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ON THE EQUIVALENCE BETWEEN CH AND THE EXISTENCE OF CERTAIN $\mathcal{I}$-LUZIN SUBSETS OF $\mathbb{R}$

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Abstract. We extend Rothberger's theorem (on the equivalence between CH and the existence of Luzin and Sierpiński-sets having power $\mathfrak{c}$) and certain paradoxical constructions due to Erdös. More precisely, by employing a suitable σ-ideal associated to the (α, β)-games introduced by Schmidt, we prove that the Continuum Hypothesis holds if and only if there exist subgroups of $(\mathbb{R}, +)$ having power $\mathfrak{c}$ and intersecting every "absolutely losing" (respectively, every meager and null) set in at most countably many points.

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1. Preliminaries

It is a well-established fact that the problem of the existence of Luzin (Sierpiński) sets of real numbers, i.e., uncountable sets meeting every meager (Lebesgue-null) set in at most countably many points – is unsolvable in ZFC.

By assuming CH and now standard transfinite methods, Mahlo (1913), Luzin (1914) and Sierpiński (1924) gave such constructions; on the other hand, Rothberger (1938) proved the reverse implication, thus establishing:

Theorem 1. *CH holds iff there exist Luzin and Sierpiński sets of power* $\mathfrak{c}$.

We omit the proof[1] and limit ourselves to pointing out how to generalise Theorem 1 in a straightforward way. To this aim, we recall some standard definitions:

Definition 2. A family $\mathcal{I} \subseteq \mathcal{P}(\mathbb{R})$ is said to be a σ-ideal (in $\mathbb{R}$) if it satisfies:

i) $I_n \in \mathcal{I}$ for all $n \in \mathbb{N} \implies \bigcup_n I_n \in \mathcal{I}$,
ii) $I \in \mathcal{I},\ H \subseteq I \implies H \in \mathcal{I}$.

We also assume the following:

iii) $x \in \mathbb{R} \implies \{x\} \in \mathcal{I}$,
iv) $\mathbb{R} \notin \mathcal{I}$.

Moreover, we call $\mathcal{I}$ invariant under affine maps if

v) $I \in \mathcal{I}$ iff $T(I) \in \mathcal{I}$ for every invertible affine map $T : \mathbb{R} \longrightarrow \mathbb{R}$.

Finally, we say that $\mathcal{I}$ is $\mathfrak{c}$-generated by $\widetilde{\mathcal{I}} \subseteq \mathcal{I}$ in case

vi) $\operatorname{card}(\widetilde{\mathcal{I}}) \leq \mathfrak{c}$ and for all $I \in \mathcal{I}$ there exists $\tilde{I} \in \widetilde{\mathcal{I}}$ such that $I \subseteq \tilde{I}$.

[1] See [8] or the second section of [4].

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Definition 3. Two σ-ideals $\mathcal{I}$ and $\mathcal{J}$ are said to be orthogonal (written as $\mathcal{I} \perp \mathcal{J}$) if $\mathbb{R} = I \cup J$, with $I \in \mathcal{I}$ and $J \in \mathcal{J}$.

As known, each of the classes $\mathcal{M}$, $\mathcal{N}$, $\mathcal{M} \cap \mathcal{N}$, $\mathcal{H}$ made up of meager, null (in the sense of Lebesgue), meager and null, 0-dimensional (in Hausdorff's sense) sets, respectively, are σ-ideals invariant under affine maps and $\mathfrak{c}$-generated by Borel sets. Moreover, $\mathcal{M} \perp \mathcal{H}$ and consequently $\mathcal{M} \perp \mathcal{N}$.[2]

Definition 4. Given a σ-ideal $\mathcal{I}$, any uncountable set intersecting every set of $\mathcal{I}$ in at most countably many points is said to be $\mathcal{I}$-Luzin.

Proposition 5. *Let $\mathcal{I}$ and $\mathcal{J}$ be σ-ideals, $\mathcal{I} \perp \mathcal{J}$. Then every $\mathcal{I}$-Luzin set A belongs to $\mathcal{J}$ (but not to $\mathcal{I}$).*

Proof. Apart from a countable set, A is contained in a certain $J \in \mathcal{J}$: by Definition 2, $A \in \mathcal{J}$. □

A close look at Miller's survey ([4], Section 2) allows us to formulate Rothberger's theorem in a more general framework:

Theorem 6. *Let $\mathcal{I}$, $\mathcal{J}$ be orthogonal, invariant under affine maps, $\mathfrak{c}$-generated σ-ideals. Then the CH holds iff there exist $\mathcal{I}$-Luzin and $\mathcal{J}$-Luzin sets having power $\mathfrak{c}$.*

Observe that by letting $\mathcal{I}$ and $\mathcal{J}$ stand for $\mathcal{M}$ and $\mathcal{N}$ we have Theorem 1 once more. Actually, an even sharper statement holds:

Theorem 7. *Let $\mathcal{I}$ and $\mathcal{J}$ be as in Theorem 6. Then the CH holds iff there exist $\mathcal{I}$-Luzin sets and $\mathcal{J}$-Luzin sets having power $\mathfrak{c}$ and being $\mathbb{Q}$-linear subspaces of $\mathbb{R}$.*

Proof. Assume $\omega_1 = \omega_c$. Let us well-order a system of generators for $\mathcal{I}$, say $\widetilde{\mathcal{I}} = \{\tilde{I}_\xi, \xi < \omega_c\}$,[3] and construct (in the way to be specified below) a strictly increasing sequence of $\mathbb{Q}$-linear spaces, V_ξ $(\xi < \omega_c)$, such that, for all $\xi < \omega_c$,

$$\operatorname{card} V_\xi = \aleph_0, \tag{1}$$

$$V_\xi \cap \left(\bigcup_{\chi<\xi} \tilde{I}_\chi \right) = \left(\bigcup_{\chi<\xi} V_\chi \right) \cap \left(\bigcup_{\chi<\xi} \tilde{I}_\chi \right). \tag{2}$$

In this way, the set $V := \bigcup_{\xi<\omega_c} V_\xi$ turns out to be $\mathcal{I}$-Luzin (in that, by (1) and (2), it intersects every $\tilde{I}_\chi$ in at most countably many points) and is also endowed with a $\mathbb{Q}$-linear structure.

Fix $\xi < \omega_c$ and suppose that $(V_\chi)_{\chi<\xi}$ is an increasing sequence of $\mathbb{Q}$-linear spaces satisfying (1). Let us find an element x_ξ in $\mathbb{R} \setminus \bigcup_{\chi<\xi} V_\chi$ such that

$$x_\xi \notin M_\xi := \bigcup_{\chi<\xi} \bigcup_{T\in\mathcal{T}_\xi} \{x : T(x) \in \tilde{I}_\chi\} = \bigcup_{T\in\mathcal{T}_\xi} T^{-1}\left(\bigcup_{\chi<\xi} \tilde{I}_\chi \right),$$

[2]The class of Liouville numbers, for instance, is 0-dimensional (hence of null Lebesgue-measure) and co-meager in $\mathbb{R}$ (see [6], Chapter 2).

[3]Without loss of generality, we may assume $\operatorname{card}(\widetilde{\mathcal{I}}) = \mathfrak{c}$.

where $\mathcal{T}_\xi$ stands for the class of all affine maps of the form

$$T(x) = qx + v$$

with $q \in \mathbb{Q} \setminus \{0\}$ and $v \in \bigcup_{\chi<\xi} V_\chi$. On the basis of Definition 2 such an element x_ξ does exist and it is therefore meaningful to define $V_\xi := \mathbb{Q}x_\xi + \bigcup_{\chi<\xi} V_\chi$.

Of course, V_ξ is denumerable and verifies (2). □

The proof above is just a generalization of Erdös' idea in [1], where he established the existence of groups in $(\mathbb{R}, +)$ with the property of being null but not meager (vice versa, meager but not null). Incidentally, it is interesting to observe that he did more than this result (recall Proposition 5 above): he constructed groups of Luzin and Sierpiński type, thus giving implicitly a formulation of CH expressed in algebraic terms. We refer the reader to pages 96–97 in [2] for other algebraic statements equivalent to the Continuum Hypothesis.

The main goal of this paper is to extend effectively Rothberger's theorem by indicating an explicit example of a $\mathfrak{c}$-generated σ-ideal $\mathcal{S}$ invariant under affine maps which also satisfies the orthogonality relation $\mathcal{S} \perp (\mathcal{M} \cap \mathcal{N})$.

Such an ideal was introduced by Schmidt in 1966 as the class of "losing" sets with respect to his (α, β)-games. He also proved the properties for $\mathcal{S}$ being invariant under diffeomorphisms and orthogonal to $\mathcal{M} \cap \mathcal{N}$ (see Theorems 1–3 in [9] and also [10], where an application of $\mathcal{S}$ is given in the number-theoretical context).

Our main contribution, motivated by the reading of Mycielski's paper [5] –in particular, Theorem 10 therein– consists in showing that $\mathcal{S}$ is $\mathfrak{c}$-generated by "absolutely losing" sets of co-Souslin type.[4] This is accomplished in the next section, after a brief introduction to Schmidt's (α, β)-games and σ-ideal, and therefore leads us to conclude:

Theorem 8. *The CH holds iff there exist $\mathbb{Q}$-linear subspaces of $\mathbb{R}$ of power $\mathfrak{c}$ and meeting every absolutely losing (dually: every meager and null) set in at most countably many points.*

It is natural to ask whether assertions analogous to Theorems 7 and 8 hold for richer algebraic structures such as fields, for instance: this is true and can be easily deduced from the analysis carried out in [10]. (Here, we limit ourselves to remarking that a sharper property for the σ-ideal $\mathcal{I}$ –in particular, for $\mathcal{S}$– is needed, i.e., its invariance with respect to local diffeomorphisms, rather than the sole affine maps.)

2. Schmidt's (α, β)-Games. The σ-Ideal $\mathcal{S}$

From now on, the constants α and β will always denote rational numbers in $\left(0, \frac{1}{2}\right)$ and $(0, 1)$, respectively. (As different from the original approach, we

[4]Mycielski's ideals associated to games played in the space of Cantor (or Baire, see [7]) turn out to be generated by losing sets of Borel type. Here, the major difficulty depends essentially on the "continuous" nature of Schmidt's games.

consider only the *rational* numbers α and β. The reason for such a choice will be clarified in the proof of Theorem 12.)

Given a compact interval K with positive length and $\delta \in (0,1)$, $B^\delta(K)$ stands for the class of all closed intervals I contained in K, whose lengths are δ times that of K (symbolically: $\ell(I) = \delta\ell(K)$).

Consider two players $\langle P \rangle$, $\langle S \rangle$ and the respective sets partitioning $\mathbb{R}$, A and $B := A^c$. For a fixed, arbitrary pair (α, β), the (α,β)-game is defined as follows.

$\langle P \rangle$ chooses a nontrivial compact interval P_1. Then $\langle S \rangle$ chooses a compact interval S_1 belonging to $B^\alpha(P_1)$. $\langle P \rangle$, in turn, selects another compact interval P_2 in $B^\beta(S_1)$, etc. In general, the rules of the game can be summarized according to the following scheme:

$$\begin{aligned} &P_1 = [a,b], && a < b,\\ &S_k \in B^\alpha(P_k), && k \in \mathbb{N},\\ &P_{k+1} \in B^\beta(S_k), && k \in \mathbb{N}. \end{aligned}$$

By a winning (α,β)-strategy (for $\langle S \rangle$) with respect to A we mean a mapping $\sigma_A = \sigma_A(\alpha,\beta)$ such that to every finite sequence of $\langle P \rangle$'s moves P_1, P_2, …, P_k ($k \in \mathbb{N}$), it associates $S_k = \sigma_A(\alpha,\beta;k;P_1,\dots,P_k)$ in such a way that

$$\bigcap_{k=1}^{\infty} S_k = \bigcap_{k=1}^{\infty} P_k = \{x\} \subseteq A^c = B. \tag{3}$$

A is said to be (α,β)-losing (and its complement B (α,β)-winning) in case there exists a winning (α,β)-strategy for $\langle S \rangle$ (in other words, (3) can always occur, independently of the sequence $(P_n)_{n\in\mathbb{N}}$).

Definition 9. A set $A \subseteq \mathbb{R}$ is said to be absolutely losing if it is (α,β)-losing for every pair (α,β). Consistently, its complement $B := A^c$ is called absolutely winning.

$\mathcal{S}$ will denote the family made up of absolutely losing sets.

Although it can be deduced from [9] in a more general setting, for the reader's convenience we give an ad hoc proof of the following theorem.

Theorem 10 (Schmidt)**.** *$\mathcal{S}$ is a σ-ideal invariant under affine maps.*

Proof. Items ii), iii), iv) of Definition 2 are obvious.

Fixing arbitrarily the constants α, β, let us prove that, for any non-singular affine map T and any (α,β)-losing A, $T(A)$ is (α,β)-losing, too. Clearly, if $\sigma_A(\alpha,\beta)$ is an (α,β)-strategy that "skips" A, then

$$\sigma_{T(A)}(\alpha,\beta;k;P_1,P_2,\dots,P_k) := T\big(\sigma_A(\alpha,\beta;k;T^{-1}(P_1),T^{-1}(P_2),\dots,T^{-1}(P_k))\big)$$

is a winning strategy with respect to $T(A)$. The invariance of $\mathcal{S}$ under affine maps is thus verified.

It remains to prove that $\mathcal{S}$ satisfies i) of Definition 2: again, fix α and β and consider a countable system of absolutely losing sets $\{A_n,\, n \in \mathbb{N}\}$. The (α,β)-strategy avoiding $\bigcup_{n=1}^{\infty} A_n$ is defined according to the following scheme: at the first, third, fifth, … move, $\langle S \rangle$ follows a winning $(\alpha,\beta\alpha\beta)$-strategy that avoids A_1. At the second, sixth, tenth,... move, $\langle S\rangle$ follows a winning $(\alpha, \beta(\alpha\beta)^3)$-strategy that misses A_2, and so on. In general, at the k^{th}-move, where

$$k = \phi(m, n) := (2m-1)2^{n-1},$$

$\langle S\rangle$ plays according to an $(\alpha, \beta(\alpha\beta)^{2^n-1})$-strategy winning with respect to A_n, as shown below:

$$\begin{aligned} &P_1 = [a, b], && a < b,\\ &S_k = \sigma_{\cup_i A_i}(\alpha, \beta; k; P_1, P_2, \dots, P_k) &&\\ &:= \sigma_{A_n}(\alpha, \beta(\alpha\beta)^{2^n-1}; m; P_{\phi(1,n)}, P_{\phi(2,n)}, \dots, P_{\phi(m,n)}) \in B^\alpha(P_k), && k \in \mathbb{N},\\ &P_{k+1} \in B^\beta(S_k), && k \in \mathbb{N}. \end{aligned}$$

Note that the expression above is valid, since for every $i, n \in \mathbb{N}$,

$$P_{\phi(i+1,n)} \in B^{\beta(\alpha\beta)^{2^n-1}}\big(S_{\phi(i,n)}\big).$$

For an arbitrary sequence $(P_k)_{k\in\mathbb{N}}$, it turns out that

$$\bigcap_{k=1}^\infty P_k = \bigcap_{k=1}^\infty S_k = \{x\} \not\subseteq \bigcup_{n=1}^\infty A_n,$$

which means that $\bigcup_{n=1}^\infty A_n$ is (α, β)-losing.

The assertion follows from the arbitrariness of α and β. □

Theorem 11 (Schmidt)**.** *$\mathcal{S} \perp (\mathcal{M} \cap \mathcal{N})$.*

See Theorem 3 and the introductory section of [9].

Theorem 12. *$\mathcal{S}$ is $\mathfrak{c}$-generated.*

Proof. It consists in showing that every absolutely winning set $B := A^c$ contains an absolutely winning set W of Souslin type, i.e., constructed by means of the Souslin operation $\mathcal{A}$.[5] Obviously, this is equivalent to stating that every absolutely losing set A is contained in an absolutely losing, co-Souslin set $L := W^c$. Since there are $\mathfrak{c}$ many co-Souslin subsets of $\mathbb{R}$, we immediately infer that $\mathcal{S}$ verifies vi) of Definition 2.

Let us firstly denote by $I_\mathbb{Q}$ the family of all compact *rational* intervals of $\mathbb{R}$ (i.e. with rational end-points) and fix the constants α and β. Since any element in $I_\mathbb{Q}$ may be viewed as a possible first move for $\langle P\rangle$, we write

$$I_\mathbb{Q} := \{P_{i_1} \mid i_1 \in \mathbb{N}\}$$

and, consequently,

$$S_{i_1} := \sigma_A(\alpha, \beta; 1; P_{i_1}) \in B^\alpha(P_{i_1}),$$

$\sigma_A(\alpha, \beta)$ denoting an (α, β)-strategy winning with respect to A. Then, denoting by $P_{i_1 i_2}$ (i_2 ranging over $\mathbb{N}$) all of the rational following legal moves for $\langle P\rangle$, that is to say the elements in $I_\mathbb{Q} \cap B^\beta(S_{i_1})$, we put

$$S_{i_1 i_2} := \sigma_A(\alpha, \beta; 2; P_{i_1}, P_{i_1 i_2}) \in B^\alpha(P_{i_1 i_2}).$$

[5]The reader may consult, e.g., Sections 11.5 and 13.1 of [3] for general reference on Souslin sets.

Proceeding inductively, for every $n \in \mathbb{N} \setminus \{1\}$ let $P_{i_1 \cdots i_n}$ $(i_n \in \mathbb{N})$ denote every rational interval in $B^\beta(S_{i_1 \cdots i_{n-1}})$ and

$$S_{i_1 \cdots i_n} := \sigma_A(\alpha, \beta; n; P_{i_1}, \ldots, P_{i_1 \cdots i_n}) \in B^\alpha(P_{i_1 \cdots i_n}).$$

Finally, put

$$\begin{aligned} W(\alpha,\beta) &:= \bigcup_{\mathbb{N}^\mathbb{N}} \bigcap_{n=1}^{\infty} S_{i_1 \cdots i_n} = \bigcup_{\mathbb{N}^\mathbb{N}} \bigcap_{n=1}^{\infty} P_{i_1 \cdots i_n}, \\ W &:= \bigcup_{\alpha,\beta} W(\alpha,\beta). \end{aligned}$$

By construction, $W \subseteq B$. Moreover, W is Souslin, since it is given as a countable union of Souslin sets.[6]

It remains to prove that W is absolutely winning. Otherwise stated: having any couple (α, β) fixed, we still have to suggest $\langle S \rangle$ an (α, β)-strategy σ_L winning with respect to $L := W^c$.

To this end, let us choose rational numbers α' and β' such that

$$\alpha < \alpha' < \frac{1}{2} \quad \text{and} \quad \alpha'\beta' = \alpha\beta. \tag{4}$$

Given the first move P_1[7] of $\langle P \rangle$, let us choose $P_1' \in I_\mathbb{Q}$ properly contained in P_1 and such that

$$\ell(P_1') > \frac{\alpha}{\alpha'}\, \ell(P_1)$$

(such a choice is ensured by (4)). Therefore

$$P_1' \in B^K(P_1) \text{ for a certain real constant } K \in \left(\frac{\alpha}{\alpha'}, 1\right). \tag{5}$$

Now put

$$S_1' := \sigma_A(\alpha', \beta'; 1; P_1') \in B^{\alpha'}(P_1')$$

and let $S_1 = \sigma_L(\alpha, \beta; 1, P_1)$ be an arbitrarily chosen compact interval in $B^k(S_1')$ with k satisfying

$$k\, K = \frac{\alpha}{\alpha'}. \tag{6}$$

After the second move $P_2 \in B^\beta(S_1)$ done by the first player $\langle P \rangle$, $\langle S \rangle$ freely chooses in $B^K(P_2)$ –K being the *same* constant as in (5)– an element $P_2' \in I_\mathbb{Q}$ (it does exist, since the ratio $\ell(P_2')/\ell(P_1') = \alpha\beta$ is rational). Again, put

$$S_2' := \sigma_A(\alpha', \beta'; 2; P_1', P_2') \in B^{\alpha'}(P_2') \tag{7}$$

and let $S_2 = \sigma_L(\alpha, \beta; 2; P_1, P_2)$ be a totally arbitrarily selected element in $B^k(S_2')$, k being the *same* constant as in (6).

[6]Of course, such an assertion is not true for uncountable unions: this makes us consider only rational α and β.

[7]It is perhaps worth remarking that indexes 1, 2, …, n, … denote here the sequential order of moves, whereas in the first part of the proof the multi-indexes $i_1 i_2 \cdots i_n$ enumerate the elements in $I_\mathbb{Q}$.

In general, for each move $P_n \in B^{\beta}(S_{n-1})$ let us select (no matter how) the corresponding $P'_n \in I_{\mathbb{Q}} \cap B^K(P_n)$ and define

$$S'_n := \sigma_A(\alpha', \beta'; n; P'_1, \ldots, P'_n) \in B^{\alpha'}(P'_n);$$

after that, $\langle S \rangle$ chooses (no matter how) $S_n = \sigma_L(\alpha, \beta; n; P_1, \ldots, P_n) \in B^k(S'_n)$. Note that, by virtue of (4) and (6), for every $n \in \mathbb{N}$ there actually hold

$$P_{n+1} \in B^{\beta}(S_n) \implies P'_{n+1} \in B^{\beta'}(S'_n),$$
$$S'_n \in B^{\alpha'}(P'_n) \implies S_n \in B^{\alpha}(P_n).$$

To sum up, for any given couple (α, β) and any sequence $(P_n)_{n\in\mathbb{N}}$ we have, by construction,

$$\bigcap_{n=1}^{\infty} P_n = \bigcap_{n=1}^{\infty} S_n = \bigcap_{n=1}^{\infty} S'_n = \bigcap_{n=1}^{\infty} P'_n = \{x\} \subseteq L^c = W,$$

which is nothing else but stating that W is absolutely winning (equivalently, that $\sigma_L = \sigma_L(\alpha, \beta)$ is a winning strategy with respect to L). □

Roughly speaking, our proof proceeds in this way: once we have constructed a Souslin set W contained in $B := A^c$ by following the winning strategy σ_A *restricted* to the rational legal choices of $\langle P \rangle$, we define a strategy σ_L winning with respect to $L := W^c$ and "modeled" on the former σ_A. The constants K and k simply "adjust the gaps".

We do not know how optimal Theorem 12 is. In other words, we do not know whether $\mathcal{S}$ is Borel generated or not.

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