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ON STRONG MAXIMAL OPERATORS CORRESPONDING TO DIFFERENT FRAMES

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ABSTRACT. The problem is posed and solved whether the conditions $f \in L(1+\ln^+ L)^2(\mathbb{R}^2)$ and $\sup_{\theta\in[0,\pi/2)} \int_{\{M_{2,\theta}(f)>1\}} M_{2,\theta}(f) < \infty$ are equivalent for functions $f \in L(\mathbb{R}^2)$ (where $M_{2,\theta}$ denotes the strong maximal operator corresponding to the frame $\{OX_\theta, OY_\theta\}$).

The results obtained represent a general solution of M. de Guzmán's problem that was previously studied by various authors.

1. NOTATION

Let B_1 be a family of all cubic intervals in $\mathbb{R}^n$. We denote by M_1 a maximal Hardy-Littlewood operator which is defined as follows:

$$M_1(f)(z) := \sup\Big\{\frac{1}{|I|}\int_I |f| : z \in I,\ I \in B_1\Big\}, \quad z \in \mathbb{R}^n,$$

for $f \in L_{loc}(\mathbb{R}^n)$.

Let B_2 be a family of all open rectangles in $\mathbb{R}^2$ whose sides are parallel to the coordinate axes; OX_θ be the straight line obtained by rotating the OX-coordinate axis through the angle θ about the point O in the positive direction (OY_θ is defined analogously); $B_{2,\theta}$ be a family of all open rectangles with the sides parallel to the straight lines OX_θ and OY_θ.

For a rectangle $I \subset \mathbb{R}^2$ we shall denote by $n(I)$ a number $\theta \in [0,\pi/2)$ for which one of the sides of I is parallel to OX_θ. The regularity factor of the rectangle I will be defined as the ratio of the length of the larger side of I to the length of the smaller side of I and will be denoted by $r(I)$.

For $\theta \in \mathbb{R}$ we shall denote by $\theta \pmod{\pi/2}$ a number such that $0 \le \theta \pmod{\pi/2} < \pi/2$ and $\theta - \theta \pmod{\pi/2} = \pi k/2$ for some $k \in \mathbb{N}$. One can easily verify that $B_{2,\theta} = B_{2,\theta \pmod{\pi/2}}$, $\theta \in \mathbb{R}$.

The sets $\{OX_\theta, OY_\theta\}$, $\theta \in [0,\pi/2)$ will be called frames.

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For $f \in L_{loc}(\mathbb{R}^2)$ let

$$M_2(f)(z) := \sup\Big\{\frac{1}{|I|}\int\limits_I |f| : z \in I,\ I \in B_2\Big\}, \quad z \in \mathbb{R}^2,$$

$$M_{2,\theta}(f)(z) := \sup\Big\{\frac{1}{|I|}\int\limits_I |f| : z \in I,\ I \in B_{2,\theta}\Big\}, \quad z \in \mathbb{R}^2.$$

M_2 is called the strong maximal operator, while $M_{2,\theta}$ is called the strong maximal operator corresponding to the frame $\{OX_{\theta\,(\mathrm{mod}\,\pi/2)}, OY_{\theta\,(\mathrm{mod}\,\pi/2)}\}$. This definition is correct because by virtue of the equality $B_{2,\theta} = B_{2,\theta\,(\mathrm{mod}\,\pi/2)}$ we have $M_{2,\theta} = M_{2,\theta\ (\mathrm{mod}\ \pi/2)}$. The latter equality implies that the family $\{M_{2,\theta}\}_{\theta\in[0,\pi/2)}$ exhausts the family of all operators $M_{2,\theta}$ ($\theta \in \mathbb{R}$).

2. Formulation of the Question

As is known, the space $L(1 + \ln^+ L)(\mathbb{R}^n)$ can be characterized by the maximal operator M_1 as follows (see [1], [2]):

Theorem 1^0. *Let $f \in L(\mathbb{R}^n)$. Then the following two conditions are equivalent:*

1. $f \in L(1 + \ln^+ L)(\mathbb{R}^n)$;
2. $\displaystyle\int\limits_{\{M_1(f)>1\}} M_1(f) < \infty.$

The implication $1 \Rightarrow 2$ was proved by Hardy and Littlewood [3] for $n = 1$ and by Wiener [4] for $n \geq 2$. The results of the reverse nature were obtained for the first time by Stein [5] and Tsereteli [6, 7]. Guzman and Welland [1, 2] improved the above results by formulating Theorem 1^0.

It is known that if $f \in L(1 + \ln^+ L)^2(\mathbb{R}^2)$ then (see [1])

$$\int\limits_{\{M_2(f)>1\}} M_2(f) < \infty. \tag{2.1}$$

Guzman (see [1]) posed the question whether it was possible to characterize the space $L(1 + \ln^+ L)^2(\mathbb{R}^2)$ by the operator M_2 as was done for the space $L(1 + \ln^+ L)$ using the operator M_1. Gogoladze [8, 9] and Bagby [10] answered this question in the negative. Their results give rise to

Theorem 2^0. *For any functions $f \notin L(1 + \ln^+ L)^2(\mathbb{R}^2)$ and $f \in L(1 + \ln^+ L)(\mathbb{R}^2)$ there exists a Lebesgue measure preserving an invertible mapping*

$\omega : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\int\limits_{\{M_2(f\circ\omega)>1\}} M_2(f \circ \omega) < \infty.$$

Now we proceed directly to formulating our problem. It is easy to verify that if $f \in L(1 + \ln^+ L)^2(\mathbb{R}^2)$ then

$$\int\limits_{\{M_{2,\theta}(f)>1\}} M_{2,\theta}(f) < \infty \quad \text{for any} \quad \theta \in [0, \pi/2), \tag{2.2}$$

and, moreover,

$$\sup_{\theta\in[0,\pi/2)} \int\limits_{\{M_{2,\theta}(f)>1\}} M_{2,\theta}(f) < \infty. \tag{2.3}$$

Clearly, conclusion (2.3) is stronger than (2.1) and hence there is a better chance for us to improve the integral properties of the function f when (2.3) is fulfilled than in the case of fulfillment of (2.1). Having given this information, we formulate the problem:

Let $f \in L(\mathbb{R}^2)$ and $\sup_{\theta\in[0,\pi/2)} \int_{\{M_{2,\theta}(f)>1\}} M_{2,\theta}(f) < \infty$. Is the inclusion $f \in L(1 + \ln^+ L)^2(\mathbb{R}^2)$ then valid?

We would like to note here that the functions constructed in [8–10] do not satisfy condition (2.2) and thus [8–10] do not provide a solution of the above problem. So we shall prove the following theorem which as a particular case contains the answer to the problem.

Theorem 1. *For any functions $f \notin L(1 + \ln^+ L)^2(\mathbb{R}^2)$ and $f \in L(1 + \ln^+ L)(\mathbb{R}^2)$ there exists a Lebesgue measure preserving an invertible mapping $\omega : \mathbb{R}^2 \to \mathbb{R}^2$ such that*

1. *the set* $\{|f \circ \omega| > 1\}$ *is a square interval*;
2. $\displaystyle\sup_{\theta\in[0,\pi/2)} \int\limits_{\{M_{2,\theta}(f\circ\omega)>1\}} M_{2,\theta}(f \circ \omega) < \infty.$

3. Auxiliary Statements

To prove Theorem 1 we shall need several lemmas. If $\Delta \subset \mathbb{R}$ is some interval and (length Δ) $< \pi/2$, then by $M^*_{2,\Delta}$ we denote the following operator: for each $f \in L_{loc}(\mathbb{R}^2)$

$$M^*_{2,\Delta}(f)(z) :=$$
$$= \sup \big\{ M_{2,\theta}(f)(z) : \theta \in [\operatorname{cen} \Delta - \pi/4, \inf \Delta] \cup [\sup \Delta, \operatorname{cen} \Delta + \pi/4) \big\}, \quad z \in \mathbb{R}^2;$$

where $\operatorname{cen}\Delta = (\inf\Delta + \sup\Delta)/2$.

Lemma 1. *Let $0 < \theta < \pi/4$, $H > 0$, $0 < \lambda < H$. It is assumed that the regularity factor of rectangle I satisfies the inequality $r(I) \geq \frac{H}{\lambda \sin^2\theta}$. Then there is a rectangle $\mathcal{I}_{I,H,\theta,\lambda}$ such that*

$$\begin{aligned}
&\big\{M^*_{2,\Delta}(H\chi_{_I}) > \lambda\big\} \subset \mathcal{I}_{I,H,\theta,\lambda},\\
&\big|\mathcal{I}_{I,H,\theta,\lambda}\big| \leq c_1 \frac{H}{\lambda}|I|,\\
&\int\limits_{\mathcal{I}_{I,H,\theta,\lambda}} M^*_{2,\Delta}(H\chi_{_I}) \leq c_2 H\Big(1 + \ln^+ \frac{H}{\lambda}\Big)|I|,
\end{aligned}$$

where $\Delta = (n(I) - \theta, n(I) + \theta)$, while c_1 and c_2 are the positive constants not depending on I, H, θ, and λ.

Proof. We begin by considering the case $\lambda = 1$, $H > 1$. Without loss of generality it will be assumed (see Fig. 1) that $n(I) = 0$, I is the rectangle $ABCD$ the sides AB and BC of which have the lengths I_1 and I_2 respectively and $I_2/I_1 \geq H/\sin^2\theta$.

Fig. 1

The strip bounded by the straight lines containing the segments AD and BC respectively will be denoted by $\widetilde{I}$. Using the convexity property of the rectangle, it is easy to prove the inequality

$$M^*_{2,\Delta}(H\chi_{_I})(z) \leq 3H\frac{I_1}{\operatorname{dist}(z,\widetilde{I})} \quad \text{for} \quad \operatorname{dist}(z,\widetilde{I}) \geq I_1. \tag{3.1}$$

A minimal number $\alpha \geq 0$ for which the straight line l is parallel to the OX_α-axis will be denoted by $n(l)$. Let l_1 $(n(l_1) = \theta)$ and l_2 $(n(l_2) = \pi - \theta)$ be the straight lines passing through the points C and D, respectively. The straight lines l_1 and l_2 divide the plane into parts. We denote the right-hand part by E_{l_1,l_2}. Due to the definition of the operator $M^*_{2,\theta}$ it is easy to show that

$$M^*_{2,\Delta}(H\chi_{_I})(z) \le \frac{H|I|}{\operatorname{dist}(z,l_1)\operatorname{dist}(z,l_2)} \quad \text{for} \quad z \in E_{l_1,l_2}. \tag{3.2}$$

Above and below the rectangle I let us draw the straight lines l_1' and l_2' which are parallel to the segment AD and situated at a distance $(3H+1)I_1$ from I. Now the rectangle $A'B'C'D'$ can be chosen so that $A'D' \subset l_1'$, $B'C' \subset l_2'$, and $\operatorname{dist}(A'B', I) = \operatorname{dist}(C'D', I) = 10I_2$. We shall prove that the rectangle $A'B'C'D'$ can be taken as $\mathcal{I}_{I,H,\theta,1}$.

The strip bounded by the straight lines l_1' and l_2' will be denoted by E; the part of E lying on the right of $C'D'$ will be denoted by E^+, while the part of E lying on the left of $A'B'$ will be denoted by E^-.

By (3.1) we conclude that

$$\big\{M^*_{2,\Delta}(H\chi_{_I}) > 1\big\} \subset E. \tag{3.3}$$

By virtue of the inequality $I_2/I_1 \ge H/\sin^2\theta$ and the definition of the rectangle $A'B'C'D'$ it is easy to show that

$$E^+ \subset E_{l_1,l_2}; \tag{3.4}$$

$$\operatorname{dist}(D', l_1) \ge \sqrt{H|I|}; \tag{3.5}$$

$$\operatorname{dist}(C', l_2) \ge \sqrt{H|I|}. \tag{3.6}$$

Obviously, for any $z \in E^+$

$$\operatorname{dist}(z, l_1) \ge \operatorname{dist}(D', l_1); \tag{3.7}$$

$$\operatorname{dist}(z, l_2) \ge \operatorname{dist}(C', l_2). \tag{3.8}$$

Using (3.2) and (3.4)–(3.8), we find that

$$\begin{aligned} M^*_{2,\Delta}(H\chi_{_I})(z) &\le \frac{H|I|}{\operatorname{dist}(z,l_1)\operatorname{dist}(z,l_2)} \le \\ &\le \frac{H|I|}{\operatorname{dist}(D',l_1)\operatorname{dist}(C',l_2)} \le \frac{H|I|}{\sqrt{H|I|}\sqrt{H|I|}} = 1 \end{aligned} \tag{3.9}$$

for any $z \in E^+$.

From (3.9) we readily obtain

$$M^*_{2,\Delta}(H\chi_{_I})(z) \le 1 \quad \text{for all} \quad z \in E^-. \tag{3.10}$$

For this it is enough to ascertain that the following equality holds:

$$M^*_{2,\Delta}(H\chi_{_I})(z) = M^*_{2,\Delta}(H\chi_{_I})(z'), \quad z \in \mathbb{R}^2,$$

where z' denotes a point symmetrical to z with respect to the center of the rectangle I.

By (3.3), (3.9), and (3.10) we have

$$\big\{M^*_{2,\Delta}(H\chi_{_I}) > 1\big\} \subset A'B'C'D'. \tag{3.11}$$

Clearly,

$$\big|A'B'C'D'\big| \le (6H+3)I_1 21 I_2 < 200H|I|. \tag{3.12}$$

It is easy to prove the inequality

$$\int\limits_{E_x} M^*_{2,\Delta}(H\chi_{_I})(x,y)dy < 15H(1+\ln H)I_1, \quad x \in \mathbb{R}, \tag{3.13}$$

where $E_x := \{(x,y) : y \in \mathbb{R}\} \cap E$, $x \in \mathbb{R}$.

By (3.13) we obtain

$$\int\limits_{A'B'C'D'} M^*_{2,\Delta}(H\chi_{_I})(x,y)dxdy \le$$
$$\le 15H(1+\ln H)I_1(\text{ length } A'D') \le 315H(1+\ln H)|I|. \tag{3.14}$$

By virtue of (3.11), (3.12), and (3.14) the lemma is proved for the case $\lambda = 1$, $H > 1$. Now it is easy to obtain the proof for the general case if we take into consideration the following obvious equalities:

$$M^*_{2,\Delta}(H\chi_{_I})(z) = \lambda M^*_{2,\Delta}\Big(\frac{H}{\lambda}\chi_{_I}\Big)(z), \quad z \in \mathbb{R}^2; \tag{3.15}$$

$$\big\{M^*_{2,\Delta}(H\chi_{_I}) > \lambda\big\} = \Big\{M^*_{2,\Delta}\Big(\frac{H}{\lambda}\chi_{_I}\Big) > 1\Big\}. \quad \square \tag{3.16}$$

As is well known (see [1]), if $f \in L(1+\ln^+ L)(\mathbb{R}^2)$, then for each $\lambda > 0$ we have the inequality

$$\big|\big\{M_2(f) > \lambda\big\}\big| \le c_3 \int\limits_{\mathbb{R}^2} \frac{|f|}{\lambda}\Big(1 + \ln^+ \frac{|f|}{\lambda}\Big), \tag{3.17}$$

where c_3 is a positive constant not depending on f and λ.

Let Γ_θ be rotation of the plane through the angle θ about the point O in the positive direction. It is easy to verify that $M_{2,\theta}(f)(z) = M_2(f \circ \Gamma_\theta)(\Gamma_\theta^{-1}(z))$. Hence we obtain $|\{M_{2,\theta}(f) > \lambda\}| = |\{M_2(f \circ \Gamma_\theta) > \lambda\}|$ which by virtue of (3.17) implies that Lemma 2 is valid.

Lemma 2. *Let $f \in L(1+\ln^+ L)(\mathbb{R}^2)$; then the inequality*

$$\big|\{M_{2,\theta}(f) > \lambda\}\big| \le c_3 \int\limits_{\mathbb{R}^2} \frac{|f|}{\lambda}\Big(1+\ln^+ \frac{|f|}{\lambda}\Big)$$

holds for each $\theta \in [0,\pi/2)$ and $\lambda > 0$.

By using this lemma we can easily prove

Lemma 3. *Let $f \in L(1+\ln^+ L)^2(\mathbb{R}^2)$; then the inequality*

$$\int\limits_{\{M_{2,\theta}(f)>1\}} M_{2,\theta}(f) \le c_4 \int\limits_{\mathbb{R}^2} |f|\big(1+\ln^+|f|\big)^2,$$

where c_4 is the positive constant not depending on f and θ, holds for each $\theta \in [0,\pi/2)$.

Let $\Omega \subset [0,\pi/2)$, $B = \cup_{\theta\in\Omega} B_{2,\theta}$ and for each $f \in L_{loc}(\mathbb{R}^2)$

$$M_B(f)(z) := \sup\Big\{\frac{1}{|I|}\int\limits_I |f| : z \in I, I \in B\Big\}, \quad z \in \mathbb{R}^2.$$

We have

Lemma 4. *Let I_k be a rectangle, $H_k > 1$, $k \in \mathbb{N}$, and $\sum_{k=1}^\infty H_k|I_k| < \infty$. It is assumed that there exists a sequence $\{\mathcal{I}_k\}$ such that $\mathcal{I}_k \cap \mathcal{I}_m = \varnothing$, $k \ne m$, $\{M_B(H_k\chi_{I_k}) > \lambda\} \subset \mathcal{I}_k$, $k \in \mathbb{N}$, where $\lambda \in (0,1]$ is a fixed number. Then*

$$\Big\{M_B\Big(\sum_{k=1}^\infty H_k\chi_{I_k}\Big) > \lambda\Big\} \subset \bigcup_{k=1}^\infty \mathcal{I}_k. \tag{3.18}$$

Proof. Let $f_k := H_k\chi_{I_k}$ $(k \in \mathbb{N})$ and $f := \sum_{k=1}^\infty H_k\chi_{I_k} = \sup_{k\in\mathbb{N}} H_k\chi_{I_k}$. Clearly, (3.18) is equivalent to the inequality

$$M_B(f)(z) \le \lambda, \quad z \notin \cup_{k=1}^\infty \mathcal{I}_k. \tag{3.19}$$

Let us assume that $z \notin \cup_{k=1}^\infty \mathcal{I}_k$, $R \ni z$, $R \in B$ and prove the inequality

$$\int\limits_{R\cap\mathcal{I}_k} f_k \le \lambda\big|R\cap\mathcal{I}_k\big|, \quad k \in \mathbb{N}. \tag{3.20}$$

Let $n(R) = \theta$ and $R_{x_\theta} := \{(\xi,\eta) : \xi_\theta = x_\theta, \eta_\theta \in \mathbb{R}\} \cap R$, for each $x_\theta \in \mathbb{R}$ (where (ξ_θ,η_θ) denotes the coordinates of the point (ξ,η) in the $X_\theta OY_\theta$-system).

We consider an arbitrarily fixed number $k \in \mathbb{N}$. The following notation is introduced:

$$
\begin{aligned}
&E_k^1 := \{x_\theta : R_{x_\theta} \cap \mathcal{I}_k \neq \varnothing, R_{x_\theta} \setminus \mathcal{I}_k \neq \varnothing\};\\
&E_k^2 := \{x_\theta : R_{x_\theta} \subset \mathcal{I}_k\};\\
&R_k^1 := \bigcup_{x_\theta \in E_k^1} \big(R_{x_\theta} \cap \mathcal{I}_k\big);\\
&R_k^2 := \bigcup_{x_\theta \in E_k^2} \big(R_{x_\theta} \cap \mathcal{I}_k\big).
\end{aligned}
$$

Clearly,

$$
R_k^1 \cap R_k^2 = \varnothing \quad \text{and} \quad R \cap \mathcal{I}_k = R_k^1 \cup R_k^2. \tag{3.21}
$$

For our further discussion we need the inequality

$$
M_B(f)(z) \leq \lambda, \quad z \in \partial \mathcal{I}_k, \tag{3.22}
$$

which immediately follows from the condition of the lemma (∂ denotes the boundary of sets).

If $x_\theta \in E_k^1$, then one can easily find that one end of the segment $R_{x_\theta} \cap \mathcal{I}_k$ belongs to $\partial \mathcal{I}_k$ and hence by virtue of (3.22) one readily obtains

$$
\int\limits_{R_{x_\theta} \cap \mathcal{I}_k} f_k(x_\theta, y_\theta) dy_\theta \leq \lambda \big|R_{x_\theta} \cap \mathcal{I}_k\big|_1,
$$

where $|\cdot|_1$ denotes the Lebesgue measure on the straight line.

The above inequality implies

$$
\begin{aligned}
\int\limits_{R_k^1} f_k = \int\limits_{E_k^1} dx_\theta \int\limits_{R_{x_\theta} \cap E_k} f_k(x_\theta, y_\theta) dy_\theta \leq\\
\leq \int\limits_{E_k^1} \lambda \big|R_{x_\theta} \cap \mathcal{I}_k\big|_1 = \lambda \big|R_k^1\big|.
\end{aligned} \tag{3.23}
$$

It is easy to prove the following facts:

$$
R_k^2 \quad \text{is a rectancle included in} \quad B; \tag{3.24}
$$

$$
R_k^2 \quad \text{has a vertex belonging to} \quad \partial \mathcal{I}_k. \tag{3.25}
$$

From (3.22), (3.24), (3.25) we obtain the inequality

$$
\int\limits_{R_k^2} f_k \leq \lambda \big|R_k^2\big|. \tag{3.26}
$$

Taking into account (3.12), (3.23), and (3.26), we have

$$\int\limits_{R\cap\mathcal{I}_k} f_k = \int\limits_{R_k^1} f_k + \int\limits_{R_k^2} f_k \le \lambda\big|R_k^1\big| + \lambda\big|R_k^2\big| = \lambda\big|R\cap\mathcal{I}_k\big|. \tag{3.27}$$

Since $k \in \mathbb{N}$ has been chosen arbitrarily, (3.27) implies the estimate

$$\int\limits_R f = \sum_{k=1}^{\infty} \int\limits_{R\cap I_k} f_k = \sum_{k=1}^{\infty} \int\limits_{R\cap\mathcal{I}_k} f_k \le \sum_{k=1}^{\infty} \lambda\big|R\cap\mathcal{I}_k\big| \le \lambda|R|;$$

Because of an arbitrarily chosen R ($z \in R$, $R \in B$) we hence conclude that

$$M_B(f)(z) \le \lambda, \quad z \notin \cup_{k=1}^{\infty}\mathcal{I}_k. \qquad \square$$

For each $\theta \in [0,\pi/2)$ let $A(\theta)$ denote a set of all $f \in L(\mathbb{R}^2)$ for which $\int_{\{M_{2,\theta}(f)>1\}} M_{2,\theta}(f) < \infty$. We have

Lemma 5. *Let f and g belong to the set $A(\theta)$. Then $f+g \in A(\theta)$ and*

$$\int\limits_{\{M_{2,\theta}(f+g)>1\}} M_{2,\theta}(f+g) \le$$

$$\le c_5\Big(\int\limits_{\{M_{2,\theta}(f)>1\}} M_{2,\theta}(f) + \int\limits_{\{M_{2,\theta}(g)>1\}} M_{2,\theta}(g)\Big) +$$

$$+c_6\big(\big\|f(1+\ln^+|f|)\big\|_1 + \big\|g(1+\ln^+|g|)\big\|_1\big), \tag{3.28}$$

where c_5 and c_6 are positive constants not depending on f, g, and θ.

Proof. One can easily verify the following inequalities:

$$\int\limits_{\{M_{2,\theta}(f+g)>1\}} M_{2,\theta}(f+g) \le$$

$$\le \int\limits_{\{M_{2,\theta}(f)>1/2\}\cup\{M_{2,\theta}(g)>1/2\}} \big(M_{2,\theta}(f) + M_{2,\theta}(g)\big); \tag{3.29}$$

$$\int\limits_{\{M_{2,\theta}(f)>1/2\}\cup\{M_{2,\theta}(g)>1/2\}} M_{2,\theta}(f) \le$$

$$\le \int\limits_{\{M_{2,\theta}(f)>1/2\}} M_{2,\theta}(f) + \frac{1}{2}\big|\{M_{2,\theta}(g) > 1/2\}\big|; \tag{3.30}$$

$$\int\limits_{\{M_{2,\theta}(g)>1/2\}\cup\{M_{2,\theta}(g)>1/2\}} M_{2,\theta}(f) \le$$

$$\leq \int\limits_{\{M_{2,\theta}(g)>1/2\}} M_{2,\theta}(g) + \frac{1}{2}\big|\{M_{2,\theta}(f) > 1/2\}\big|. \tag{3.31}$$

By virtue of (3.29), (3.30), (3.31) and Lemma 2 we conclude that (3.28) is valid. □

Let A denote a set of all functions $f \in L(\mathbb{R}^2)$ satisfying condition (2.3). Lemma 5 immediately gives rise to

Lemma 6. *Let f and g belong to the set A. Then $f + g \in A$.*

4. Proof of Theorem 1

It is assumed without loss of generality that f is positive, $|\{f > 1\}| = 1$. Let $E_k := \{k-1 \leq f < k\}$, $k \in \mathbb{N}$. Clearly, there exists $k_0 \geq 3$ for which

$$\sum_{k=k_0}^{\infty} c_1 k|E_k| < 1. \tag{4.1}$$

Choose a sequence $\{m_k\} \subset \mathbb{N}$ such that

$$\frac{k\ln^2 k|E_k|}{m_k} \leq 1, \quad k \in \mathbb{N}. \tag{4.2}$$

Let $N_0 := \{k \in \mathbb{N} : k \geq k_0, |E_k| > 0\}$ and the sequence $\{\Delta_{k,m}\}_{k\in N_0, m=\overline{1,m_k}}$ consist of pairwise nonintersecting intervals lying on the segment $[0, \pi/2)$.

For each $k \in N_0$ and $m \in [1, m_k]$ choose a rectangle $I_{k,m}$ such that

$$\big|I_{k,m}\big| = \frac{|E_k|}{m_k}; \tag{4.3}$$

$$n\big(I_{k,m}\big) = \operatorname{cen}\Delta_{k,m}; \tag{4.4}$$

$$r\big(I_{k,m}\big) \geq \frac{k}{\sin^2|\Delta_{k,m}|_1/2}\ . \tag{4.5}$$

By virtue of (4.5) and Lemma 1 there exists a rectangle $\mathcal{I}_{k,m}$ such that

$$\big\{M^*_{2,\Delta_{k,m}}\big(k\chi_{I_{k,m}}\big) > 1\big\} \subset \mathcal{I}_{k,m}; \tag{4.6}$$

$$\big|\mathcal{I}_{k,m}\big| \leq c_1 k\big|I_{k,m}\big|; \tag{4.7}$$

$$\int\limits_{\mathcal{I}_{k,m}} M^*_{2,\Delta_{k,m}}\big(k\chi_{I_{k,m}}\big) \leq 2c_2 k\ln k\big|I_{k,m}\big|. \tag{4.8}$$

Let $\{Q_{k,m}\}_{k\in N_0, m=\overline{1,m_k}}$ be a sequence of pairwise rectangular intervals lying on $[0,1]^2$ and each having height equal to 1, and

$$|Q_{k,m}| = \big|\mathcal{I}_{k,m}\big|, \quad k \in N_0, \quad m = \overline{1,m_k} \tag{4.9}$$

(such a sequence exists by virtue of (4.1), (4.3) and (4.7)).

For each $k \in N_0$ and $m \in [1, m_k]$ we complete the rectangle $Q_{k,m}$ with pairwise nonintersecting rectangles $\{\mathcal{I}_{k,m,q}\}$ which are homothetic to the rectangle $\mathcal{I}_{k,m}$, i.e., we have

$$\mathcal{I}_{k,m,q} = P_{k,m,q}(\mathcal{I}_{k,m}) \quad \text{where } P_{k,m,q} \text{ is the homothety } (q \in \mathbb{N}); \tag{4.10}$$

$$\mathcal{I}_{k,m,q} \subset Q_{k,m}, \quad q \in \mathbb{N}; \tag{4.11}$$

$$\mathcal{I}_{k,m,q} \cap \mathcal{I}_{k,m,q'} = \varnothing, \quad q \neq q'; \tag{4.12}$$

$$\Big| Q_{k,m} \setminus \bigcup_{q \in \mathbb{N}} \mathcal{I}_{k,m,q} \Big| = 0. \tag{4.13}$$

Let $I_{k,m,q} := P_{k,m,q}(I_{k,m})$, $k \in N_0$, $m = \overline{1, m_k}$, $q \in \mathbb{N}$. Clearly,

$$\sum_{q \in \mathbb{N}} \big| I_{k,m,q} \big| = \big| I_{k,m} \big| = \frac{|E_k|}{m_k}\,, \quad k \in N_0, \ \ m \in [1, m_k]. \tag{4.14}$$

Since $P_{k,m,q}$ is a homotopy, by (4.6)–(4.8) we conclude that for each $k \in N_0$, $m \in [1, m_k]$, and $q \in \mathbb{N}$

$$\big\{ M^*_{2,\Delta_{k,m}}\big(k\chi_{I_{k,m,q}}\big) > 1 \big\} \subset \mathcal{I}_{k,m,q}; \tag{4.15}$$

$$\big|\mathcal{I}_{k,m,q}\big| \leq c_1 k \big| I_{k,m,q} \big|; \tag{4.16}$$

$$\int\limits_{\mathcal{I}_{k,m,q}} M^*_{2,\Delta_{k,m}}\big(k\chi_{I_{k,m,q}}\big) \leq 2c_2 k \ln k \big| I_{k,m,q} \big|. \tag{4.17}$$

We introduce the notation $g_{k,m} := \sup_{q \in \mathbb{N}} (k\chi_{I_{k,m,q}})$, $k \in N_0$, $m_0 = \overline{1, m_k}$, and $g := \sup\{g_{k,m} : k \in N_0, m \in [1, m_k]\}$. Let us prove that $g \in A$.

It is assumed that $\theta \in [0, \pi/2)$ is an arbitrary fixed number. Two cases are possible:

(a) $\theta \in \underset{k,m}{\cup} \Delta_{k,m}$;

(b) $\theta \notin \underset{k,m}{\cup} \Delta_{k,m}$;

(a) Let $\theta \in \Delta_{k(\theta),m(\theta)}$. We introduce the notation $T := \{(k, m, q) : k \in N_0, m \in [1, m_k], q \in \mathbb{N}, (k, m) \neq (k(\theta), m(\theta))\}$.

Since $\theta \notin \Delta_{k,m}$ for $(k, m) \neq (k(\theta), m(\theta))$, by the definition of $M^*_{2,\Delta}$ and (4.15) we obtain

$$\big\{ M_{2,\theta}\big(k\chi_{I_{k,m,q}}\big) > 1 \big\} \subset \big\{ M^*_{2,\Delta_{k,m}}\big(k\chi_{I_{k,m,q}}\big) > 1 \big\} \subset \mathcal{I}_{k,m,q}, \quad (k, m, q) \in T.$$

Hence by (4.12) and Lemma 4 we conclude that

$$\big\{M_{2,\theta}\big(g-g_{k(\theta),m(\theta)}\big)>1\big\}\subset\bigcup_{(k,m,q)\in T}\mathcal{I}_{k,m,q}, \tag{4.18}$$

$$M_{2,\theta}\big(g-g_{k(\theta),m(\theta)}\big)(z)\le M_{2,\theta}\big(k\chi_{I_{k,m,q}}\big)(z)+1$$
$$\text{for}\quad z\in\mathcal{I}_{k,m,q},\quad (k,m,q)\in T. \tag{4.19}$$

On account of (4.14) and (4.16)–(4.19) we write

$$\int\limits_{\{M_{2,\theta}(g-g_{k(\theta),m(\theta)})>1\}} M_{2,\theta}\big(g-g_{k(\theta),m(\theta)}\big)\le$$
$$\le\sum_{(k,m,q)\in T}\int\limits_{\mathcal{I}_{k,m,q}} M_{2,\theta}\big(g-g_{k(\theta),m(\theta)}\big)\le$$
$$\le\sum_{(k,m,q)\in T}\int\limits_{\mathcal{I}_{k,m,q}}\big[M_{2,\theta}\big(k\chi_{I_{k,m,q}}\big)+1\big]\le$$
$$\le\sum_{(k,m,q)\in T}\big[2c_2k\ln k|I_{k,m,q}|+|\mathcal{I}_{k,m,q}|\big]\le$$
$$\le 1+\sum_{k\in N_0}2c_2k\ln k|E_k|\le 1+8c_2\|f\ln^+ f\|_1. \tag{4.20}$$

By (4.2), (4.14) and Lemma 3 we have

$$\int\limits_{\{M_{2,\theta}(g_{k(\theta),m(\theta)})>1\}} M_{2,\theta}\big(g_{k(\theta),m(\theta)}\big)\le c_4\int\limits_{\mathbb{R}^2} g_{k(\theta),m(\theta)}\big(1+\ln^+ g_{k(\theta),m(\theta)}\big)^2=$$
$$=c_4\sum_{q\in\mathbb{N}}k(\theta)\big(1+\ln k(\theta)\big)^2\big|I_{k(\theta),m(\theta),q}\big|=$$
$$=c_4k(\theta)\big(1+\ln k(\theta)\big)^2\frac{|E_{k(\theta)}|}{m_{k(\theta)}}\le 4c_4. \tag{4.21}$$

From the construction we easily find that

$$\big\|\big(g-g_{k(\theta),m(\theta)}\big)\big(1+\ln^+\big(g-g_{k(\theta),m(\theta)}\big)\big\|_1\le$$
$$\le\big\|g(1+\ln^+ g)\big\|_1\le 4\|f\ln^+ f\|_1, \tag{4.22}$$

Analogously,

$$\big\|g_{k(\theta),m(\theta)}\big(1+\ln^+ g_{k(\theta),m(\theta)}\big)\big\|_1\le 4\|f\ln^+ f\|_1 \tag{4.23}$$

Using the representation $g = (g - g_{k(\theta),m(\theta)}) + g_{k(\theta),m(\theta)}$, by virtue of (4.20)–(4.23) and Lemma 5 we obtain

$$\int\limits_{\{M_{2,\theta}(g)>1\}} M_{2,\theta}(g) \leq$$
$$\leq c_5\bigl(1 + 8c_2\|f \ln^+ f\|_1 + 4c_4\bigr) + 8\|f \ln^+ f\|_1 c_6. \tag{4.24}$$

In proving the case (b) we have no "dangerous" term $g_{k(\theta),m(\theta)}$ and therefore, applying the same reasoning as for (4.20), we can write

$$\int\limits_{\{M_{2,\theta}(g)>1\}} M_{2,\theta}(g) \leq$$
$$\leq \sum_{(k,m,q)\in T_0} \int\limits_{\mathcal{I}_{k,m,q}} M_{2,\theta}(g) \leq \sum_{(k,m,q)\in T_0} \int\limits_{\mathcal{I}_{k,m,q}} \bigl[M_{2,\theta}(k\chi_{I_{k,m,q}}) + 1\bigr] \leq$$
$$\leq \sum_{(k,m,q)\in T_0} \bigl[2c_2 k \ln k |I_{k,m,q}| + |\mathcal{I}_{k,m,q}|\bigr] \leq$$
$$\leq 1 + \sum_{k\in N_0} 2c_2 k \ln k |E_k| \leq 1 + 8c_2\|f \ln^+ f\|,$$
$$\text{where} \quad T_0 := \bigl\{(k,m,q) : k \in N_0,\ m = \overline{1, m_k},\ q \in \mathbb{N}\bigr\}. \tag{4.25}$$

Since $\theta \in [0, \pi/2)$ was chosen arbitrarily, by virtue of (4.24) and (4.25) we conclude that $g \in A$.

We shall now find the desired mapping of ω. From the construction it easily follows that

$$E_k \cap E_{k'} = \varnothing, \quad k \neq k', \tag{4.26}$$

$$\Bigl(\bigcup_{(m,q)\in T_k} I_{k,m,q}\Bigr) \bigcap \Bigl(\bigcup_{(m,q)\in T_{k'}} I_{k',m,q}\Bigr) = \varnothing, \quad k = k', \tag{4.27}$$

where $T_k := \{(m,q) : m \in [1, m_k], q \in \mathbb{N}\}$ for $k \in N_0$;

$$|E_k| = \Bigl|\bigcup_{(m,q)\in T_k} I_{k,m,q}\Bigr| > 0, \quad k \in N_0, \tag{4.28}$$

$$\Bigl|\{f > 1\} \setminus \bigcup_{k\in N_0} E_k\Bigr| = \Bigl|(0,1)^2 \setminus \bigcup_{(k,m,q)\in T_0} I_{k,m,q}\Bigr| > 0. \tag{4.29}$$

By conditions (4.26)–(4.29) and one familiar result on measure-preserving transformations (see, e.g., [11, Chapter: Uniform Approximation]) we conclude that there exists a Lebesgue measure preserving an invertible mapping

$\omega : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\omega\Big(\bigcup_{(m,q)\in T_k} I_{k,m,q}\Big) = E_k, \quad k \in N_0; \tag{4.30}$$

$$\omega\Big((0,1)^2 \backslash \bigcup_{(k,m,q)\in T_0} I_{k,m,q}\Big) = \{f > 1\} \backslash \bigcup_{k\in N_0} E_k; \tag{4.31}$$

$$\omega\big(\mathbb{R}^2 \backslash (0,1)^2\big) = \{f \le 1\}. \tag{4.32}$$

(4.30) and (4.31) clearly imply

$$\{f \circ \omega > 1\} = \omega^{-1}\big(\{f > 1\}\big) = (0,1)^2. \tag{4.33}$$

From the construction and conditions (4.30)–(4.32) we obtain

$$(f \circ \omega)\chi_{\underset{k\in N_0}{\cup} E_k} \le g,$$

$$(f \circ \omega)\chi_{\mathbb{R}^2 \backslash \underset{k\in N_0}{\cup} E_k} \in L\big(1 + \ln^+ L\big)^2(\mathbb{R}^2).$$

Hence, taking into account $f \circ \omega = (f \circ \omega)\chi_{\underset{k\in N_0}{\cup} E_k} + (f \circ \omega)\chi_{\mathbb{R}^2 \backslash \underset{k\in N_0}{\cup} E_k}$, inclusions $g \in A$, and Lemmas 3 and 6, we conclude that $f \circ \omega \in A$, i.e.,

$$\sup_{\theta\in[0,\pi/2)} \int\limits_{\{M_{2,\theta}(f\circ\omega)>1\}} M_{2,\theta}(f \circ \omega) < \infty. \tag{4.34}$$

By (4.33) and (4.34) ω is the desired mapping. $\square$

5. Remarks

Remark 1. On overcoming certain technical difficulties, we can prove by a technique similar to that used to prove Theorem 1 the following generalization.

Theorem 2. *Let a function $f \notin L(1+\ln^+ L)^2(\mathbb{R}^2)$, $f \in L(1+\ln^+ L)(\mathbb{R}^2)$. It is assumed that a set G_1, $|G_1| > 0$, is such that $f\chi_{\mathbb{R}^2\backslash G_1} \in L(1 + \ln^+ L)^2(\mathbb{R}^2)$. Then for any set G_2, $|G_2| = |G_1|$, there exists a Lebesgue measure preserving an invertible mapping $\omega : \mathbb{R}^2 \to \mathbb{R}^2$ such that*

1. $\omega(G_1) = G_2$,
2. $\{z : \omega(z) \ne z\} \subset G_1 \cup G_2$,
3. $\displaystyle\sup_{\theta\in[0,\pi/2)} \int\limits_{\{M_{2,\theta}(f\circ\omega)>1\}} M_{2,\theta}(f \circ \omega) < \infty.$

Theorem 2 yields as a corollary

Theorem 3. *Let a function $f \notin L(1+\ln^+ L)^2(\mathbb{R}^2)$, $f \in L(1+\ln^+ L)(\mathbb{R}^2)$. It is assumed that a set G is such that $f\chi_{\mathbb{R}^2\setminus G} \in L(1+\ln^+ L)^2(\mathbb{R}^2)$. Then there exists a Lebesgue measure preserving an invertible mapping $\omega : \mathbb{R}^2 \to \mathbb{R}^2$ such that*

1. $\{z : \omega(z) \neq z\} \subset G$,
2. $\sup\limits_{\theta\in[0,\pi/2)} \int\limits_{\{M_{2,\theta}(f\circ\omega)>1\}} M_{2,\theta}(f\circ\omega) < \infty.$

Remark 2. Let X be a set and R an equivalence relation on X. A subset $Y \subset X$ will be called an R-set if the fact that $y \in Y$ implies that $[y]_R \subset Y$, where $[y]_R$ denotes a set of all elements from X R-equaivalent to y. The following problem is posed in [6]: Given an equivalence relation R on the set X, characterize the set $E \subset X$ from the standpoint of R, i.e., give in explicit terms the kernel $\underline{E}(R)$ (the greatest R-set contained in E) and the hull $\overline{E}(R)$ (the least R-set containing E).

Consider an arbitrary set $G \subset \mathbb{R}^2$ and choose $|G| > 0$, X_G, R_G and E_G in the following manner: $X_G := \{f \in L(\mathbb{R}^2), \operatorname{supp} f \subset G\}$, $E_G := \{f \in X_G, f \in A\}$, f and $g \in R_G$ will be called R_G-equivalent if there exists a Lebesgue measure preserving an invertible mapping $\omega : \mathbb{R}^2 \to \mathbb{R}^2$, $\{z : \omega(z) \neq z\} \subset G$, such that $g = f \circ \omega$.

Let us agree that $\varphi(L)(G)$ denotes a class of functions $f : \mathbb{R}^2 \to \mathbb{R}$ with the following properties: $\operatorname{supp} f \subset G$, $\int_G \varphi(|f|) < \infty$. One can easily show that $\underline{E}_G(R_G) = L(1 + \ln^+ L)^2(G)$, while by virtue of (2.3) and Theorem 3 we have the equality $\overline{E}_G(R_G) = L(1 + \ln^+ L)(G)$. Thus the next theorem is valid.

Theorem 4. *For each $G \subset \mathbb{R}^2$, $|G| > 0$, we have*

$$\underline{E}_G(R_G) = L(1 + \ln^+ L)^2(G) \quad \text{and} \quad \overline{E}_G(R_G) = L(1 + \ln^+ L)(G).$$

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References

1. M. de Guzmán, Differentiation of integrals in $\mathbb{R}^n$. *Lecture Notes in Mathematics* 481, *Springer,* 1975.

2. M. de Guzmán and G. V. Welland, On the differentiation of integrals. *Rev. Un. Mat. Argentina* **25**(1971), 253–276.

3. G. H. Hardy and J. E. Littlewood, A maximal theorem with function-theoretic applications. *Acta Math.* **54**(1930), 81–116.

4. N. Wiener, The ergodic theorem. *Duke Math. J.* **5**(1939), 1–18.

5. E. M. Stein, Note on class $L \log L$. *Studia Math.* **31**(1969), 305–310.

6. O. D. Tsereteli, On the inversion of some Hardy and Littlewood theorems. (Russian) *Bull. Acad. Sci. Georgian SSR* **56**(1969), No. 2, 269–271.

7. O. D. Tsereteli, A metric characteristic of the set of functions whose maximal functions are summable. (Russian) *Trudy Tbilissk. Mat. Inst.* **42**(1972), 104–118.

8. L. D. Gogoladze, The Hardy-Littlewood maximal functions. (Russian) *Bull. Acad. Sci. Georgian SSR* **111**(1983), 257–259.

9. L. D. Gogoladze, On the summation of multiple trigonometric series and conjugate functions. (Russian) *Dissertation for the Doctor of Science Degree, Tbilisi,* 1984.

10. R. J. Bagby, A note on the strong maximal function. *Proc. Amer. Math. Soc.* **88**(1983), 648–650.

11. P. R. Halmos, Lectures on the ergodic theory. *Math. Soc. Japan, Tokyo,* 1956.

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