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ON A RELATIONSHIP BETWEEN THE INTEGRABILITIES OF VARIOUS MAXIMAL FUNCTIONS

L. EPHREMIDZE

Abstract. It is shown that the right-sided, left-sided, and symmetric maximal functions of any measurable function can be integrable only simultaneously. The analogous statement is proved for the ergodic maximal functions.

Introduction. We deal with integrable functions on $\mathbb{T}=[0,2\pi)$ and assume that they are extended to 2π-periodic functions on the whole line $\mathbb{R}$. The class of such functions will be denoted by L. One can also consider the functions of L to be defined on the unit circle in the complex plane.

If a measurable set $E\subset\mathbb{R}$ is such that $\mathbb{I}_E$ is a 2π-periodic function and $f\in L$, then we assume that $|E|=\nu E=\nu(E\cap\mathbb{T})$ and

$$\int\limits_E f d\nu=\int\limits_{E\cap\pi} f d\nu$$

(ν denotes the Lebesgue measure on the line).

We shall say that a subset $\Delta\subset\mathbb{R}$ is a segment of $\mathbb{T}$ if it is the preimage of an open arc of the unit circle by the exponential function. The set of such segments is denoted by $\mathcal{E}$. If $\Delta\in\mathcal{E}$, $\Delta\neq\mathbb{R}$ and (a,b) is a connected component of Δ, then we shall write $\Delta=(a,b)$, which should not cause any confusion. Obviously, in that case $|\Delta|=b-a$.

Let $x\in\mathbb{T}$. We introduce the following notations of subsets of $\mathcal{E}$:

$$\begin{aligned}
&\mathcal{E}_0(x)=\{(a,b)\in\mathcal{E}: a<x<b\},\\
&\mathcal{E}_1(x)=\{(a,b)\in\mathcal{E}: b=x\},\\
&\mathcal{E}_2(x)=\{(a,b)\in\mathcal{E}: a=x\},\\
&\mathcal{E}_3(x)=\{(a,b)\in\mathcal{E}: \frac{a+b}{2}=x\}.
\end{aligned}$$

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Consider the maximal operators M_j, $j = 0, 1, 2, 3$, defined by the equalities

$$M_j(f)(x) = \sup_{\Delta \in \mathcal{E}_j(x)} \frac{1}{|\Delta|} \Big| \int\limits_{\Delta} f d\nu \Big|, \quad f \in L.$$

It is wellknown that $f \in L \lg^+ L \Rightarrow M_j(f) \in L$, $j = 0, 1, 2, 3$, and if $f \geq 0$, then the inverse implication is true (see [1], [2]). But, in general, one cannot write explicitly the set of functions f for which $M_j(f)$ is integrable (in connection with this see [2], [3]). In this paper we shall show that for an arbitrary $f \in L$ the functions $M_j(f)$, $j = 0, 1, 2, 3$, can be integrable only simultaneously. An analogous statement is proved for the ergodic maximal functions in §2.

The author's interest in this investigation was due to the question posed by Prof. L. Gogoladze (personal communication).

§ 1. Obviously, $M_0(f) \geq M_j(f)$, $j = 1, 2, 3$. We shall prove the following theorems.

Theorem 1. *Let $f \in L$ and $M_1(f) \notin L$. Then $M_3(f) \notin L$.*

Theorem 2. *Let $f \in L$. Then*

$$M_1(f) \notin L \Leftrightarrow M_2(f) \notin L.$$

Since $M_0(f) \leq M_1(f) + M_2(f)$, Theorems 1 and 2 enable us to conclude that the functions $M_j(f)$, $j = 1, 2, 3$, are nonintegrable whenever $M_0(f)$ is nonintegrable.

We begin by proving some lemmas. Their proofs are given in the form simplifying their extension to the ergodic case.

Let M be the operator

$$M(f)(x) = \sup_{a<x} \frac{1}{x-a} \int\limits_a^x f d\nu, \quad f \in L.$$

Evidently, $\{x \in \mathbb{R} : M(f)(x) > t\} = (M(f) > t)$ is an open subset of $\mathbb{R}$ for each t.

Lemma 1. *Let $f \in L$, $t > 0$, and let (a, b) be a finite (i.e., $a \neq -\infty$, $b \neq \infty$) connected component of $(M(f) > t)$. Then we have*

$$\frac{1}{x-a} \int\limits_a^x f d\nu > t \tag{1}$$

for each $x \in (a, b)$.

This lemma was actually proved in [4] but we give it here for the sake of completeness.

Proof. Suppose $h(x) = \int_a^x f d\nu - t(x-a)$, $x \in \mathbb{R}$. Note that whenever $y < x$ we have $h(y) < h(x) \Leftrightarrow \frac{1}{x-y}\int_y^x f d\nu > t$. Evidently, $h(a) = 0$ and $h(x) \geq 0$ for $x < a$, since, by the assumption, $M(f)(a) \leq t$. We have to show that $h(x) > 0$ for each $x \in (a,b)$. Indeed, otherwise there would exist a point $x \in (a,b)$ for which $h(x) = \inf_{y\in[a,x)} h(y)$. Then we would have $h(y) \leq h(x)$ for each $y < x$, which is impossible, since $M(f)(x) > t$. □

If E is an open subset of $\mathbb{R}$ not containing any neighborhood of $-\infty$ and if the representation of E by the union of disjoint connected components has the form

$$E = \underset{n=1}{\cup}(a_n, b_n), \tag{2}$$

then we suppose

$$E^- = \underset{n=1}{\cup}(2a_n - b_n, a_n)$$

(each component is rotated with respect to the left origin).

Lemma 2. *Let E be an open proper subset of $\mathbb{R}$ for which $\mathbb{I}_E$ is a 2π-periodic function. Then*

$$|E^-| \geq \frac{1}{2}|E|.$$

Proof. Assume in representation (2) of E that $a_n \in [0, 2\pi)$ and $(a_n, b_n) = \cup_{k\in\mathbb{Z}}(a_n + 2\pi k, b_n + 2\pi k)$ (i.e., $(a_n, b_n) \in \mathcal{E}$), $n = 1, 2, \dots$.

If $I \subset \mathcal{E}$ and Δ_I is a segment from $\mathcal{E}$ such that $\Delta_I \in I$ and

$$|\Delta_I| = \sup_{\Delta\in I}|\Delta|,$$

then we shall say that $\Delta_I = \max(I)$. If there are several segments with such properties, then one of them (for our proof it does not matter which one) will be called $\max(I)$. Also, I' will denote the set of segments from I which are included in the rotated $\max(I)$, i.e.,

$$\Delta \in I' \Leftrightarrow \Delta \in I, \quad \Delta \subset \max(I)^-,$$

and $S(I)$ will denote the subset of I $I\backslash(I' \cup \{\max(I)\})$. (The case $S(I) = \varnothing$ is not excluded.)

Suppose $I_0 = \{(a_n, b_n) : n = 1, 2, \dots\}$, $I_n = S(I_{n-1})$, $n = 1, 2, \dots$ and $\Delta_n = \max(I_n)$. Obviously, $I_0 \supset I_1 \supset \dots$ and

$$I_0 = \underset{n=0}{\cup}(\{\Delta_n\} \cup I_n'), \tag{3}$$

since each segment of I_0 will at some moment become maximal or be excluded.

Since each $\Delta_n \in I_0$, we have

$$\Delta_n^- \subset E^-, \quad n = 0, 1, \dots . \tag{4}$$

If now $i<j$, then $|\Delta_i|\geq|\Delta_j|$, $\Delta_i\cap\Delta_j=\varnothing$ and $\Delta_j\not\subset\Delta_i^-$, which imply

$$\Delta_i^-\cap\Delta_j^-=\varnothing. \tag{5}$$

Hence $\{\Delta_n^- : n=0,1,\dots\}$ is a set of pairwise disjoint segments.

We also have the inequality

$$|\Delta_n|\geq\sum_{\Delta\in I_n'}|\Delta|,\quad n=0,1,\dots\ . \tag{6}$$

Using (4), (5), (6), and (3), we conclude that

$$|E^-|\geq|\underset{n=0}{\cup}\Delta_n^-|=\sum_{n=0}|\Delta_n^-|=\sum_{n=0}|\Delta_n|\geq$$
$$\geq\sum_{n=0}\frac{1}{2}\Big(|\Delta_n|+\sum_{\Delta\in I_n'}|\Delta|\Big)=\frac{1}{2}\sum_{\Delta\in I_0}|\Delta|=\frac{1}{2}|E|.\quad\square$$

Proof of Theorem 1. Let us show that if t is so large that $(M(f)>t)\neq\mathbb{R}$ (for instance, whenever $t>\frac{1}{2\pi}\|f\|=\frac{1}{2\pi}\int_{\mathbb{T}}|f|d\nu$), then

$$\nu(M(f)>t)\leq 2\nu(M_3(f)>t). \tag{7}$$

Indeed, if the representation of $(M(f)>t)$ by the union of connected components has the form

$$(M(f)>t)=\underset{n=1}{\cup}(a_n,b_n), \tag{8}$$

then each $x\in(a_n,\frac{1}{2}(a_n+b_n))$ belongs to $(M_3(f)>t)$, since by Lemma 1

$$\frac{1}{2(x-a_n)}\int\limits_{a_n}^{2x-a_n}fd\nu>t.$$

Hence $\cup_{n=1}(a_n,\frac{1}{2}(a_n+b_n))\subset(M_3(f)>t)$ and (7) holds.

If now $M_1(f)\notin L$, then we can assume without loss of generality that $M(f)\notin L$, since

$$M_1(f)\leq\max(M(f),M(-f)).$$

Thus the left term in inequality (7) will not be integrable as a function of t in a neighborhood of ∞. This implies that neither will the right term, and consequently $M_3(f)\notin L$. $\square$

Proof of Theorem 2. It is enough to show that

$$M_1(f)\notin L\Rightarrow M_2(f)\notin L, \tag{9}$$

since the inverse implication will be obtained by applying (9) to the function $x\longmapsto f(-x)$.

Let us show that

$$\nu(M(f) > t) \le 2\nu\Big(M_2(f) > t/4\Big), \tag{10}$$

for $t > \frac{1}{2\pi}\|f\|$. Indeed, if $a < x$ and (1) holds, then

$$\max\Big(\frac{1}{x-a}\Big|\int\limits_{2a-x}^{a} f d\nu\Big|, \quad \frac{1}{2(x-a)}\Big|\int\limits_{2a-x}^{x} f d\nu\Big|\Big) > \frac{t}{4},$$

since otherwise

$$\begin{aligned}\int\limits_{a}^{x} f d\nu \le \Big|\int\limits_{2a-x}^{a} f d\nu\Big| + \Big|\int\limits_{2a-x}^{x} f d\nu\Big| \le \\ \le \frac{t}{4}(x-a) + \frac{t}{2}(x-a) < t(x-a).\end{aligned}$$

Therefore, taking into account the representation of $(M(f) > t)$ by form (8) and Lemma 1, we conclude that

$$(M(f) > t)^{-} \subset \Big(M_2(f) > t/4\Big).$$

Thus (10) holds by Lemma 2.

If now $M_1(f) \notin L$, then we we can assume, as in the proof of Theorem 1, that $M(f) \notin L$. Therefore (10) implies $M_2(f) \notin L$. $\square$

Remark. Theorem 2 shows that the functions $t \longmapsto \nu(M_i(f) > t)$, $t > 0$, $i = 1, 2$, can be integrable only simultaneously. There naturally arises the question whether the inequality

$$\int\limits_{0}^{\infty} \big|\nu(M_1(f) > t) - \nu(M_2(f) > t)\big| dt < \infty \tag{11}$$

is satisfied.

The following example shows that (11) may not be valid even for a positive integrable function f.

Let $f \in L$ be a continuous function with the properties: $f(x) > 0$ for $0 < x \le \pi$, $f(x) = 0$ for $\pi < x \le 2\pi$, f is monotonically decreasing on $(0, \pi]$, and

$$\int\limits_{0}^{\pi} f(x) \lg \frac{\int_0^x f d\nu}{x f(x)}\, dx = \infty$$

(the class of such functions is considered in [5]).

Clearly, $M(f)(x)=\frac{1}{x}\int_0^x f\,d\nu$ for each $x\in(0,2\pi)$. Thus

$$\int\limits_{(f>0)} f\lg\frac{M(f)}{f}\,d\nu=\infty.$$

For $t>l=\frac{1}{2\pi}\|f\|$ let x_t be the point from $(0,\pi]$ for which $f(x_t)=t$, let y_t be the point from $(0,2\pi]$ for which $M_1(f)(y_t)=t$, and let z_t be the point from $[-\pi,0)$ for which $\frac{1}{x_t-z_t}\int_0^{x_t}f\,d\nu=t$. Then it is not difficult to show that $(M_1(f)>t)=(0,y_t)$ and $(M_2(f)>t)=(z_t,x_t)$. Hence

$$\nu(M_1(f)>t)=\frac{1}{t}\int\limits_{(M(f)>t)} f\,d\nu$$

and

$$\nu(M_2(f)>t)=\frac{1}{t}\int\limits_{(f>t)} f\,d\nu.$$

By Fubini's theorem we now obtain

$$\int\limits_l^\infty\big|\nu(M_1(f)>t)-\nu(M_2(f)>t)\big|dt=\int\limits_l^\infty\frac{dt}{t}\int\limits_{(M(f)>t)\setminus(f>t)} f\,d\nu=$$
$$=\int\limits_{(M(f)>l)\cap(f\le l)} f\lg\frac{M(f)}{l}\,d\nu+\int\limits_{(f>l)} f\lg\frac{M(f)}{f}\,d\nu=\infty.$$

§ 2. This section will be devoted to proving analogous theorems for ergodic maximal operators.

Let $(X,\mathbb{S},\mu)$ be a σ-finite measure space and let $T:X\to X$ be an invertible measure-preserving ergodic transformation.

To emphasize the analogues we shall retain the notions of the preceding section, which should not cause misunderstanding.

Let L be the class of integrable functions (with respect to the measure μ) on X. As usual, the functions distinct from each other on a set of measure 0 are identified.

By $\mathcal{E}$ we shall denote the class of subsets of $\mathbb{Z}$ of the type $\{m,m+1,\dots,m+k\}$, $m\in\mathbb{Z}$, $k=0,1.\dots$. If $\Delta\in\mathcal{E}$, then it is assumed that $|\Delta|=\operatorname{card}(\Delta)$. Let $\mathcal{E}_j$, $j=0,1,2,3$, be the following subclasses of $\mathcal{E}$:

$$\mathcal{E}_0=\big\{\{m,m+1,\dots,m+k\}:m\le0,\ \ m+k\ge0\big\},$$
$$\mathcal{E}_1=\big\{\{m,m+1,\dots,m+k\}:m+k=0\big\},$$
$$\mathcal{E}_2=\big\{\{m,m+1,\dots,m+k\}:m=0\big\},$$
$$\mathcal{E}_3=\big\{\{m,m+1,\dots,m+k\}:-m=m+k\big\},$$

and let M_j, $j = 0, 1, 2, 3$, be the corresponding ergodic maximal operators:

$$M_j(f)(x) = \sup_{\Delta \in \mathcal{E}_j} \frac{1}{|\Delta|} \Big| \sum_{n \in \Delta} f \circ T^n(x) \Big|.$$

It is wellknown that if $f \geq 0$, then $f \in L \lg^+ L \Leftrightarrow M_j(f) \in L$ when $\mu(X) < \infty$ and $M_j(f) \in L \Leftrightarrow f \equiv 0$ when $\mu(X) = \infty$ (see [6]). But a necessary and sufficient condition for $M_j(f)$ to be integrable does not exist on f in general (in this connection see [7]). We shall show that for arbitrary $f \in L$ the functions $M_j(f)$, $j = 0, 1, 2, 3$, can be integrable only simultaneously. To this end, as in Section 1, it is sufficient to prove theorems which formally look like Theorems 1 and 2. They will be called Theorems $1'$ and $2'$. It can be said that Lemmas 3 and 4 to be used in proving these theorems are ergodic analogues of Lemmas 1 and 2.

Let M be the operator

$$M(f)(x) = \sup_{N \geq 0} \frac{1}{N+1} \sum_{n=0}^{N} f \circ T^{-n}(x), \quad f \in L.$$

We shall say that a measurable set $A \subset X$ is a tower with the base B if $B = \cup_{m=0}^{\infty} B_n$, the sets $T^n B_m$, $m = 0, 1, 2, \ldots$, $0 \leq n \leq m$, are paiwise disjoint, and

$$A = \overset{\infty}{\underset{m=0}{\cup}} \overset{m}{\underset{n=0}{\cup}} T^n B_m. \tag{12}$$

$C_m = \cup_{n=0}^{m} T^n B_m$ is said to be a column of height m.

Lemma 3. *Let $f \in L$ and let $(M(f) > t) \neq X$. Then the set $(M(f) > t)$ can be represented as a tower*

$$(M(f) > t) = \overset{\infty}{\underset{m=0}{\cup}} \overset{m}{\underset{n=0}{\cup}} T^n B_m, \tag{13}$$

such that

$$\frac{1}{N+1} \sum_{n=0}^{N} f \circ T^n(x) > t, \quad N = 0, 1, \ldots, m, \tag{14}$$

for each $x \in B_m$.

Proof. Let

$$B = T(M(f) \leq t) \cap (M(f) > t).$$

Since T is ergodic, we have

$$\mu(B) > 0.$$

For each $x \in B$ let $m(x)$ be the maximum value of m for which (14) holds. (We can easily check, and it also follows from the reasoning below, that $x \in B$ implies $f(x) > t$. Therefore $m(x)$ is correctly defined. It can be formally assumed that $m(x)$ is not defined whenever $f(x) \leq t$.) Suppose

$$B_m = \{x \in B : m(x) = m\}, \quad m = 0, 1, \dots .$$

Then the set A defined by equality (12) will be the tower with the desired property. Let us now show that

$$A = (M(f) > t).$$

To this end it is sufficient to prove

$$(M(f) > t) \subset A,$$

since the inverse inclusion directly follows from the construction of A.

Suppose $x \in (M(f) > t)$. Let $\bar{m}$ be a nonnegative integer for which $x, T^{-1}x, \dots, T^{-\bar{m}}x \in (M(f) > t)$ and $T^{-\bar{m}-1}x \notin (M(f) > t)$. Then $\bar{x} = T^{-\bar{m}}(x) \in B$ and

$$M(f)(T^{-1}\bar{x}) \leq t. \tag{15}$$

We shall show that

$$m(\bar{x}) \geq \bar{m},$$

which, by the definition of A, implies that $T^{\bar{m}}\bar{x} = x \in A$.

Consider the function

$$h(k) = \operatorname{sign}(k)\Big(\sum_{n \in \Delta_k} f \circ T^n(\bar{x}) - t|\Delta_k|\Big), \quad k \in \mathbb{Z},$$

where $\Delta_k = \{0, 1, \dots, k-1\}$ when $k > 0$, $\Delta_0 = \varnothing$ and $\Delta_k = \{k, k+1, \dots, -1\}$ when $k < 0$. Note that if $p < k$, then $h(p) < h(k) \Leftrightarrow \frac{1}{p-k}\sum_{n=1}^{p-k} f \circ T^{-n}(T^k\bar{x}) > t$. We have $h(0) = 0$ and, due to (15), $h(k) \geq 0$ when $k < 0$. We need to show that the inequality $h(k) > 0$ holds for each $k = 0, 1, \dots, \bar{m}$. Indeed, otherwise there would exist $k \in \{0, 1, \dots, \bar{m}\}$ for which $h(k) = \inf_{1 \leq p \leq k} h(p)$. Then we would have $h(k) \leq h(p)$ for each $p \leq k$, which is impossible, since $M(f)(T^k\bar{x}) > t$. □

Proof of Theorem $1'$. As in proving Theorem 1, we assume that $M(f) \notin L$ and it is sufficient to show that

$$\mu(M(f) > t) \leq 2\mu(M_3(f) > t) \tag{16}$$

whenever t is so large that $(M(f) > t) \neq X$.

Representing, by virtue of Lemma 3, the set $(M(f) > t)$ in the form (13) and assuming that $x \in B_m$, $m = 0, 1, \dots$, on account of (14) we have

$$\frac{1}{2N+1}\sum_{n=-N}^{N} f \circ T^n(T^N x) > t$$

for all nonnegative integers N which do not exceed $\frac{m}{2}$. Thus

$$\bigcup_{m=0}^{\infty} \bigcup_{n=0}^{\underline{m}} T^n B_m \subset (M_3(f) > t),$$

where $\underline{m}$ denotes $\frac{m}{2}$ for even m and $\frac{m+1}{2}$ for odd m, and (16) holds, since representation (13) has a tower construction. $\square$

If A is a tower, then A^- will denote the union of rotated columns with respect to the base, i.e., if A has the form (12), then

$$A^- = \bigcup_{m=0}^{\infty} \bigcup_{n=0}^{m} T^{-n} B^m.$$

Lemma 4. *Let A be a tower. Then*

$$\mu(A^-) \geq \frac{1}{2}\mu(A).$$

(*The case where both sides of this inequality are infinite is not excluded.*)

Proof. Suppose that A is a tower whose height is finite, i.e., it has the form

$$A = \bigcup_{m=0}^{k} \bigcup_{n=0}^{m} T^n B_m.$$

The lemma will be obtained if k is made to tend to ∞.

For everyone of such towers A we shall use the following notation. Let $C(A)$ be a column of the maximum height and $C(A)^-$ be its rotation, i.e.,

$$C(A) = \bigcup_{n=0}^{k} T^n B_k, \quad C(A)^- = \bigcup_{n=0}^{k} T^{-n} B_k.$$

Since $C(A)$ is a column, the sets $B_k, T^{-1}B_k, \dots, T^{-k}B_k$ will be pairwise disjoint and hence

$$\mu(C(A)) = \mu(C(A)^-). \tag{17}$$

Let A' be the union of parts of columns of height less than k whose ground floors are contained in $C(A)^-$, i.e.,

$$A' = \bigcup_{m=0}^{k-1} \bigcup_{n=0}^{m} T^n (B_m \cap C(A)^-),$$

and let

$$S(A) = A \backslash (A' \cup C(A)).$$

Obviously, $A' \cap C(A) = \varnothing$ and

$$A' \subset C(A)^-. \tag{18}$$

Suppose $A_0 = A$ and

$$A_n = S(A_{n-1}), \quad n = 1, 2, \dots .$$

We shall therefore have a sequence of imbedded towers $A_0 \supset A_1 \supset \dots$. Clearly,

$$A = \underset{n=0}{\cup}(C(A_n) \cup A'_n), \tag{19}$$

since all columns of A will split into several parts everyone of which will, at some moment, be either maximum or excluded.

Because of $C(A_n) \subset A$ we have

$$C(A_n)^- \subset A^-, \quad n = 0, 1, \dots . \tag{20}$$

If $i < j$, then the height of $C(A_i)$ exceeds that of $C(A_j)$, $C(A_i) \cap C(A_j) = \varnothing$ and the intersection of $C(A_i)^-$ with the base of $C(A_j)$ is also empty. This enables us to conclude that

$$C(A_i)^- \cap C(A_j)^- = \varnothing, \tag{21}$$

i.e., $C(A_1)^-, C(A_2)^-, \dots$ are pairwise disjoint.

By virtue of (18)

$$\mu(A'_n) \le \mu(C(A_n)), \quad n = 0, 1, \dots . \tag{22}$$

Taking (20), (21), (17), (22), and (19) into account, we have

$$\begin{aligned} \mu(A^-) &\ge \mu(\underset{n=0}{\cup} C(A_n)^-) = \sum_{n=0} \mu(C(A_n)^-) = \\ &= \sum_{n=0} \mu(C(A_n)) \ge \sum_{n=0} \frac{1}{2}\Big(\mu(C(A_n)) + \mu(A'_n)\Big) = \\ &= \sum_{n=0} \frac{1}{2}\mu\Big(C(A_n) \cup A'_n\Big) = \frac{1}{2}\mu(A). \quad \square \end{aligned}$$

Proof of Theorem $2'$. It is sufficient to show that

$$M_1(f) \notin L \Rightarrow M_2(f) \notin L. \tag{23}$$

The inverse implication will be obtained by applying (23) to the transformation T^{-1}.

Assume without loss of generality that $M(f) \notin L$. Then we have

$$\int_{E(f)}^{\infty} \mu(M(f) > t)dt = \infty,$$

where $E(f) = \frac{1}{\mu(X)}\left|\int_X f d\mu\right|$ for $\mu(X) < \infty$ and $E(f) = 0$ for $\mu(X) = \infty$. Thus the proof will be completed as soon as we show that

$$\mu(M(f) > t) \le 2\mu\Big(M_2(f) > t/4\Big) \tag{24}$$

for $t > E(f)$.

First we note that $(M(f) > t) \neq X$, since by the ergodic theorem

$$\limsup_{N\to\infty}\Big(\sum_{n=0}^{N-1} f\circ T^n(x) - Nt\Big) \leq 0$$

for almost all $x \in X$.

If x and N are such that (14) holds, then

$$\max\Big(\frac{1}{N}\Big|\sum_{n=0}^{N-1} f\circ T^n(T^{-N}x)\Big|, \quad \frac{1}{2N+1}\Big|\sum_{n=0}^{2N} f\circ T^n(T^{-N}x)\Big|\Big) > \frac{t}{4},$$

since otherwise

$$\sum_{n=0}^{N} f\circ T^n(x) \leq \Big|\sum_{n=0}^{2N} f\circ T^n(T^{-N}x)\Big| +$$
$$+\Big|\sum_{n=0}^{N-1} f\circ T^n(T^{-N}x)\Big| < \frac{Nt}{4} + \frac{(2N+1)t}{4} < Nt.$$

Hence by Lemma 3

$$(M(f) > t)^- \subset \Big(M_2(f) > t/4\Big)$$

and by Lemma 4 equality (24) holds. $\square$

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Author's address:
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, Z. Rukhadze St., Tbilisi 380093
Republic of Georgia