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ON A DARBOUX PROBLEM FOR A THIRD ORDER HYPERBOLIC EQUATION WITH MULTIPLE CHARACTERISTICS

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ABSTRACT. A Darboux type problem for a model hyperbolic equation of the third order with multiple characteristics is considered in the case of two independent variables. The Banach space $\overset{\circ}{C}{}^{2,1}_{\alpha}$, $\alpha \geq 0$, is introduced where the problem under consideration is investigated. The real number α_0 is found such that for $\alpha > \alpha_0$ the problem is solved uniquely and for $\alpha < \alpha_0$ it is normally solvable in Hausdorff's sense. In the class of uniqueness an estimate of the solution of the problem is obtained which ensures stability of the solution.

§ 1. Formulation of the Problem

In the plane of independent variables x, y let us consider a third order hyperbolic equation

$$u_{xxy} = f, \tag{1.1}$$

where f is the given and u the unknown real function.

Straight lines $y = const$ form a double family of characteristics of (1.1), while $x = const$ a single family.

Let $\gamma_i : \varphi = \varphi_i$, $0 \leq r < +\infty$, $i = 1, 2$, $0 \leq \varphi_1 < \varphi_2 \leq \frac{\pi}{2}$, be two rays coming out of the origin of the coordinates $O(0,0)$ written in terms of the polar coordinates r, φ. The angle formed by these rays will be denoted by $D : \varphi_1 < \varphi < \varphi_2$, $0 < r < +\infty$. Let P_1^0 and P_2^0 be the points at which γ_1 and γ_2 intersect respectively the characteristics $L_1(P^0) : x = x_0$ and $L_2(P^0) : y = y_0$ coming out of an arbitrarily taken point $P^0(x_0, y_0) \in D$. Equation (1.1) will be considered in the rectangular domain $D_0 : 0 < x <$

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x_0, $0 < y < y_0$, bounded by the characteristics $x = 0$, $x = x_0$ and $y = 0$, $y = y_0$.

Equations of segments OP_1^0, OP_2^0 of the rays γ_1, γ_2 will be written in terms of rectangular coordinates x, y as $OP_1^0 : y = \rho_1 x$, $0 \le x \le x_0$; $OP_2^0 : x = \rho_2 y$, $0 \le y \le y_0$, where $\rho_1 = \operatorname{tg}\varphi_1$, $\rho_2 = \operatorname{ctg}\varphi_2$ and $0 \le \rho_1\rho_2 = \frac{\operatorname{tg}\varphi_1}{\operatorname{tg}\varphi_2} < 1$.

For equation (1.1) we shall consider a Darboux type problem formulated as follows: Find in $\overline{D}_0$ a regular solution $u : \overline{D}_0 \to R$, $R \equiv (-\infty, +\infty)$, of equation (1.1), satisfying on the segments OP_1^0 and OP_2^0 the conditions

$$(M_1 u_{xx} + N_1 u_{xy} + P_1 u_x + Q_1 u_y + S_1 u)|_{OP_1^0} = f_1, \tag{1.2}$$

$$(M_2 u_{xx} + N_2 u_{xy} + P_2 u_x + Q_2 u_y + S_2 u)|_{OP_2^0} = f_2, \tag{1.3}$$

$$(M_3 u_{xx} + N_3 u_{xy} + P_3 u_x + Q_3 u_y + S_3 u)|_{OP_2^0} = f_3, \tag{1.4}$$

where $M_i,\ N_i,\ P_i,\ Q_i,\ S_i,\ f_i$, $i = 1, 2, 3$, are given continuous real functions.

A function $u : \overline{D}_0 \to R$ continuous in $\overline{D}_0$ together with its partial derivatives $D_x^i D_y^j u$, $i = 0, 1, 2$, $j = 0, 1$, $i + j > 0$, $D_x \equiv \frac{\partial}{\partial x}$, $D_y \equiv \frac{\partial}{\partial y}$ and satisfying equation (1.1) in $\overline{D}_0$ and conditions (1.2)–(1.4) is called a regular solution of problem (1.1)– (1.4).

It should be noted that the boundary value problem (1.1)–(1.4) is the natural development of the well-known classical formulations of Goursat and Darboux problems (see, for example, [1]–[4]) for second-order linear hyperbolic equations. Variants of Goursat and Darboux problems for one hyperbolic equation and systems of second order, and also for systems of first order, are investigated in some papers (see, for example, [5]–[13]). Note that the results obtained in [7]–[9] are new even for one equation and in a certain sense bear a complete character.

Initial-boundary and characteristic problems for a wide class of hyperbolic equations of third and higher order with dominating derivatives are treated in [14]–[19] and other papers.

Remark 1.1. Conditions (1.2)–(1.4) take into account the hyperbolic nature of problem (1.1)–(1.4) as they contain only the derivatives dominated by the derivative $D_x^2 D_y u$.

Remark 1.2. Since the family of characteristics $y = const$ is the double one for the hyperbolic equation (1.1), two conditions (1.3), (1.4) are given on the segment OP_2^0.

Remark 1.3. The above problem could be formulated also for an angular domain bounded by the rays γ_1, γ_2 and the characteristics $L_1(P^0)$, $L_2(P^0)$ of (1.1) under the same boundary conditions (1.2)–(1.4), but, as is well known, the solution $u(x,y)$ of the thus formulated problem continues into the domain D_0 as the solution of the original problem (1.1)–(1.4). The problem formulated in the form of (1.1)–(1.4) is convenient for further investigations provided that equation (1.1) contains dominated lowest terms and the Riemann function is effectively used.

Let us introduce the functional spaces

$$\overset{\circ}{C}_\alpha(\overline{D}_0) \equiv \{u : u \in C(\overline{D}_0), u(0) = 0, \sup_{z\neq 0, z\in \overline{D}_0} |z|^{-\alpha}|u(z)| < \infty\},$$
$$\alpha \geq 0, z = x + iy,$$
$$\overset{\circ}{C}_\alpha[0,d] \equiv \{\varphi : \varphi \in C[0,d], \varphi(0) = 0, \sup_{0<t\leq d} t^{-\alpha}|\varphi(t)| < \infty\},$$
$$\alpha \geq 0, d > 0.$$

For $\alpha = 0$ the above classes will be denoted by $\overset{\circ}{C}(\overline{D}_0)$ and $\overset{\circ}{C}[0,d]$, respectively.

Obviously, $\overset{\circ}{C}_\alpha(\overline{D}_0)$ and $\overset{\circ}{C}_\alpha[0,d]$ will be the Banach spaces with respect to the norms

$$\|u\|_{\overset{\circ}{C}_\alpha(\overline{D}_0)} = \sup_{z\neq 0, z\in\overline{D}_0} |z|^{-\alpha}|u(z)|, \quad \|\varphi\|_{\overset{\circ}{C}_\alpha[0,d]} = \sup_{0<t\leq d} t^{-\alpha}|\varphi(t)|,$$

respectively.

It is easy to see that the belonging of the functions $u \in \overset{\circ}{C}(\overline{D}_0)$ and $\varphi \in \overset{\circ}{C}[0,d]$ to the spaces $\overset{\circ}{C}_\alpha(\overline{D}_0)$ and $\overset{\circ}{C}_\alpha[0,d]$, respectively, is equivalent to the fulfillment of the following inequalities:

$$|u(z)| \leq c_1|z|^\alpha, \quad z \in \overline{D}_0, \tag{1.5}$$
$$|\varphi(t)| \leq c_2 t^\alpha, \quad t \in [0,d], \quad c_i \equiv const > 0, \quad i = 1, 2. \tag{1.6}$$

The boundary value problem (1.1)–(1.4) will be investigated in the space

$$\overset{\circ}{C}{}^{2,1}_{\alpha}(\overline{D}_0) \equiv \{u : D_x^i D_y^j u \in \overset{\circ}{C}_\alpha(\overline{D}_0),\ i = 0,1,2,\ j = 0,1\},$$

which is Banach with respect to the norm

$$\|u\|_{\overset{\circ}{C}{}^{2,1}_{\alpha}(\overline{D}_0)} = \sum_{i=0}^{2}\sum_{j=0}^{1} \|D_x^i D_y^j u\|_{\overset{\circ}{C}_\alpha(\overline{D}_0)}.$$

To consider (1.1)–(1.4) in the class $\overset{\circ}{C}{}^{2,1}_{\alpha}(\overline{D}_0)$ it is required that $f \in \overset{\circ}{C}_{\alpha}(\overline{D}_0)$, $M_1, N_1, P_1, Q_1, S_1 \in C[0,x_0]$, $M_i, N_i, P_i, Q_i, S_i \in C[0,y_0]$, $i=2,3$, $f_1 \in \overset{\circ}{C}_{\alpha}[0,x_0]$, $f_i \in \overset{\circ}{C}_{\alpha}[0,y_0]$, $i=2,3$.

§ 2. Integral Representation of a Regular Solution of the Class $\overset{\circ}{C}{}^{2,1}_{\alpha}(\overline{D}_0)$ of Equation (1.1)

The integration of equation (1.1) along the characteristics enables us to prove that the following lemma is valid.

Lemma 2.1. *The formula*

$$u(x,y)=\int\limits_0^x (x-\xi)\varphi(\xi)d\xi+\int\limits_0^y \psi(\eta)d\eta+x\int\limits_0^y \nu(\eta)d\eta+$$
$$+\int\limits_0^x\int\limits_0^y (x-\xi)f(\xi,\eta)d\xi d\eta,\quad (x,y)\in\overline{D}_0, \tag{2.1}$$

establishes the one-to-one correspondence between regular solutions $u(x,y)$ of the class $\overset{\circ}{C}{}^{2,1}_{\alpha}(\overline{D}_0)$ of equation (1.1) *and values $\varphi \in \overset{\circ}{C}_{\alpha}[0,x_0]$, $\psi,\nu \in \overset{\circ}{C}_{\alpha}[0,y_0]$. Note that $\varphi(x)=u_{xx}(x,0)$, $0\le x\le x_0$, $\psi(y)=u_y(0,y)$, $\nu(y)=u_{xy}(0,y)$, $0\le y\le y_0$.*

Proof. Introduce the notation $u(x,0)\equiv\varphi_1(x)$, $0\le x\le x_0$, $u(0,y)\equiv\psi_1(y)$, $u_x(0,y)\equiv\nu_1(y)$, $0\le y\le y_0$. It is obvious that $\varphi_1(0)=\varphi_1'(0)=\psi_1(0)=\nu_1(0)=0$. Further, by integrating (1.1) twice and one time along the characteristics $y=const$ and $x=const$, respectively, we obtain

$$u(x,y)=\varphi_1(x)+\psi_1(y)+x\nu_1(y)+$$
$$+\int\limits_0^x\int\limits_0^y (x-\xi)f(\xi,\eta)d\xi d\eta,\quad (x,y)\in\overline{D}_0. \tag{2.2}$$

Let $\varphi_1''(x)\equiv\varphi(x)$, $0\le x\le x_0$, $\psi_1'(y)\equiv\psi(y)$, $\nu_1'(y)\equiv\nu(y)$, $0\le y\le y_0$. Then (2.2) takes the form of (2.1).

From $u \in \overset{\circ}{C}{}^{2,1}_{\alpha}(\overline{D}_0)$ it follows that $\varphi \in \overset{\circ}{C}_{\alpha}[0,x_0]$, $\psi,\nu \in \overset{\circ}{C}_{\alpha}[0,y_0]$. To prove the converse statement note that by (1.5), (1.6) we have the estimates

$$|\varphi(x)|\le c_3x^{\alpha},\ x\in[0,x_0],\ |\psi(y)|\le c_4y^{\alpha},\ |\nu(y)|\le c_5y^{\alpha},\ y\in[0,y_0],$$
$$|f(x,y)|\le c_6|z|^{\alpha},\ z\equiv(x,y)\in\overline{D}_0,\ c_{2+i}\equiv const>0,\ i=1,2,3.$$

Setting $\zeta = (\xi, \eta)$, from formula (2.1) we have

$$|u(x,y)| \leq c_3 x_0 \int\limits_0^x \xi^\alpha d\xi + c_4 \int\limits_0^y \eta^\alpha d\eta + c_5 x_0 \int\limits_0^y \eta^\alpha d\eta +$$
$$+ c_6 x_0 \int\limits_0^x \int\limits_0^y |\zeta|^\alpha d\xi d\eta \leq c_7 |z|^\alpha,$$

where $c_7 \equiv \frac{1}{\alpha+1}(c_3 x_0^2 + c_4 y_0 + c_5 x_0 y_0) + c_6 x_0^2 y_0 > 0$, $z \in \overline{D}_0$.

Hence $u \in \overset{\circ}{C}_\alpha (\overline{D}_0)$. In a similar but relatively simpler manner it is proved that $D_x^i D_y^j u \in \overset{\circ}{C}_\alpha (\overline{D}_0)$, $i = 0,1,2$, $j = 0,1$, $i + j > 0$, and therefore $u \in \overset{\circ}{C}{}_\alpha^{2,1}(\overline{D}_0)$. $\square$

§ 3. Reducing Problem (1.1)–(1.4) to a System of Integro-Functional Equations

Substituting (2.1) into conditions (1.2)–(1.4) gives us

$$\begin{cases} M_1(x)\varphi(x) + \int_0^x [P_1(x) + S_1(x)(x-\xi)]\varphi(\xi)d\xi + Q_1(x)\psi(\rho_1 x) + \\ \qquad + S_1(x) \int_0^{\rho_1 x} \psi(\eta)d\eta + [N_1(x) + xQ_1(x)]\nu(\rho_1 x) + [P_1(x) + \\ \qquad + xS_1(x)] \int_0^{\rho_1 x} \nu(\eta)d\eta = \widetilde{f}_1(x), \ \ 0 \leq x \leq x_0, \\ Q_i(y)\psi(y) + S_i(y) \int_0^y \psi(\eta)d\eta + [N_i(y) + \rho_2 y Q_i(y)]\nu(y) + \\ \qquad + [P_i(y) + \rho_2 y S_i(y)] \int_0^y \nu(\eta)d\eta + M_i(y)\varphi(\rho_2 y) + \int_0^{\rho_2 y} [P_i(y) + \\ \qquad + S_i(y)(\rho_2 y - \xi)]\varphi(\xi)d\xi = \widetilde{f}_i(y), \ \ i = 2,3, \ \ 0 \leq y \leq y_0, \end{cases} \tag{3.1}$$

where

$$\begin{cases} \widetilde{f}_1(x) \equiv f_1(x) - M_1(x) \int_0^{\rho_1 x} f(x,\eta)d\eta - N_1(x) \int_0^x f(\xi, \rho_1 x)d\xi - \\ \qquad - P_1(x) \int_0^x \int_0^{\rho_1 x} f(\xi,\eta)d\xi\, d\eta - Q_1(x) \int_0^x (x-\xi) f(\xi, \rho_1 x)d\xi - \\ \qquad - S_1(x) \int_0^x \int_0^{\rho_1 x} (x-\xi) f(\xi,\eta)d\xi\, d\eta, \ 0 \leq x \leq x_0, \\ \widetilde{f}_i(y) \equiv f_i(y) - M_i(y) \int_0^y f(\rho_2 y, \eta)d\eta - N_i(y) \int_0^{\rho_2 y} f(\xi, y)d\xi - \\ \qquad - P_i(y) \int_0^{\rho_2 y} \int_0^y f(\xi,\eta)d\xi\, d\eta - Q_i(y) \int_0^{\rho_2 y} (\rho_2 y - \xi) f(\xi, y)d\xi - \\ \qquad - S_i(y) \int_0^{\rho_2 y} \int_0^y (\rho_2 y - \xi) f(\xi,\eta)d\xi\, d\eta, \ i = 2,3, \ 0 \leq y \leq y_0. \end{cases} \tag{3.2}$$

Transferring all the integral terms contained in system (3.1) to the right-hand side, we have

$$M_1(x)\varphi(x)+Q_1(x)\psi(\rho_1x)+R_1(x)\nu(\rho_1x)=F_1(x),\ 0\le x\le x_0, \quad (3.3)$$
$$Q_2(y)\psi(y)+R_2(y)\nu(y)+M_2(y)\varphi(\rho_2y)=F_2(y),\ 0\le y\le y_0, \quad (3.4)$$
$$Q_3(y)\psi(y)+R_3(y)\nu(y)+M_3(y)\varphi(\rho_2y)=F_3(y),\ 0\le y\le y_0, \quad (3.5)$$

where

$$F_1(x)\equiv \widetilde{f}_1(x)-\int_0^x[P_1(x)+S_1(x)(x-\xi)]\varphi(\xi)d\xi-S_1(x)\int_0^{\rho_1x}\psi(\eta)d\eta-$$
$$-[P_1(x)+xS_1(x)]\int_0^{\rho_1x}\nu(\eta)d\eta,\ \ 0\le x\le x_0,$$
$$F_i(y)\equiv \widetilde{f}_i(y)-S_i(y)\int_0^y\psi(\eta)d\eta-[P_i(y)+\rho_2yS_i(y)]\int_0^y\nu(\eta)d\eta-$$
$$-\int_0^{\rho_2y}[P_i(y)+S_i(y)(\rho_2y-\xi)]\varphi(\xi)d\xi,\ \ i=2,3,\ \ 0\le y\le y_0,$$
$$R_1(x)\equiv N_1(x)+xQ_1(x),\ 0\le x\le x_0,$$
$$R_i(y)\equiv N_i(y)+\rho_2yQ_i(y),\ i=2,3,\ 0\le y\le y_0.$$

Rewrite equations (3.4) and (3.5) as follows:

$$Q_i(y)\psi(y)+R_i(y)\nu(y)=F_i(y)-M_i(y)\varphi(\rho_2y),$$
$$i=2,3,\ \ 0\le y\le y_0. \quad (3.6)$$

Assuming that

$$\Delta(y)\equiv\begin{vmatrix} Q_2(y) & N_2(y)\\ Q_3(y) & N_3(y)\end{vmatrix}\ne 0,\ 0\le y\le y_0, \quad (3.7)$$

we find from system (3.6) that

$$\psi(y)=a_1(y)-b_1(y)\varphi(\rho_2y),$$
$$\nu(y)=a_2(y)-b_2(y)\varphi(\rho_2y),\ 0\le y\le y_0, \quad (3.8)$$

where

$$a_1(y)\equiv\Delta^{-1}(y)[F_2(y)R_3(y)-F_3(y)R_2(y)],$$
$$b_1(y)\equiv\Delta^{-1}(y)[M_2(y)R_3(y)-M_3(y)R_2(y)],$$
$$a_2(y)\equiv\Delta^{-1}(y)[F_3(y)Q_2(y)-F_2(y)Q_3(y)],$$
$$b_2(y)\equiv\Delta^{-1}(y)[M_3(y)Q_2(y)-M_2(y)Q_3(y)],\ 0\le y\le y_0.$$

Note that here and in what follows the upper index -1 means the inverse value.

Let

$$M_1(x) \neq 0, \;\; 0 \leq x \leq x_0. \tag{3.9}$$

If the obtained expressions for the functions $\psi(y)$, $\nu(y)$, $0 \leq y \leq y_0$, are substituted from (3.8) into equality (3.3), then we shall have

$$\varphi(x) - a(x)\varphi(\tau_0 x) = F(x), \;\; 0 \leq x \leq x_0, \tag{3.10}$$

where $a(x) \equiv M_1^{-1}(x)[Q_1(x)b_1(\rho_1 x)+R_1(x)b_2(\rho_1 x)]$, $F(x) \equiv M_1^{-1}(x)[F_1(x) - Q_1(x)a_1(\rho_1 x) - R_1(x)a_2(\rho_1 x)]$, $0 \leq x \leq x_0$, $\tau_0 \equiv \rho_1\rho_2$.

Simple calculations lead to

$$\begin{cases} F(x) = \int_0^x K_1(x,\xi)\varphi(\xi)d\xi + \int_0^{\tau_0 x} K_2(x,\xi)\varphi(\xi)d\xi + \\ \qquad +K_3(x)\int_0^{\rho_1 x}\psi(\eta)d\eta + K_4(x)\int_0^{\rho_1 x}\nu(\eta)d\eta + F_4(x), \; 0 \leq x \leq x_0, \\ a_1(y) = \int_0^{\rho_2 y} K_5(\xi,y)\varphi(\xi)d\xi + K_6(y)\int_0^y \psi(\eta)d\eta + \\ \qquad +K_7(y)\int_0^y \nu(\eta)d\eta + F_5(y), \; 0 \leq y \leq y_0, \\ a_2(y) = \int_0^{\rho_2 y} K_8(\xi,y)\varphi(\xi)d\xi + K_9(y)\int_0^y \psi(\eta)d\eta + \\ \qquad +K_{10}(y)\int_0^y \nu(\eta)d\eta + F_6(y), \; 0 \leq y \leq y_0, \end{cases} \tag{3.11}$$

where $K_1(x,\xi)$, $0 \leq x \leq x_0$, $0 \leq \xi \leq x$, $K_2(x,\xi)$, $0 \leq x \leq x_0$, $0 \leq \xi \leq \tau_0 x$, $K_i(x)$, $0 \leq x \leq x_0$, $i = 3,4$, $K_i(\xi,y)$, $0 \leq \xi \leq \rho_2 y$, $0 \leq y \leq y_0$, $i = 5,8$, $K_i(y)$, $0 \leq y \leq y_0$, $i = 6,7,9,10$, expressed in terms of the M_1, N_i, P_i, Q_i, S_i, ρ_1, ρ_2, $i = 1,2,3$, are continuous kernels of the integral terms contained on the right-hand sides of system (3.11), while the functions F_i, $i = 4,5,6$, denote the following values:

$$\begin{cases} F_4(x) \equiv M_1^{-1}(x)\widetilde{f}_1(x) + [M_1^{-1}(x)R_1(x)\Delta^{-1}(\rho_1 x)Q_3(\rho_1 x) - \\ \qquad -M_1^{-1}(x)Q_1(x)\Delta^{-1}(\rho_1 x)R_3(\rho_1 x)]\widetilde{f}_2(\rho_1 x) + \\ \qquad +[M_1^{-1}(x)Q_1(x)\Delta^{-1}(\rho_1 x)R_2(\rho_1 x) - \\ \qquad -M_1^{-1}(x)R_1(x)\Delta^{-1}(\rho_1 x)Q_2(\rho_1 x)]\widetilde{f}_3(\rho_1 x), \; 0 \leq x \leq x_0, \\ F_5(y) \equiv \Delta^{-1}(y)R_3(y)\widetilde{f}_2(y) - \Delta^{-1}(y)R_2(y)\widetilde{f}_3(y), \; 0 \leq y \leq y_0, \\ F_6(y) \equiv \Delta^{-1}(y)Q_2(y)\widetilde{f}_3(y) - \Delta^{-1}(y)Q_3(y)\widetilde{f}_2(y), \; 0 \leq y \leq y_0. \end{cases} \tag{3.12}$$

Introducing the notation

$$(K\varphi)(x) \equiv \varphi(x) - a(x)\varphi(\tau_0 x), \;\; 0 \leq x \leq x_0, \tag{3.13}$$

equalities (3.8), (3.10) due to (3.11) we can write as

$$\begin{cases}(K\varphi)(x) = \int_0^x K_1(x,\xi)\varphi(\xi)d\xi + \int_0^{\tau_0 x} K_2(x,\xi)\varphi(\xi)d\xi + \\ \quad +K_3(x)\int_0^{\rho_1 x}\psi(\eta)d\eta + K_4(x)\int_0^{\rho_1 x}\nu(\eta)d\eta + F_4(x),\ 0 \le x \le x_0, \\ \psi(y) = \int_0^{\rho_2 y} K_5(\xi,y)\varphi(\xi)d\xi + K_6(y)\int_0^y \psi(\eta)d\eta + \\ \quad +K_7(y)\int_0^y \nu(\eta)d\eta - b_1(y)\varphi(\rho_2 y) + F_5(y),\ 0 \le y \le y_0, \\ \nu(y) = \int_0^{\rho_2 y} K_8(\xi,y)\varphi(\xi)d\xi + K_9(y)\int_0^y \psi(\eta)d\eta + \\ \quad +K_{10}(y)\int_0^y \nu(\eta)d\eta - b_2(y)\varphi(\rho_2 y) + F_6(y),\ 0 \le y \le y_0.\end{cases} \tag{3.14}$$

Remark 3.1. It is obvious that if conditions (3.7), (3.9) are fulfilled, then in the class $\overset{\circ}{C}{}_{\alpha}^{2,1}(\overline{D}_0)$ problem (1.1)–(1.4) is equivalent to the system of equations (3.14) with respect to the unknowns $\varphi \in \overset{\circ}{C}_{\alpha}[0,x_0]$, $\psi,\nu \in \overset{\circ}{C}_{\alpha}[0,y_0]$.

§ 4. Invertibility of the Functional Operator K Defined by Equality (3.13)

Assume that conditions (3.7), (3.9) are fulfilled. Set $\sigma \equiv a(0)$ and $\alpha_0 \equiv -\log|\sigma|/\log\tau_0$ $(\sigma \neq 0)$.

Lemma 4.1. *Let either γ_1 or γ_2 be the characteristic of equation* (1.1) *(i.e., $\tau_0 = 0$). Then the equation*

$$(K\varphi)(x) = g(x),\ \ 0 \le x \le x_0, \tag{4.1}$$

has a unique solution in the space $\overset{\circ}{C}_{\alpha}[0,x_0]$ for all $\alpha \ge 0$.

The proof follows from the fact that under the assumption of the lemma K is the identity operator in the space $\overset{\circ}{C}_{\alpha}[0,x_0]$.

Lemma 4.2. *Let the straight lines γ_1, γ_2 be not the characteristics of equation* (1.1) *(i.e., $0 < \tau_0 < 1$) and $\sigma \neq 0$. Then for $\alpha > \alpha_0$ equation* (4.1) *has a unique solution in the space $\overset{\circ}{C}_{\alpha}[0,x_0]$ and for the inverse operator K^{-1} we have the estimate*

$$|(K^{-1}g)(x)| \le cx^{\alpha}\|g\|_{\overset{\circ}{C}_{\alpha}[0,x]}, \tag{4.2}$$

where the positive constant c does not depend on the function g.

Proof. We introduce into consideration the operators

$$(\Gamma\varphi)(x) = a(x)\varphi(\tau_0 x),\ 0 \le x \le x_0,\ \ K^{-1} = I + \sum_{j=1}^{\infty} \Gamma^j, \tag{4.3}$$

where I is the identity operator. It is easy to see that the operator K^{-1} is formally inverse to the operator K. Thus it enough for us to prove that the Neuman series $I + \sum_{j=1}^{\infty} \Gamma^j$ converges in the space $\overset{\circ}{C}_\alpha [0, x_0]$.

By the definition of the operator Γ from (4.3) we have $(\Gamma^j\varphi)(x) = a(x)a(\tau_0 x)\dots a(\tau_0^{j-1}x)\varphi(\tau_0^j x)$, $0 \le x \le x_0$. The condition $\alpha > \alpha_0$ is equivalent to the inequality $\tau_0^\alpha|\sigma| < 1$. Therefore by virtue of the continuity of the function a and the equality $a(0) = \sigma$ there are positive numbers ε ($\varepsilon < x_0$), δ and q such that the inequalities

$$|a(x)| \le |\sigma| + \delta,\ \ \tau_0^\alpha(|\sigma| + \delta) \equiv q < 1 \tag{4.4}$$

will hold for $0 \le x \le \varepsilon$.

It is obvious that the sequence $\{\tau_0^j x\}_{j=0}^{\infty}$ uniformly converges to zero as $j \to \infty$ on the segment $[0, x_0]$. Therefore there is a natural number j_0 such that

$$\tau_0^j x \le \varepsilon,\ \text{ for }\ 0 \le x \le x_0,\ j \ge j_0. \tag{4.5}$$

We can take as j_0, say, $j_0 = \left[\frac{\log\ \varepsilon x_0^{-1}}{\log\ \tau_0}\right] + 1$, where $[p]$ denotes the integral part of the number p.

Let $\max\limits_{0\le x\le x_0} |a(x)| \equiv \beta$. By virtue of (4.4), (4.5) the following estimates hold for $j > j_0$, $g \in \overset{\circ}{C}_\alpha [0, x_0]$:

$$\begin{aligned} |(\Gamma^j g)(x)| &= |a(x)a(\tau_0 x)\dots a(\tau_0^{j_0-1}x)| \cdot |a(\tau_0^{j_0}x)\dots a(\tau_0^{j-1}x)| \cdot |g(\tau_0^j x)| \le \\ &\le \beta^{j_0}(|\sigma| + \delta)^{j-j_0}(\tau_0^j x)^\alpha \|g\|_{\overset{\circ}{C}_\alpha[0,x]} \le \\ &\le \beta^{j_0}(|\sigma| + \delta)^{-j_0}\Big(\tau_0^\alpha(|\sigma| + \delta)\Big)^j x^\alpha \|g\|_{\overset{\circ}{C}_\alpha[0,x]} = c_0 q^j x^\alpha \|g\|_{\overset{\circ}{C}_\alpha[0,x]}, \end{aligned} \tag{4.6}$$

where $c_0 \equiv \beta^{j_0}(|\sigma| + \delta)^{-j_0}$.

For $1 \le j \le j_0$ we have

$$|(\Gamma^j g)(x)| \le \beta^j(\tau_0^j x)^\alpha \|g\|_{\overset{\circ}{C}_\alpha[0,x]} \le \beta^j x^\alpha \|g\|_{\overset{\circ}{C}_\alpha[0,x]}. \tag{4.7}$$

Now by (4.6) and (4.7) we eventually have

$$|\varphi(x)| = |(K^{-1}g)(x)| \leq |g(x)| + \Big|\sum_{j=1}^{j_0}(\Gamma^j g)(x)\Big| + \Big|\sum_{j=j_0+1}^{\infty}(\Gamma^j g)(x)\Big| \leq$$
$$\leq \Big(1 + \sum_{j=1}^{j_0}\beta^j + c_0\sum_{j=j_0+1}^{\infty}q^j\Big)x^\alpha\|g\|_{\overset{\circ}{C}_\alpha[0,x]} =$$
$$= \Big(1 + \sum_{j=1}^{j_0}\beta^j + c_0\frac{q^{j_0+1}}{1-q}\Big)x^\alpha\|g\|_{\overset{\circ}{C}_\alpha[0,x]},$$

from which we obtain the continuity of the operator K^{-1} in the space $\overset{\circ}{C}_\alpha\,[0,x_0]$ and the validity of estimate (4.2). □

Remark 4.1. If $\sigma = 0$, then the inequality $\tau_0^\alpha|\sigma| < 1$ is fulfilled for any $\alpha \geq 0$ and, as seen from the proof, in that case Lemma 4.2 holds for all $\alpha \geq 0$.

Lemma 4.3. *Let the straight lines γ_1, γ_2 be not the characteristics of equation* (1.1) (*i.e.,* $0 < \tau_0 < 1$) *and $\sigma \neq 0$. Then equation* (4.1) *is solvable in the space $\overset{\circ}{C}_\alpha\,[0,x_0]$ for $\alpha < \alpha_0$ and the homogeneous equation corresponding to* (4.1) *has in the said space an infinite number of linearly independent solutions, i.e.,* $\dim \operatorname{Ker} K = \infty$.

Proof. The condition $\alpha < \alpha_0$ is equivalent to the inequality $\tau_0^\alpha|\sigma| > 1$. Therefore, as in proving Lemma 4.2, there are positive numbers ε_1 ($\varepsilon_1 < x_0$), δ_1 and q_1 such that the inequalities

$$|a^{-1}(x)| \leq (|\sigma| - \delta_1)^{-1},\ |\sigma| - \delta_1 > 0,\ \tau_0^\alpha(|\sigma| - \delta_1) \equiv q_1^{-1} > 1 \quad (4.8)$$

will hold for $0 \leq x \leq \varepsilon_1$.

It is easy to see that the operator Γ from (4.3) is invertible and

$$(\Gamma^{-1}\varphi)(x) = a^{-1}(\tau_0^{-1}x)\varphi(\tau_0^{-1}x),\ \ 0 \leq x \leq \tau_0\varepsilon_1.$$

Rewrite (4.3) in the equivalent form

$$\varphi(x) - (\Gamma^{-1}\varphi)(x) = -(\Gamma^{-1}g)(x),\ \ 0 \leq x \leq \tau_0\varepsilon_1. \quad (4.9)$$

Obviously, for any x from the interval $0 < x < \tau_0\varepsilon_1$ there exists a unique natural number $n_1 = n_1(x)$ satisfying the inequalities

$$\tau_0\varepsilon_1 < \tau_0^{-n_1}x \leq \varepsilon_1.$$

It is easy to verify that

$$n_1(x) = \left[\frac{\log\ \varepsilon_1^{-1}x}{\log\ \tau_0}\right] \geq \frac{\log\ \varepsilon_1^{-1}x}{\log\ \tau_0} - 1. \tag{4.10}$$

Similarly, for $\varepsilon_1 < x \leq x_0$ there exists a unique natural number $n_2 = n_2(x)$ satisfying the inequalities

$$\tau_0\varepsilon_1 \leq \tau_0^{n_2}x < \varepsilon_1.$$

Clearly, $n_2(x) = \left[1 - \frac{\log\ \varepsilon_1^{-1}x}{\log\ \tau_0}\right]$.

One can easily verify that any continuous solution on the half-interval $0 < x \leq x_0$ of equation (4.1) or (4.9) is given by the formula

$$\varphi(x) = \begin{cases} \varphi^0(x), \quad \tau_0\varepsilon_1 \leq x \leq \varepsilon_1, \\ (\Gamma^{-n_1(x)}\varphi^0)(x) - \sum_{j=1}^{n_1(x)}(\Gamma^{-j}g)(x), \ \ 0 < x < \tau_0\varepsilon_1, \\ (\Gamma^{n_2(x)}\varphi^0)(x) - \sum_{j=0}^{n_2(x)-1}(\Gamma^{j}g)(x), \ \ x > \varepsilon_1, \end{cases} \tag{4.11}$$

where φ^0 is an arbitrary function from the class $C[\tau_0\varepsilon_1, \varepsilon_1]$, satisfying the condition $\varphi^0(\varepsilon_1) - a(\varepsilon_1)\varphi^0(\tau_0\varepsilon_1) = g(\varepsilon_1)$.

Let us show that the function φ given by (4.11) belongs to the class $\overset{\circ}{C}_\alpha\,[0, x_0]$ for $g \in \overset{\circ}{C}_\alpha\,[0, x_0]$. The arbitrariness of φ^0 implies that Lemma 4.3 holds for equation (4.1).

By (4.8), (4.10) the estimates

$$\begin{aligned} |(\Gamma^{-n_1(x)}\varphi(x)| &= |a^{-1}(\tau_0^{-1}x)a^{-1}(\tau_0^{-2}x)\dots a^{-1}(\tau_0^{-n_1(x)}x)\varphi(\tau_0^{-n_1(x)}x)| \leq \\ &\leq (|\sigma| - \delta_1)^{-n_1(x)}\|\varphi^0\|_{C[\tau_0\varepsilon_1,\varepsilon_1]} \leq \tau_0^{\alpha n_1(x)}\|\varphi^0\|_{C[\tau_0\varepsilon_1,\varepsilon_1]} \leq \\ &\leq \tau_0^{\left(\frac{\log\ \varepsilon_1^{-1}x}{\log\ \tau_0} - 1\right)\alpha}\|\varphi^0\|_{C[\tau_0\varepsilon_1,\varepsilon_1]} = \tau_0^{-\alpha}\varepsilon_1^{-\alpha}x^\alpha\|\varphi^0\|_{C[\tau_0\varepsilon_1,\varepsilon_1]} \end{aligned} \tag{4.12}$$

hold for $0 < x < \tau_0\varepsilon_1$.

In a similar manner for $0 < x < \tau_0\varepsilon_1$ and $1 \leq j \leq n_1(x)$ we have

$$\begin{aligned} |(\Gamma^{-j}g)(x)| &\leq (|\sigma| - \delta_1)^{-j}(\tau_0^{-j}x)^\alpha\|g\|_{\overset{\circ}{C}_\alpha[0,x_0]} = \\ &= [\tau_0^\alpha(|\sigma| - \delta_1)]^{-j}x^\alpha\|g\|_{\overset{\circ}{C}_\alpha[0,x_0]} = q_1^j x^\alpha\|g\|_{\overset{\circ}{C}_\alpha[0,x_0]}. \end{aligned}$$

Hence it follows that

$$\begin{aligned} \Big|\sum_{j=1}^{n_1(x)}(\Gamma^{-j}g)(x)\Big| &\leq \Big(\sum_{j=1}^{n_1(x)} q_1^j\Big)x^\alpha\|g\|_{\overset{\circ}{C}_\alpha[0,x_0]} \leq \\ &\leq \frac{q_1}{1 - q_1}x^\alpha\|g\|_{\overset{\circ}{C}_\alpha[0,x_0]}. \end{aligned} \tag{4.13}$$

By (4.12) and (4.13) we conclude that the function φ given by (4.11) and being a solution of equation (4.1) belongs to the class $\overset{\circ}{C}_\alpha\,[0,x_0]$. $\square$

§ 5. Proof of the Main Results

Theorem 5.1. *If at least either γ_1 or γ_2 is the characteristic of equation* (1.1) (*i.e.*, $\tau_0=0$) *and conditions* (3.7), (3.9) *are fulfilled, then problem* (1.1)–(1.4) *has a unique solution in the class $\overset{\circ}{C}{}^{2,1}_{\alpha}(\overline{D}_0)$ for all $\alpha\geq 0$.*

Theorem 5.2. *Let conditions* (3.7), (3.9) *be fulfilled and the straight lines γ_1, γ_2 be not the characteristics of equation* (1.1) (*i.e.*, $0<\tau_0<1$). *If the equality $\sigma=0$ holds, then problem* (1.1)–(1.4) *is uniquely solvable in the class $\overset{\circ}{C}{}^{2,1}_{\alpha}(\overline{D}_0)$ for all $\alpha\geq 0$. If however $\sigma\neq 0$, then problem* (1.1)–(1.4) *is uniquely solvable in the class $\overset{\circ}{C}{}^{2,1}_{\alpha}(\overline{D}_0)$ for $\alpha>\alpha_0$, while for $\alpha<\alpha_0$ problem* (1.1)–(1.4) *is normally solvable in Hausdorff's sense in the class $\overset{\circ}{C}{}^{2,1}_{\alpha}(\overline{D}_0)$ and its index $\varkappa=+\infty$. In particular, the homogeneous problem corresponding to* (1.1)–(1.4) *has an infinite number of linearly independent solutions.*

Proof. Rewrite system (3.14) in terms of the new unknown functions

$$\psi(y)+b_1(y)\varphi(\rho_2 y)\equiv\omega(y),\ \nu(y)+b_2(y)\varphi(\rho_2 y)\equiv\lambda(y),\ 0\leq y\leq y_0,$$

as

$$\begin{cases}(K\varphi)(x)=\int_0^x K_{11}(x,\xi)\varphi(\xi)d\xi+K_3(x)\int_0^{\rho_1 x}\omega(\eta)d\eta+\\ \qquad +K_4(x)\int_0^{\rho_1 x}\lambda(\eta)d\eta+F_4(x),\ 0\leq x\leq x_0,\\ \omega(y)=\int_0^{\rho_2 y}K_{12}(\xi,y)\varphi(\xi)d\xi+K_6(y)\int_0^y\omega(\eta)d\eta+\\ \qquad +K_7(y)\int_0^y\lambda(\eta)d\eta+F_5(y),\ 0\leq y\leq y_0,\\ \lambda(y)=\int_0^{\rho_2 y}K_{13}(\xi,y)\varphi(\xi)d\xi+K_9(y)\int_0^y\omega(\eta)d\eta+\\ \qquad +K_{10}(y)\int_0^y\lambda(\eta)d\eta+F_6(y),\ 0\leq y\leq y_0,\end{cases}\tag{5.1}$$

where

$$K_{11}(x,\xi)\equiv K_1(x,\xi)+K_1^*(x,\xi),$$

$$K_1^*(x,\xi)\equiv\begin{cases}K_2(x,\xi)+\rho_2^{-1}[K_3(x)b_1(\rho_2^{-1}\xi)+K_4(x)b_2(\rho_2^{-1}\xi)],\ 0\leq\xi\leq\tau_0 x,\\ 0,\quad \tau_0 x<\xi\leq x,\end{cases}$$

$$K_{12}(\xi,y)\equiv K_5(\xi,y)+\rho_2^{-1}[K_6(y)b_1(\rho_2^{-1}\xi)+K_7(y)b_2(\rho_2^{-1}\xi)],$$

$$K_{13}(\xi,y)\equiv K_8(\xi,y)+\rho_2^{-1}[K_9(y)b_1(\rho_2^{-1}\xi)+K_{10}(y)b_2(\rho_2^{-1}\xi)].\quad\square$$

Remark 5.1. If $\rho_2 = 0$, then $\psi(y) \equiv \omega(y)$, $\nu(y) \equiv \lambda(y)$, $0 \le y \le y_0$, and the introduction of the new unknown functions ω and λ is superfluous.

We can rewrite system (5.1) in terms of the new independent variables $x = x_0 t$, $y = y_0 t$, $\xi = x_0\tau$, $\eta = y_0\tau$, $0 \le t, \tau \le 1$, as

$$\begin{cases} (K\widetilde{\varphi})(t) = \int_0^t \widetilde{K}_{11}(t,\tau)\widetilde{\varphi}(\tau)d\tau + \widetilde{K}_3(t)\int_0^{\tau_1 t}\widetilde{\omega}(\tau)d\tau + \\ \qquad +\widetilde{K}_4(t)\int_0^{\tau_1 t}\widetilde{\lambda}(\tau)d\tau + \widetilde{F}_4(t),\ 0 \le t \le 1,\ 0 < \frac{\rho_1 x_0}{y_0} \equiv \tau_1 < 1, \\ \widetilde{\omega}(t) = \int_0^{\tau_2 t}\widetilde{K}_{12}(\tau,t)\widetilde{\varphi}(\tau)d\tau + \widetilde{K}_6(t)\int_0^t \widetilde{\omega}(\tau)d\tau + \\ \qquad +\widetilde{K}_7(t)\int_0^t \widetilde{\lambda}(\tau)d\tau + \widetilde{F}_5(t),\ 0 \le t \le 1,\ 0 < \frac{\rho_2 y_0}{x_0} \equiv \tau_2 < 1, \\ \widetilde{\lambda}(t) = \int_0^{\tau_2 t}\widetilde{K}_{13}(\tau,t)\widetilde{\varphi}(\tau)d\tau + \widetilde{K}_9(t)\int_0^t \widetilde{\omega}(\tau)d\tau + \\ \qquad +\widetilde{K}_{10}(t)\int_0^t \widetilde{\lambda}(\tau)d\tau + \widetilde{F}_6(t),\ 0 \le t \le 1, \end{cases} \tag{5.2}$$

where the functions with waves are expressions of the corresponding functions in terms of the variables t and τ, for example, $\varphi(x) = \varphi(x_0 t) \equiv \widetilde{\varphi}(t)$, $\omega(y) = \omega(y_0 t) \equiv \widetilde{\omega}(t)$, $\widetilde{K}_{11}(t,\tau) \equiv x_0 K_{11}(x_0 t, x_0\tau)$, $\widetilde{K}_{12}(\tau,t) \equiv x_0 K_{12}(x_0\tau, y_0 t)$, $\widetilde{K}_3(t) \equiv y_0 K_3(x_0 t)$, $\widetilde{K}_6(t) \equiv y_0 K_6(y_0 t)$, $0 \le t,\ \tau \le 1$.

Let $T_i(\widetilde{\varphi},\widetilde{\omega},\widetilde{\lambda})$, $i = 1,2,3$, be the linear integral operators acting by the formulas

$$\begin{cases} T_1(\widetilde{\varphi},\widetilde{\omega},\widetilde{\lambda})(t) \equiv \int_0^t \widetilde{K}_{11}(t,\tau)\widetilde{\varphi}(\tau)d\tau + \widetilde{K}_3(t)\int_0^{\tau_1 t}\widetilde{\omega}(\tau)d\tau + \\ \qquad +\widetilde{K}_4(t)\int_0^{\tau_1 t}\widetilde{\lambda}(\tau)d\tau,\ 0 \le t \le 1,\ 0 < \tau_1 < 1, \\ T_2(\widetilde{\varphi},\widetilde{\omega},\widetilde{\lambda})(t) \equiv \int_0^{\tau_2 t}\widetilde{K}_{12}(\tau,t)\widetilde{\varphi}(\tau)d\tau + \widetilde{K}_6(t)\int_0^t \widetilde{\omega}(\tau)d\tau + \\ \qquad +\widetilde{K}_7(t)\int_0^t \widetilde{\lambda}(\tau)d\tau,\ 0 \le t \le 1,\ 0 < \tau_2 < 1, \\ T_3(\widetilde{\varphi},\widetilde{\omega},\widetilde{\lambda})(t) \equiv \int_0^{\tau_2 t}\widetilde{K}_{13}(\tau,t)\widetilde{\varphi}(\tau)d\tau + \widetilde{K}_9(t)\int_0^t \widetilde{\omega}(\tau)d\tau + \\ \qquad +\widetilde{K}_{10}(t)\int_0^t \widetilde{\lambda}(\tau)d\tau,\ 0 \le t \le 1. \end{cases} \tag{5.3}$$

Remark 5.2. The integral operators T_i, $i = 1,2,3$, acting by formulas (5.3) are Volterra type operators.

To prove Theorems 5.1 and 5.2 we shall solve system (5.2) for the unknown functions $\widetilde{\varphi} \in \overset{\circ}{C}_\alpha$, $\widetilde{\omega} \in \overset{\circ}{C}_\alpha$, $\widetilde{\lambda} \in \overset{\circ}{C}_\alpha$, using the method of successive approximations.

Set $\widetilde{\varphi}_0(t) \equiv 0$, $\widetilde{\omega}_0(t) \equiv 0$, $\widetilde{\lambda}_0(t) \equiv 0$, $0 \le t \le 1$, and for $n \ge 1$,

$$(K\widetilde{\varphi}_n)(t) = \int_0^t \widetilde{K}_{11}(t,\tau)\widetilde{\varphi}_{n-1}(\tau)d\tau + \widetilde{K}_3(t)\int_0^{\tau_1 t}\widetilde{\omega}_{n-1}(\tau)d\tau +$$

$$+\widetilde{K}_4(t)\int_0^{\tau_1 t}\widetilde{\lambda}_{n-1}(\tau)d\tau + \widetilde{F}_4(t),\ 0 \le t \le 1, \tag{5.4}$$

$$\widetilde{\omega}_n(t) = \int\limits_0^{\tau_2 t} \widetilde{K}_{12}(\tau,t)\widetilde{\varphi}_{n-1}(\tau)d\tau + \widetilde{K}_6(t)\int\limits_0^t \widetilde{\omega}_{n-1}(\tau)d\tau +$$

$$+ \widetilde{K}_7(t)\int\limits_0^t \widetilde{\lambda}_{n-1}(\tau)d\tau + \widetilde{F}_5(t),\ 0 \le t \le 1, \tag{5.5}$$

$$\widetilde{\lambda}_n(t) = \int\limits_0^{\tau_2 t} \widetilde{K}_{13}(\tau,t)\widetilde{\varphi}_{n-1}(\tau)d\tau + \widetilde{K}_9(t)\int\limits_0^t \widetilde{\omega}_{n-1}(\tau)d\tau +$$

$$+ \widetilde{K}_{10}(t)\int\limits_0^t \widetilde{\lambda}_{n-1}(\tau)d\tau + \widetilde{F}_6(t),\ 0 \le t \le 1, \tag{5.6}$$

where the operator K acts by (3.13).

Using estimate (4.2) and taking into account Remark 5.2, we shall prove below that under the assumptions of Lemma 4.1 or Lemma 4.2 we have the estimates

$$|\widetilde{\varphi}_{n+1}(t) - \widetilde{\varphi}_n(t)| \le M\frac{L^n}{n!}t^{n+\alpha}, \tag{5.7}$$

$$|\widetilde{\omega}_{n+1}(t) - \widetilde{\omega}_n(t)| \le M\frac{L^n}{n!}t^{n+\alpha}, \tag{5.8}$$

$$|\widetilde{\lambda}_{n+1}(t) - \widetilde{\lambda}_n(t)| \le M\frac{L^n}{n!}t^{n+\alpha}, \tag{5.9}$$

where $M = M(M_i, N_i, P_i, Q_i, S_i, f_i, i = 1,2,3, f, c, \rho_1, \rho_2) > 0$, $L = L(M_i, N_i, P_i, Q_i, S_i, i = 1,2,3, c, \rho_1, \rho_2) > 0$ are sufficiently large positive numbers which do not depend on n and which are to be defined, while c is the constant from (4.2).

Proof of Estimates (5.7)–(5.9). Since a restriction is imposed on f, f_i, $i = 1,2,3$, we have $\widetilde{F}_{3+i} \in \overset{\circ}{C}_\alpha$, $i = 1,2,3$. Indeed, by (1.6) we have $|f_1(x)| \le k_1 x^\alpha$, $k_1 > 0$, $\alpha \ge 0$, $x \in [0, x_0]$. Further, the first of equalities (3.2) gives

$$|\widetilde{f}_1(x)| \le k_1 x^\alpha + k_4 c_6 \int\limits_0^{\rho_1 x} (x^2+\eta^2)^{\alpha/2} d\eta + k_4 c_6 \int\limits_0^x (\xi^2 + \rho_1^2 x^2)^{\alpha/2} d\xi +$$

$$+ k_4 c_6 \int\limits_0^x \int\limits_0^{\rho_1 x} (\xi^2+\eta^2)^{\alpha/2} d\xi d\eta \le k_5 x^\alpha,\ x \in [0, x_0],\ \alpha \ge 0,$$

where $k_4 \equiv k_4(M_1, N_1, P_1, Q_1, S_1)$, $k_5 \equiv k_5(k_1, k_4, c_6, x_0, \alpha, \rho_1)$ are the completely defined positive numbers.

Hence we conclude that $\widetilde{f}_1 \in \overset{\circ}{C}_\alpha$. The case $\widetilde{f}_2, \widetilde{f}_3 \in \overset{\circ}{C}_\alpha$ is proved similarly.

Taking into account the expressions of the functions F_{3+i}, $i = 1, 2, 3$, from (3.12), it is now easy to establish that $\widetilde{F}_{3+i} \in \overset{\circ}{C}_\alpha$, $i = 1, 2, 3$. Therefore by (1.6) the following estimates hold: $|\widetilde{F}_{3+i}(t)| \leq \theta_{3+i} t^\alpha$ or $t^{-\alpha}|\widetilde{F}_{3+i}(t)| \leq \theta_{3+i}$, $i = 1, 2, 3$, $\alpha \geq 0$, $t \in [0, 1]$. If in this inequality t is replaced by $s \in [0, t]$, then by the definition of a norm in the space $\overset{\circ}{C}_\alpha\,[0, t]$ we shall have

$$\|\widetilde{F}_{3+i}\|_{\overset{\circ}{C}_\alpha[0,t]} \leq \theta_{3+i} \quad i = 1, 2, 3, \ \ \forall t \in [0, 1]. \tag{5.10}$$

Since $\widetilde{\varphi}_0(t) \equiv \widetilde{\omega}_0(t) \equiv \widetilde{\lambda}_0(t) \equiv 0$, $0 \leq t \leq 1$, and under the assumptions of Lemma 4.2 estimate (4.2) holds, from (5.4), (5.10) we shall have

$$\begin{aligned} |\widetilde{\varphi}_1(t) - \widetilde{\varphi}_0(t)| = |\widetilde{\varphi}_1(t)| = |(K^{-1}\widetilde{F}_4)(t)| \leq \\ \leq ct^\alpha \|\widetilde{F}_4\|_{\overset{\circ}{C}_\alpha[0,t]} \leq c\theta_4 t^\alpha. \end{aligned} \tag{5.11}$$

(5.5), (5.10) in turn imply

$$|\widetilde{\omega}_1(t) - \widetilde{\omega}_0(t)| = |\widetilde{\omega}_1(t)| = |\widetilde{F}_5(t)| \leq \theta_5 t^\alpha. \tag{5.12}$$

Similarly, (5.6), (5.10) give

$$|\widetilde{\lambda}_1(t) - \widetilde{\lambda}_0(t)| = |\widetilde{\lambda}_1(t)| = |\widetilde{F}_6(t)| \leq \theta_6 t^\alpha. \tag{5.13}$$

Assuming that estimates (5.7)–(5.9) hold for n, $n > 0$, let us prove that they are valid for $n + 1$ for sufficiently large M and L.

Denote by $\widetilde{K}$ the largest of the numbers $\sup\limits_{(t,\tau)\in[0,1]\times[0,1]} |\widetilde{K}_{1i}(t, \tau)|$, $i = 1, 2, 3$, $\sup\limits_{t\in[0,1]} |\widetilde{K}_i(t)|$, $i = 3, 4, 6, 7, 9, 10$.

From (5.4) we have

$$K(\widetilde{\varphi}_{n+2} - \widetilde{\varphi}_{n+1})(t) = T(\widetilde{\varphi}_{n+1} - \widetilde{\varphi}_n, \widetilde{\omega}_{n+1} - \widetilde{\omega}_n, \widetilde{\lambda}_{n+1} - \widetilde{\lambda}_n)(t), \tag{5.14}$$

where

$$T(\widetilde{\varphi}_{n+1} - \widetilde{\varphi}_n, \widetilde{\omega}_{n+1} - \widetilde{\omega}_n, \widetilde{\lambda}_{n+1} - \widetilde{\lambda}_n)(t) \equiv \int\limits_0^t \widetilde{K}_{11}(t, \tau)(\widetilde{\varphi}_{n+1} - \widetilde{\varphi}_n)(\tau)d\tau +$$

$$+\widetilde{K}_3(t) \int\limits_0^{\tau_1 t} (\widetilde{\omega}_{n+1} - \widetilde{\omega}_n)(\tau)d\tau + \widetilde{K}_4(t) \int\limits_0^{\tau_1 t} (\widetilde{\lambda}_{n+1} - \widetilde{\lambda}_n)(\tau)d\tau.$$

Further, for the right-hand side of (5.14) we have the estimate

$$T(\widetilde{\varphi}_{n+1}-\widetilde{\varphi}_n,\widetilde{\omega}_{n+1}-\widetilde{\omega}_n,\widetilde{\lambda}_{n+1}-\widetilde{\lambda}_n)(t)\le \widetilde{K}M\frac{L^n}{n!}\int\limits_0^t \tau^{n+\alpha}d\tau+$$
$$+\widetilde{K}M\frac{L^n}{n!}\int\limits_0^{\tau_1 t}\tau^{n+\alpha}d\tau+\widetilde{K}M\frac{L^n}{n!}\int\limits_0^{\tau_1 t}\tau^{n+\alpha}d\tau\le$$
$$\le 3\widetilde{K}M\frac{L^n}{(n+1)!}t^{n+1+\alpha}. \tag{5.15}$$

As in deriving inequality (5.10), we shall have

$$\|T(\widetilde{\varphi}_{n+1}-\widetilde{\varphi}_n,\widetilde{\omega}_{n+1}-\widetilde{\omega}_n,\widetilde{\lambda}_{n+1}-\widetilde{\lambda}_n)\|_{\overset{\circ}{C}_\alpha[0,t]}\le 3\widetilde{K}M\frac{L^n}{(n+1)!}t^{n+1}.$$

Now (4.2), (5.14), and (5.15) imply

$$|(\widetilde{\varphi}_{n+2}-\widetilde{\varphi}_{n+1})(t)|=|\{K^{-1}T(\widetilde{\varphi}_{n+1}-\widetilde{\varphi}_n,\widetilde{\omega}_{n+1}-\widetilde{\omega}_n,\widetilde{\lambda}_{n+1}-\widetilde{\lambda}_n)\}(t)|\le$$
$$\le ct^\alpha\|T(\widetilde{\varphi}_{n+1}-\widetilde{\varphi}_n,\widetilde{\omega}_{n+1}-\widetilde{\omega}_n,\widetilde{\lambda}_{n+1}-\widetilde{\lambda}_n)\|_{\overset{\circ}{C}_\alpha[0,t]}\le$$
$$\le 3c\widetilde{K}M\frac{L^n}{(n+1)!}t^{n+1+\alpha}. \tag{5.16}$$

Similarly, from (5.5) and (5.6) we find

$$\begin{aligned}|(\widetilde{\omega}_{n+2}-\widetilde{\omega}_{n+1})(t)|&\le 3c\widetilde{K}M\frac{L^n}{(n+1)!}t^{n+1+\alpha}.\\ |(\widetilde{\lambda}_{n+2}-\widetilde{\lambda}_{n+1})(t)|&\le 3c\widetilde{K}M\frac{L^n}{(n+1)!}t^{n+1+\alpha}.\end{aligned} \tag{5.17}$$

From (5.11)–(5.13), (5.16), and (5.17) it immediately follows that if we set

$$M=\max\{c\theta_4,\theta_5,\theta_6\},\;\; L=\max\{3c\widetilde{K},3\widetilde{K}\}, \tag{5.18}$$

then estimates (5.7)–(5.9) shall be valid for any integer $n\ge 0$.

(5.7)–(5.9) imply that the series

$$\begin{cases}\widetilde{\varphi}(t)=\lim_{n\to\infty}\widetilde{\varphi}_n(t)=\sum_{n=0}^\infty(\widetilde{\varphi}_{n+1}(t)-\widetilde{\varphi}_n(t)),\\ \widetilde{\omega}(t)=\lim_{n\to\infty}\widetilde{\omega}_n(t)=\sum_{n=0}^\infty(\widetilde{\omega}_{n+1}(t)-\widetilde{\omega}_n(t)),\\ \widetilde{\lambda}(t)=\lim_{n\to\infty}\widetilde{\lambda}_n(t)=\sum_{n=0}^\infty(\widetilde{\lambda}_{n+1}(t)-\widetilde{\lambda}_n(t)),\;0\le t\le 1,\end{cases} \tag{5.19}$$

converge in the space $\overset{\circ}{C}_\alpha\,[0,1]$ and by virtue of (5.4)–(5.6) the limit functions $\widetilde{\varphi}$, $\widetilde{\omega}$, $\widetilde{\lambda}$ satisfy system (5.2). Returning to the previous variables x, y and the functions φ, ψ, ν, we thus conclude that these values satisfy system

(3.14). Further, by Lemma 2.1 the function u represented by formula (2.1) belongs to the class $\overset{\circ}{C}{}^{2,1}_{\alpha}(\overline{D}_0)$. It is thereby shown that in the plane of the variables x, y the function u is a solution of problem (1.1)–(1.4) belonging to the class $\overset{\circ}{C}{}^{2,1}_{\alpha}(\overline{D}_0)$.

We shall now show that problem (1.1)–(1.4) has no other solutions in the class $\overset{\circ}{C}{}^{2,1}_{\alpha}(\overline{D}_0)$. Indeed, let the function u^0 be a solution of the homogeneous problem corresponding to (1.1)–(1.4) belonging to the class $\overset{\circ}{C}{}^{2,1}_{\alpha}(\overline{D}_0)$. Then the functions $\widetilde{\varphi}^0(t) \equiv \varphi^0(x_0t) = \varphi^0(x) \equiv u^0_{xx}(x,0)$, $\widetilde{\omega}^0(t) \equiv \omega^0(y_0t) = \omega^0(y) \equiv \psi^0(y) + b_1(y)\varphi^0(\rho_2y) \equiv u^0_y(0,y) + b_1(y)u^0_{xx}(\rho_2y,0)$, $\widetilde{\lambda}^0(t) \equiv \lambda^0(y_0t) = \lambda^0(y) \equiv \nu^0(y) + b_2(y)\varphi^0(\rho_2y) \equiv u^0_{xy}(0,y) + b_2(y)u^0_{xx}(\rho_2y,0)$ satisfy the homogeneous system of equations

$$\begin{cases} (K\widetilde{\varphi}^0)(t) = \int_0^t \widetilde{K}_{11}(t,\tau)\widetilde{\varphi}^0(\tau)d\tau + \widetilde{K}_3(t)\int_0^{\tau_1 t}\widetilde{\omega}^0(\tau)d\tau + \\ \qquad +\widetilde{K}_4(t)\int_0^{\tau_1 t}\widetilde{\lambda}^0(\tau)d\tau, \quad 0 \le t \le 1, \\ \widetilde{\omega}^0(t) = \int_0^{\tau_2 t}\widetilde{K}_{12}(\tau,t)\widetilde{\varphi}^0(\tau)d\tau + \widetilde{K}_6(t)\int_0^t \widetilde{\omega}^0(\tau)d\tau + \\ \qquad +\widetilde{K}_7(t)\int_0^t \widetilde{\lambda}^0(\tau)d\tau, \quad 0 \le t \le 1, \\ \widetilde{\lambda}^0(t) = \int_0^{\tau_2 t}\widetilde{K}_{13}(\tau,t)\widetilde{\varphi}^0(\tau)d\tau + \widetilde{K}_9(t)\int_0^t \widetilde{\omega}^0(\tau)d\tau + \\ \qquad +\widetilde{K}_{10}(t)\int_0^t \widetilde{\lambda}^0(\tau)d\tau, \quad 0 \le t \le 1. \end{cases} \tag{5.20}$$

Apply the method of successive approximations to system (5.20), taking the functions $\widetilde{\varphi}^0$, $\widetilde{\omega}^0$, $\widetilde{\lambda}^0$ themselves as zero approximations. Since these functions satisfy system (5.20), each next aprroximation will coincide with the latter, i.e., $\widetilde{\varphi}^0_n(t) \equiv \widetilde{\varphi}^0(t)$, $\widetilde{\omega}^0_n(t) \equiv \widetilde{\omega}^0(t)$, $\widetilde{\lambda}^0_n(t) \equiv \widetilde{\lambda}^0(t)$, $0 \le t \le 1$. Recalling that these functions satisfy estimates of form (1.6), by a reasoning similar to that used in deriving inequalities (5.7)–(5.9) we obtain

$$\begin{aligned} |\widetilde{\varphi}^0(t)| &= |\widetilde{\varphi}^0_{n+1}(t)| \le M_0\frac{L_0^n}{n!}t^{n+\alpha}, \\ |\widetilde{\omega}^0(t)| &= |\widetilde{\omega}^0_{n+1}(t)| \le M_0\frac{L_0^n}{n!}t^{n+\alpha}, \\ |\widetilde{\lambda}^0(t)| &= |\widetilde{\lambda}^0_{n+1}(t)| \le M_0\frac{L_0^n}{n!}t^{n+\alpha}, \end{aligned}$$

where M_0 and L_0 are positive constants defined as M and L. When $n \to \infty$ we obtain $\widetilde{\varphi}^0 \equiv \widetilde{\omega}^0 \equiv \widetilde{\lambda}^0 \equiv 0$ or, which is the same, $\widetilde{\varphi}^0 \equiv \widetilde{\psi}^0 \equiv \widetilde{\nu}^0 \equiv 0$. Finally, by (2.1) we have $u^0(x,y) \equiv 0$ everywhere in $\overline{D}_0$.

We have thus proved Theorem 5.1 and also the first part of Theorem 5.2. To prove the second part of Theorem 5.2, rewrite system (5.2) as a single equation

$$K_1\widetilde{\chi} + T_4\widetilde{\chi} = \widetilde{F}, \tag{5.21}$$

where $\widetilde{\chi} \equiv (\widetilde{\varphi}, \widetilde{\omega}, \widetilde{\lambda}) \in \overset{\circ}{C}_\alpha \times \overset{\circ}{C}_\alpha \times \overset{\circ}{C}_\alpha$, $K_1\widetilde{\chi} \equiv (K\widetilde{\varphi}, I\widetilde{\omega}, I\widetilde{\lambda})$, $T_4\widetilde{\chi} \equiv \big(T_1(\widetilde{\varphi}, \widetilde{\omega}, \widetilde{\lambda}),\ T_2(\widetilde{\varphi}, \widetilde{\omega}, \widetilde{\lambda}),\ T_3(\widetilde{\varphi}, \widetilde{\omega}, \widetilde{\lambda})\big)$, $\widetilde{F} \equiv (\widetilde{F}_4, \widetilde{F}_5, \widetilde{F}_6)$, and the operators T_i, $i = 1, 2, 3$, are defined by (5.3).

It is obvious that the operator T_4 is compact in the space $\overset{\circ}{C}{}^3_\alpha[0,1] \equiv \overset{\circ}{C}_\alpha [0,1] \times \overset{\circ}{C}_\alpha [0,1] \times \overset{\circ}{C}_\alpha [0,1]$, since each of the operators T_i, $i = 1, 2, 3$, is represented as the sum of completely defined linear integral operators of the Volterra type in the space $\overset{\circ}{C}_\alpha [0,1]$.

Now let us show that under the assumptions of the second part of Theorem 5.2, i.e., under

$$\sigma \neq 0, \quad \alpha < \alpha_0, \tag{5.22}$$

the equation

$$K_1\widetilde{\chi} = \widetilde{\Phi} \tag{5.23}$$

is normally solvable in Hausdorff's sense (see, for example, [6]) in the space $\overset{\circ}{C}{}^3_\alpha[0,1]$ and its index $\varkappa = +\infty$, i.e., the image of the space $\overset{\circ}{C}{}^3_\alpha[0,1]$ at the mapping K_1 is closed in this very space and $\dim \operatorname{Ker} K_1 = +\infty$, $\dim \operatorname{Ker} K_1^* < +\infty$, where K_1^* is the conjugate operator of K_1. To this end, as is easy to see, it is enough to show that the equation

$$K\varphi = \widetilde{\Phi}_1 \tag{4.1$'$}$$

possesses the said property in the space $\overset{\circ}{C}_\alpha [0,1]$.

Lemma 4.3 implies by (5.22) that in the space $\overset{\circ}{C}_\alpha [0,1]$ equation (4.1$'$) is normally solvable in Hausdorff's sense and its index $\varkappa = +\infty$, $\dim \operatorname{Ker} K = +\infty$, $\dim \operatorname{Ker} K^* = 0$. Therefore equation (5.23), too, is also normally solvable in Hausdorff's sense in Banach space $\overset{\circ}{C}{}^3_\alpha[0,1]$ and its index $\varkappa = +\infty$. Hence, in turn, it follows that equation (5.21), too, possesses this property in the space $\overset{\circ}{C}{}^3_\alpha[0,1]$, since the operator T_4 is compact and the property of the equation to be normally solvable and to have an index equal to $+\infty$ is stable at compact perturbations (see, for example, [20]).

The latter arguments prove the second part of Theorem 5.2, since in the class $\overset{\circ}{C}{}^{2,1}_\alpha(\overline{D}_0)$ problem (1.1)–(1.4) is equivalently reduced to equation (5.21) in the space $\overset{\circ}{C}{}^3_\alpha[0,1]$. $\square$

Remark 5.3. It should be noted that the value α_0 appearing in the solvability conditions of problem (1.1)–(1.4) depends only on the value at the point $O(0,0)$ of the coefficients M_i, N_i, Q_i, $i = 1, 2, 3$, and on the value $\tau_0 = \rho_1\rho_2$,
since a simple verification shows that $a(0) = M_1^{-1}\Delta^{-1}(M_2N_3Q_1 - M_3N_2Q_1 + M_3N_1Q_2 - M_2N_1Q_3)(0)$.

§ 6. Estimation of a Regular Solution of Problem (1.1)–(1.4) Belonging to the Class $\overset{\circ}{C}{}^{2,1}_{\alpha}(\overline{D}_0)$

Below it will be shown that if the conditions of Theorems 5.1 and 5.2 guaranteeing the unique solvability of problem (1.1)–(1.4) are fulfilled, then for the solution u of this problem belonging to the class $\overset{\circ}{C}{}^{2,1}_{\alpha}(\overline{D}_0)$ we have the estimate

$$\|u\|_{\overset{\circ}{C}{}^{2,1}_{\alpha}(\overline{D}_0)} \leq C\big(\|f_1\|_{\overset{\circ}{C}_{\alpha}[0,x_0]} + \|f_2\|_{\overset{\circ}{C}_{\alpha}[0,y_0]} + \\ + \|f_3\|_{\overset{\circ}{C}_{\alpha}[0,y_0]} + \|f\|_{\overset{\circ}{C}_{\alpha}(\overline{D}_0)}\big) \equiv CC^*(f_1,f_2,f_3,f), \tag{6.1}$$

where C is a positive constant not depending on f, f_i, $i=1,2,3$.

The proof of the above statement will be divided into two parts. First we shall prove that for the solution u of the class $\overset{\circ}{C}{}^{2,1}_{\alpha}(\overline{D}_0)$ of problem (1.1)–(1.4) we have the estimate

$$\|u\|_{\overset{\circ}{C}{}^{2,1}_{\alpha}(\overline{D}_0)} \leq C_1 C^*(\varphi,\psi,\nu,f), \tag{6.2}$$

where C_1 is a positive constant not depending on f, $\varphi(x) = u_{xx}(x,0)$, $0 \leq x \leq x_0$, $\psi(y) = u_y(0,y)$, $\nu(y) = u_{xy}(0,y)$, $0 \leq y \leq y_0$.

Indeed, similarly to the proof of Lemma 2.1, by virtue of the definition of norms in the spaces $\overset{\circ}{C}_{\alpha}[0,d]$, $\overset{\circ}{C}_{\alpha}(\overline{D}_0)$ formula (2.1) yields

$$|u(x,y)| \leq x_0\|\varphi\|_{\overset{\circ}{C}_{\alpha}[0,x_0]} \int\limits_0^x \xi^{\alpha} d\xi + \|\psi\|_{\overset{\circ}{C}_{\alpha}[0,y_0]} \int\limits_0^y \eta^{\alpha} d\eta + \\ + x_0\|\nu\|_{\overset{\circ}{C}_{\alpha}[0,y_0]} \int\limits_0^y \eta^{\alpha} d\eta + x_0\|f\|_{\overset{\circ}{C}_{\alpha}(\overline{D}_0)} \int\limits_0^x \int\limits_0^y |\zeta|^{\alpha} d\xi d\eta \leq \\ \leq \frac{x_0^2}{\alpha+1}\|\varphi\|_{\overset{\circ}{C}_{\alpha}[0,x_0]}|z|^{\alpha} + \frac{y_0^2}{\alpha+1}\|\psi\|_{\overset{\circ}{C}_{\alpha}[0,y_0]}|z|^{\alpha} + \\ + \frac{x_0y_0}{\alpha+1}\|\nu\|_{\overset{\circ}{C}_{\alpha}[0,y_0]}|z|^{\alpha} + x_0^2y_0\|f\|_{\overset{\circ}{C}_{\alpha}(\overline{D}_0)}|z|^{\alpha},$$

from which it follows that

$$\|u\|_{\overset{\circ}{C}_{\alpha}(\overline{D}_0)} \leq C_{0,0} C^*(\varphi,\psi,\nu,f), \tag{6.3}$$

where $C_{0,0} \equiv \max\Big\{\frac{x_0^2}{\alpha+1}, \frac{y_0}{\alpha+1}, \frac{x_0y_0}{\alpha+1}, x_0^2y_0\Big\}$. In a similar manner one can prove the estimates

$$\|D_x^i D_y^j u\|_{\overset{\circ}{C}_{\alpha}(\overline{D}_0)} \leq C_{i,j} C^*(\varphi,\psi,\nu,f), \tag{6.4}$$

where $C_{i,j}$, $i=0,1,2$, $j=0,1$, $i+j>0$, are positive constants not depending on the functions f, φ, ψ, ν.

Estimates (6.3), (6.4) immediately imply (6.2) where $C_1 \equiv \sum\limits_{i=0}^{2}\sum\limits_{j=0}^{1} C_{ij}$.

Now let us prove the second part of the statement. From (5.7) and (5.19) we immediately have

$$|\varphi(x)| \le \sum_{n=0}^{\infty} |\varphi_{n+1}(x) - \varphi_n(x)| \le M\Big(\frac{x}{x_0}\Big)^{\alpha} \sum_{n=0}^{\infty} \frac{L^n}{n!}\Big(\frac{x}{x_0}\Big)^{n} =$$
$$= M\Big(\frac{x}{x_0}\Big)^{\alpha} e^{L\frac{x}{x_0}}, \quad 0 \le x \le x_0.$$

Hence we easily obtain

$$\|\varphi\|_{\overset{\circ}{C}_\alpha[0,x_0]} \le M\gamma, \tag{6.5}$$

where $\gamma \equiv x_0^{-\alpha} e^{L}$.

Similar estimates hold for the functions ψ, ν as well:

$$\|\psi\|_{\overset{\circ}{C}_\alpha[0,y_0]} \le M\widetilde{\gamma}, \ \ \|\nu\|_{\overset{\circ}{C}_\alpha[0,y_0]} \le M\widetilde{\gamma}, \tag{6.6}$$

where $\widetilde{\gamma} \equiv y_0^{-\alpha} L$.

By (5.18) and the proof of inequality (5.10) it is easy to see that we can take as M the value

$$M = C_2 C^*(f_1, f_2, f_3, f), \tag{6.7}$$

where C_2 is a sufficiently large positive constant not depending on f, f_i, $i=1,2,3$.

Finally, with regard to (6.7) inequalities (6.2), (6.5), (6.6) give estimate (6.1), where the positive constant C is expressed in terms of C_1, C_2, γ, $\widetilde{\gamma}$.

Estimate (6.1) immediately implies that the solution of problem (1.1)–(1.4) is stable.

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