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$C(X)$ IN THE WEAK TOPOLOGY

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ABSTRACT. Some relations between cardinal invariants of X and $C(X)$ are established in the weak topology, where $C(X)$ is the space of continuous real-valued functions on X in the compact-open topology.

Let X be a compact space. Denote by $C(X)$ the space of continuous real-valued functions on X in the compact-open topology, by $C'(X) \equiv (C(X))'$ the vector space dual to $C(X)$, i.e. the space of continuous linear forms on $C(X)$, by $C_\omega(X)$ ($C'_\omega(X)$) the space $C(X)$ ($C'(X)$) in $C'(X)$-topology ($C(X)$-topology), and by $C_p(X)$ the space $C(X)$ in the topology of pointwise convergence.

Symbols $|X|, \omega, \chi, d, \pi\omega, \pi\chi, p\omega, n\omega$ denote the cardinality, weight, character, density, π-weight, π-character, pseudo-weight, and network weight, respectively (see, e.g., [1]).

In this paper we shall establish some relationship between cardinal invariants of X and $C_\omega(X)$.

Proposition 1. $\omega(C_\omega(X)) \le \exp \omega(X)$.

Proof. Let $\omega(X) = \tau$. As is well known, $\omega(C(X)) = \tau$ and $|C(C(X))| \le \exp\tau$. Since $C'(X) \subseteq C_p(C(X))$, we have $|C'(X)| \le \exp\tau$. Since $C_\omega(X)$ is a subspace of $C_p(C'(X))$, it follows that $\omega(C_\omega(X)) \le \omega(C_p(C'(X))) \le |C'(X)|$. And finally, $\omega(C_\omega(X)) \le \exp\omega(X)$, which completes the proof. □

To get further estimates we need the following general proposition ($\sigma(\cdot,\cdot)$ stands below for weak topology [2]).

Proposition 2. *Let E be a Banach space, $E_\omega = (E, \sigma(E, E'))$, and S_ω be the unit closed ball in E_ω. Then $d(E') \le \pi\chi(S_\omega) \le \pi\chi(E_\omega) \le \chi(E)$.*

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Proof. Let $\pi = \{V_\alpha : \alpha \in A\}$ be a π-base at the point $0 \in S_\omega$. One may assume that $|A| = \pi\chi(S_\omega)$ and $V_\alpha = S_\omega \cap \{x \in E : |f_{\alpha i}(x)| < \varepsilon_\alpha,\ i \le n(\alpha),\ f_{\alpha i} \in E'\}$. Let Z' be the linear hull of the set $\{f_{\alpha i} : \alpha \in A, i \le n(\alpha)\}$. Prove that Z' is dense in E'. Assume the contrary: there exists a form $f \in E' \setminus [Z']$. By the Hahn–Banach theorem there exists a linear form $g \in E''$ such that $g(f) = 1$ and $g(Z') = 0$. There is no loss of generality in assuming that $g \in S''$. Let $V = \{x \in E : f(x) < 2^{-1}\}$. There exists $\alpha \in A$ such that $S_\omega \cap V \subseteq V_\alpha$. The neighborhood

$$V(f_{\alpha i}, f, \varepsilon_\alpha, 2^{-1}, i \le n(\alpha)) = \left\{g' \in E'' : |g'(f_{\alpha i})| < \varepsilon_\alpha, |g'(f) - 1| < 2^{-1}\right\}$$

of the point g in $\sigma(E'', E')$-topology of the space E'' contains some element $k(x)$, where $x \in S_\omega$ and $k : E \to E'$ is a canonical embedding (this follows from Goldstine's theorem [3]). But then $x \in V_\alpha \setminus V$, which contradicts the choice of V_α. We have proved that $[Z'] = E'$; it follows that $\alpha(E') \le |Z'| = |A| \le \pi\chi(S_\omega)$. $\square$

Corollary 1. *For $E = C(X)$ the following is valid:*

$$|X| \le d(C'(X)) \le \pi\chi(S_\omega).$$

Indeed, the canonical mapping $q : q(t)(x) = x(t)$ transfers the set X onto a discrete subset of the space $C'(X)$ (since $|q(t) - q(t')| = 2$ for any points t, t' from X). It follows that $|X| \le d(C'(X))$, and we can apply Proposition 2.

Question 1. Is it true that $|X| = d(C'(X))$?

One more assertion. Let $F \subseteq E'$. Set $E_F = (E, \sigma(E, F))$.

Proposition 3. $\chi(E_F) = \omega(E_F)$.

Proof. Let $\{V_\alpha : \alpha \in A\}$ be a fundamental family of $\sigma(E, F)$-neighborhoods of zero in E_F such that $|A| \le \chi(E_F)$. We can suppose that $V_\alpha = \{x \in E : |f_{\alpha i}(x)| < \varepsilon_\alpha,\ f_{\alpha i} \in F,\ i \le n(\alpha)\}$. Fix some countable base π in $\mathbb{R}$. Put $\gamma = \{f_{\alpha i}^{-1}(V) : \alpha \in A,\ i \le n(\alpha),\ V \in \pi\}$. Then $|\gamma| = |A|$ and it suffices to show that γ is a subbase for the topology $\sigma(E, F)$. Let x_0 be an arbitrary point of E and let V be a $\sigma(E, F)$-neighborhood of x_0. The set $V - x_0$ is $\sigma(E, F)$-neighborhood of zero, so that there exists $\alpha \in A$ such that $V_\alpha \subseteq V - x_0$. Choose the sets $W_{\alpha i} \in \pi$ such that $f_{\alpha i}(x_0) \in W_{\alpha i} \subseteq (f_{\alpha i}(x_0) - \varepsilon_\alpha, f_{\alpha i}(x_0) + \varepsilon_\alpha)$, $i \le n(\alpha)$. Set $\Gamma = \cap\{f_{\alpha i}^{-1}(W_{\alpha i}) : i \le n(\alpha)\}$ and prove that $\Gamma \subseteq V$. Let $x \in \Gamma$. Then $f_{\alpha i} \in W_{\alpha i}$. Hence either $|f_{\alpha i}(x) - f_{\alpha i}(x_0)| < \varepsilon_\alpha$ or $|f_{\alpha i}(x - x_0)| < \varepsilon_\alpha$ for $i \le n(\alpha)$, and $x - x_0 \in V_\alpha \subseteq V - x_0$ or $x \in V$. We have proved that γ is a subbase and so $\omega(E_F) \le |\gamma| = |A| \le \chi(E_F)$. $\square$

Summing up the above arguments, we arrive at the following

Theorem 1.

$$\omega(X) = d(C(X)) \leq \pi\chi(C_\omega(X)) = \pi\omega(C_\omega(X)) = \\ = \chi(C_\omega(X)) = \omega(C_om(X)) \leq \exp\omega(X).$$

Proof. It is necessary here to employ Proposition 1, Corollary 1, Proposition 3, and the well-known fact that $\pi\chi = \chi$ and $\pi\omega = \omega$ for every topological group. □

Theorem 2.

$$|X| \leq \pi\chi(S_\omega) = \pi\omega(S_\omega) = \chi(S_\omega) = \omega(S_\omega) = d(C'(X)) \leq \exp\omega(X).$$

Proof. Since $d(C'(X)) \leq \omega(C'(X))$ and $n\omega(C_p(C'(X))) \leq \omega(C'(X))$, we have $n\omega(S''_\omega) \leq d(C'(X))$, where $C''_\omega = ((C'_\omega(X))', \sigma(C''(X), C'(X))) = (C''(X), \sigma(C''(X), C'(X)))$. As S'_ω is compact, $\omega(S''_\omega) = n\omega(S''_\omega)$. Since S_ω is embedded topologically in S''_ω, $\omega(S_\omega) \leq \omega(S''_\omega) = n\omega(S''_\omega) \leq d(C'(X))$. By Corollary 1, $\pi\chi(S_\omega) \geq d(C'(X))$, so that $\pi\chi(S_\omega) = \pi\omega(S_\omega) = \chi(S_\omega) = \omega(S_\omega) = d(C'(X))$. □

Question 2. Is it true that $\omega(E_\omega) \geq \omega(E)$?

Question 3. Is it true that $\omega(E') \geq \omega(E'_\omega)$?

Note that the weight of $C_\omega(X)$ does not necessarily coincide with the weight of S_ω. Indeed, let X be a convergent sequence of real numbers. Then $C(X)$ is isomorphic to the space C of all convergent sequences of real numbers. $C' = \ell_1$ is separable, hence its unit closed ball is metrizable in $C(X)$-topology, i.e., $\chi(S_\omega) = \omega(S_\omega) = \omega_0$. But $C_\omega(X)$ is nonmetrizable, so that $\chi(C_\omega(X)) > \omega(S_\omega)$.

The problem of ψ-characters for $C_p(X)$ was solved in [4]:

$$\psi(C_p(X)) \leq d(X). \tag{1}$$

Since $C_\omega(X)$ maps onto $C_p(X)$ one-to-one, from (1), it follows that

$$\psi(C_\omega(X)) \leq d(X). \tag{2}$$

Moreover, the following proposition is true.

Proposition 4. $\psi(C_\omega(X)) = p\omega(C_\omega(X)) = d(C'_\omega(X))$.

Proof. Let $\{V_\alpha : \alpha \in A\}$ be a subbase of $C_\omega(X)$ in zero having the least cardinality and consisting of standard sets $V_\alpha = \{x \in C(X) : f_{\alpha i}(x) < \varepsilon_\alpha, i \leq n(\alpha)\}$. Let T be the $C(X)$-closure of the linear hull of the set $\{f_{\alpha i} : \alpha \in A, i \leq n(\alpha)\}$. Prove that $T = C'(X)$. Suppose the contrary: there exists a point $g \in C'(X) \backslash T$. By the theorem on separation of convex sets there exists a $C(X)$-continuous linear form f such that $f(T) = 0$ and $f(y) > 0$. By virtue of the $C(X)$-continuity, $f \in C(X)$. For all αi we have $f_{\alpha i}(f) = 0$.

Hence $f \in V_\alpha$ for all α, but $\cap\{V_\alpha : \alpha \in A\} = \{0\}$. We have a contradiction. Consequently, $T = C'(X)$ and $\psi(C_\omega(X)) \geq d(C'_\omega)$.

Now let $M = \{g_\alpha : \alpha \in A\}$ be a $C'(X)$-dense set of cardinality $d(C'(X))$. Then M is $C(X)$-separating family and the diagonal of mappings g_α produces a condensation of $C_\omega(X)$ in the product space $\prod\{R_\alpha : \alpha \in A\}$. From this it follows that $p\omega(C_\omega(X)) \leq \omega(\prod\{R_\alpha : \alpha \in A\}) = |A|$. $\square$

Proposition 4 underlies the following assertion.

Theorem 3. *If X is an Eberline compactum, then $\psi(C_\omega(X)) = d(X)$.*

Proof. Let $Y \subseteq C_p(X)$ be an X-separating compactum. Then $\omega(Y) = d(Y) = d(X) = \omega(X)$. One may suppose that Y lies in S_ω. By the Grothendieck theorem [5], Y is compact in $C_\omega(X)$. Then $\omega(\gamma) = p\omega(Y) \leq p\omega(C_\omega(X)) = \psi(C_\omega(X))$ or $d(X) \leq \psi(C_\omega(X))$. Reference to Proposition 4 completes the proof. $\square$

Proposition 5. *The following statements are equivalent:*
(1) $C_\omega(X)$ *is a k-space;*
(2) $C_\omega(X)$ *is sequential;*
(3) $C_\omega(X)$ *is a Frechet–Urysohn space;*
(4) $C_\omega(X)$ *is metrizable;*
(5) $C_\omega(X)$ *is finite.*

Proof. It suffices to show that (1)⇒(5).

Suppose that X is infinite. Then $\dim C(X) = \infty$ and there exists a set A in $C(X)$ such that $0 \in [A]_\omega$, where $[A]_\omega$ is the weak closure of A, but the intersection of A with any bounded set is finite (see, e.g., [6]).

Let K be an arbitrary compact set in $C_\omega(X)$. Then K is a Frechet–Urysohn space. Hence, if $x \in [K \cap A]_\omega$, there exists a sequence $\{x_n\}$ of elements of A which converges to x. Any weak convergent sequence is always bounded. For the definition of A it follows that the set $\cup\{x_n : n \in \omega\}$ is finite, i.e., the sequence $\{x_n\}$ is stationary: $x_n \equiv x$ beginning with some n. Hence $x \in A$ and $K \cap A$ is weakly closed. But this means that $C_\omega(X)$ is not a k-space. $\square$

The situation with S_ω is somehow different. Here there are other criteria for metrizability.

Proposition 6. *S_ω is metrizable iff X is countable.*

Proof. If S_ω is metrizable, then X is countable by Theorem 2. If S_ω is countable, then $C_p(X)$ is metrizable. As X is compact, X is scattered. By the theorem from [7], S_ω is homeomorphic to S_p, hence S_ω is metrizable. $\square$

Then the following proposition is valid.

Proposition 7. *If X is a scattered compactum, then S_ω has the Frechet–Urysohn (FU) property.*

Proof. If X is scattered, then $C_p(X)$ satisfies the FU-property. S_ω is homeomorphic to S_p, hence S_ω has the FU-property. □

Consequently, if X is an uncountable scattered compactum, then S_ω is a nonmetrizable Frechet–Urysohn space.

In conclusion we shall prove the formula

$$d(C_\omega(X)) = hd(C_\omega(X)) = n\omega(C_\omega(X)) = \omega(X). \tag{3}$$

Proof. Obviously, $d(Z) \le hd(Z) \le n\omega(Z)$ for any Z. But $d(C_p(X)) = d(C(X)) = \omega(X)$ [8], from which it follows that $d(C_\omega(X)) = \omega(X)$. But $n\omega(C_\omega(X)) = \omega(X)$, which completes the proof. □

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