)\Big| \le M^*_{j_1+j_2}\frac{M^n_{*j_1+j_2}}{n!}(1+|\xi|)^n \times \\ \times\, e^{-N|\xi|}y_1^{n+k+\alpha-j_1-j_2-1}, \tag{92}$$

$$\Big|\Big[\frac{d^{j_1+j_2}}{dy_1^{j_1+j_2}}(\psi_{n+1}-\psi_n)\Big](y_1)\Big| \le M^*_{j_1+j_2}\frac{M^n_{*j_1+j_2}}{n!}(1+|\xi|)^n \times \\ \times\, e^{-N|\xi|}y_1^{n+k+\alpha-j_1-j_2-1}, \tag{93}$$

where $\partial^{j_1,j_2}=\partial^{j_1+j_2}/\partial x_1^{j_1}\partial y_1^{j_2}$, $0\le j_1+j_2\le k-1$; $\widetilde{B}_p$, M_p^*, and M_{*p}, $p=1,\dots,k-1$, are positive integers not depending on ξ.

By (89) we find that for $0\le j_1+j_2\le k-1$ the series

$$\widehat{u}_{j_1,j_2}(x_1,y_1)=\lim_{n\to\infty}[\partial^{j_1,j_2}\widehat{u}_n](x_1,y_1)=\sum_{n=1}^{\infty}\big[\partial^{j_1,j_2}(\widehat{u}_n-\widehat{u}_{n-1})\big](x_1,y_1)$$

converges uniformly in $\overline{\Omega}_{1Q}$ and for the sum of this series we have the estimate

$$\big|\widehat{u}_{j_1,j_2}(x_1,y_1)\big| \le M^*_{j_1+j_2}e^{M_{*j_1+j_2}(1+|\xi|)y_1}e^{-N|\xi|}y_1^{k+\alpha-j_1-j_2-1}. \tag{94}$$

This estimate implies that $\widehat{u}_{j_1,j_2}\in\overset{\circ}{\Phi}_{k-1+\alpha-j_1-j_2}(\overline{\Omega}_{1Q})$, since, as is easy to verify, the operator of multiplication by the function

$$e^{M_{*j_1+j_2}(1+|\xi|)y_1}$$

transforms the space $\overset{\circ}{\Phi}_{k-1+\alpha-j_1-j_2}(\overline{\Omega}_{1Q})$ into itself. This in turn implies that $\widehat{u}^1(x_1,y_1)\equiv\widehat{u}_{0,0}(x_1,y_1)\in\overset{\circ}{\Phi}{}^{k-1}_{\alpha}(\overline{\Omega}_{1Q})$ and $\widehat{u}_{j_1,j_2}(x_1,y_1)\equiv\partial^{j_1,j_2}\widehat{u}^1(x_1,y_1)$.

By (90)–(93) we find in a similar manner that the series

$$w_p^1(x_1,y_1)=\lim_{n\to\infty}w_{p,n}(x_1,y_1)=\sum_{n=1}^{\infty}\big(w_{p,n}(x_1,y_1)-w_{p,n-1}(x_1,y_1)\big),\ p=1,2,$$

$$\widehat{v}_q^1(x_1,y_1)=\lim_{n\to\infty}\widehat{v}_{q,n}(x_1,y_1)=\sum_{n=1}^{\infty}\big(\widehat{v}_{q,n}(x_1,y_1)-\widehat{v}_{q,n-1}(x_1,y_1)\big),\ q=3,4,$$

converge in the space $\overset{\circ}{\Phi}{}^{k-1}_{\alpha}(\overline{\Omega}_{1Q})$, and the series

$$\varphi^1(y_1)=\lim_{n\to\infty}\varphi_n(y_1)=\sum_{n=1}^{\infty}\big(\varphi_n(y_1)-\varphi_{n-1}(y_1)\big),$$

$$\psi^1(y_1)=\lim_{n\to\infty}\psi_n(y_1)=\sum_{n=1}^{\infty}\big(\psi_n(y_1)-\psi_{n-1}(y_1)\big)$$

converge in the spaces $\overset{\circ}{\Phi}{}_{\alpha}^{k-1}[0,d_1]$ and $\overset{\circ}{\Phi}{}_{\alpha}^{k-1}[0,d_2]$, respectively. Hence by virtue of (77)–(83) it follows that the limit fnctions $\widehat{u}^1$, w^1, $\widehat{v}_3^1$, $\widehat{v}_4^1$, φ^1, and ψ^1 satisfy the system of equations (58)–(65), (71), (72). Thus to prove Lemma 1 it only remains to show that the system of equations (58)–(65), (71), (72) has no other solutions in the classes considered above. Indeed, let us assume that the functions $\widehat{u}^*$, w^*, $\widehat{v}_3^*$, $\widehat{v}_4^*$, φ^*, and ψ^* from the classes proved above satisfy the homogeneous system of equations corresponding to (58)–(65), (71), (72), i.e., for

$$\widetilde{F}_2=0, \quad \widetilde{F}_3=0, \quad g_1=0, \quad g_2=0. \tag{95}$$

To this system with the homogeneous conditions (95) we apply the method of successive approximations, assuming that the functions $\widehat{u}^*$, w^*, $\widehat{v}_3^*$, $\widehat{v}_4^*$, φ^*, and ψ^* are zero approximations. Since this system of functions satisfy the homogeneous system of equations, each of the following approximations will coincide with it, i.e.,

$$\widehat{u}_n^*(x_1,y_1)\equiv\widehat{u}^*(x_1,y_1),\ \ w_n^*(x_1,y_1)\equiv w^*(x_1,y_1),\ \ \widehat{v}_{3,n}^*(x_1,y_1)\equiv\widehat{v}_3^*(x_1,y_1),$$
$$\widehat{v}_{4,n}^*(x_1,y_1)\equiv\widehat{v}_4^*(x_1,y_1),\ \ \varphi_n^*(y_1)\equiv\varphi^*(y_1),\ \ \psi_n^*(y_1)\equiv\psi^*(y_1).$$

By the same reasoning as for estimates (84)–(88), for $|\xi|>\widetilde{B}^*$ we obtain

$$|\widehat{u}^*(x_1,y_1)|=|\widehat{u}_n^*(x_1,y_1)|\le\widetilde{M}^*\frac{\widetilde{M}_*^n}{n!}(1+|\xi|)^n e^{-N|\xi|}y_1^{n+k+\alpha-1},$$
$$|w_p^*(x_1,y_1)|=|w_{p,n}^*(x_1,y_1)|\le\widetilde{M}^*\frac{\widetilde{M}_*^n}{n!}(1+|\xi|)^n e^{-N|\xi|}y_1^{n+k+\alpha-1},\ \ p=1,2,$$
$$|\widehat{v}_q^*(x_1,y_1)|=|\widehat{v}_{q,n}^*(x_1,y_1)|\le\widetilde{M}^*\frac{\widetilde{M}_*^n}{n!}(1+|\xi|)^n e^{-N|\xi|}y_1^{n+k+\alpha-1},\ \ q=3,4,$$
$$|\varphi^*(y_1)|=|\varphi_n^*(y_1)|\le\widetilde{M}^*\frac{\widetilde{M}_*^n}{n!}(1+|\xi|)^n e^{-N|\xi|}y_1^{n+k+\alpha-1},$$
$$|\psi^*(y_1)|=|\psi_n^*(y_1)|\le\widetilde{M}^*\frac{\widetilde{M}_*^n}{n!}(1+|\xi|)^n e^{-N|\xi|}y_1^{n+k+\alpha-1}.$$

Hence, passing to the limit as $n\to\infty$, we find that

$$\widehat{u}^*\equiv 0,\ \ w_1^*\equiv 0,\ \ w_2^*\equiv 0,\ \ \widehat{v}_3^*\equiv 0,\ \ \widehat{v}_4^*\equiv 0,\ \ \varphi^*\equiv 0,\ \ \psi^*\equiv 0,$$

which completes the proof of the lemma.

Let $Q_0(x_1^0,x_2^0,y_1^0,y_2^0\in D_1$. Denote by

$$D_{1Q_0}: -k_2y_1<x_1<k_1y_1,\ \ y_1-x_1<y_1^0-x_1^0,\ \ y_1+x_1<y_1^0+x_1^0$$

a subdomain of the domain D_1 bounded by the surfaces S_1^0, S_2^0 and the characteristic hyperplanes $\widetilde{S}_1: y_1-x_1=y_1^0-x_1^0$ and $\widetilde{S}_2: y_1+x_1=y_1^0+x_1^0$ passing through the point Q_0.

Since problem (1), (17), (18) is equivalently reduced to problem (44)–(53), Lemma 1 immediately implies

Theorem 2. *Let conditions* (69), (70) *be fulfilled. Then for* $k+\alpha > -\frac{\log|\sigma|}{\log\tau}+1$ *problem* (1), (17), (18) *is uniquely solvable in the class* $\overset{\circ}{\Phi}{}^k_\alpha(\overline{D}_1)$ *for any* $F\in\overset{\circ}{\Phi}{}^{k-1}_\alpha(\overline{D}_1)$ *and* $f_i\in\overset{\circ}{\Phi}{}^{k-1}_\alpha(S_i^0)$, $i=1,2$*; the domain of the dependence of the solution* u *of this problem for the point* $Q_0\in D_1$ *is contained in* $\overline{D}_{1Q_0}$.

Note that if conditions (69), (70) or the inequality $k+\alpha > -\frac{\log|\sigma|}{\log\tau}+1$ are invalid, then problem (1), (17), (18) may turn out to be formulated incorrectly.

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(Recieved 29.12.1995)

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