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ON THE SOLVABILITY OF A NONLOCAL PROBLEM ARISING IN DYNAMICS OF MOISTURE TRANSFER

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Abstract. In the recent years, evolution problems with an integral term in the boundary conditions have received a great deal of attention. Such problems, in general, are nonself-adjoint, and this poses the basic source of difficulty, which can considerably complicate the application of standard functional and numerical techniques. To avoid these complications, we have introduced a nonclassical function space to establish a priori estimates without any additional complication as compared to the classical evolution problems. As an example of the applicability of this way of solving problems of this type, we investigate an initial-boundary value problem for a pseudoparabolic equation which combines Neumann and integral conditions.

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1. Introduction

We consider the following problem: find a function $\theta = \theta(x,t)$ satisfying:

$$\frac{\partial \theta}{\partial t} - \frac{\partial^2 \theta}{\partial x^2} - \kappa \frac{\partial^3 \theta}{\partial t \partial x^2} = \mathrm{f}(x,t), \qquad (x,t) \in (0,1) \times (0,T), \tag{1.1}$$

$$\theta(x,0) = \theta_0(x), \qquad x \in (0,1), \tag{1.2}$$

$$\frac{\partial \theta(0,t)}{\partial x} = \mu(t), \qquad t \in (0,T), \tag{1.3}$$

$$\int_0^1 \theta(x,t)\,dx = E(t), \qquad t \in (0,T), \tag{1.4}$$

and the compatibility conditions

$$\theta_0'(0) = \mu(0), \qquad \int_0^1 \theta_0(x)dx = E(0), \tag{1.5}$$

where f, θ_0, μ and E are the known functions, T and κ are given positive constants.

Problem (1.1)–(1.4) can be used to construct a mathematical model of the dynamics of moisture transfer in a porous subsoil $0 < x < 1$; then θ is a moisture distribution in the subsoil layer $0 < x < 1$, θ_0 is the initial moisture content, $\mu(t)$ and $E'(t)$ represent, respectively, the moisture flow at depth $x = 0$ and the moisture flux in the region $0 < x < 1$ [6].

In this paper, we prove the existence, uniqueness and continuous dependence upon the data of the solution of a generalization of problem (1.1)–(1.4):

$$\mathcal{L}\theta = \frac{\partial \theta}{\partial t} - \frac{\partial}{\partial x}\left(p(x,t)\frac{\partial \theta}{\partial x}\right) - \frac{\partial^2}{\partial t \partial x}\left(q(x,t)\frac{\partial \theta}{\partial x}\right) + r(x,t)\theta = \mathrm{f}\,(x,t)\,, \qquad (x,t) \in Q, \tag{1.6}$$

$$\ell\theta = \theta(x,0) = \theta_0(x), \qquad x \in (a,b)\,, \tag{1.7}$$

$$\frac{\partial \theta(0,t)}{\partial x} = \mu(t), \qquad t \in (0,T)\,, \tag{1.8}$$

$$\int_a^b \theta\,(x,t)\,dx = E(t), \qquad t \in (0,T)\,, \tag{1.9}$$

with

$$\theta_0'(a) = \mu(0), \qquad \int_a^b \theta_0(x)dx = E(0), \tag{1.10}$$

where the coefficients p, q, r satisfy the following conditions: $p(x,t) \in C^{2,2}\left(\overline{Q}\right)$, $q(x,t) \in C^{2,2}\left(\overline{Q}\right)$, $r(x,t) \in C^{0,1}\left(\overline{Q}\right)$. Furthermore, $\mathrm{f}(x,t) \in L^2\left(0,T; B_2^1\left(a,b\right)\right)$ (which will be defined later) and $u_0 \in H^1\left(a,b\right)$. Here $Q = (a,b) \times (0,T)$.

Mixed problems with integral conditions for other equations have been studied by several authors. The reader is referred, for instance, to [1, 5].

The present work is organized as follows. In Section 2, we reduce problem (1.6)–(1.9) to an equivalent problem with homogeneous boundary conditions, and we give assumptions on the coefficients of equation (1.6); then we introduce appropriate function spaces needed in the rest of the paper; lastly, we establish an abstract formulation of the studied problem. Section 3 shows that operator L is continuous and has a continuous inverse on its range $R(L)$. In Section 4, we prove the existence of a solution. Finally, in Section 5 we give more results on the generalization of problem (1.6)–(1.9).

2. Preliminaries

We start by introducing a new unknown function $u(x,t)$ defined by

$$u(x,t) = \theta(x,t) - U(x,t)$$

so that $u(x,t)$ represents the deviation of the function $\theta(x,t)$ from the function $U(x,t)$ defined by

$$U(x,t) = (x-a)\mu(t) + 3\frac{(x-a)^2}{(b-a)}\left(\frac{E(t)}{(b-a)^2} - \frac{1}{2}\mu(t)\right).$$

Then problem (1.6)-(1.9) can be reduced to the following problem with homogeneous boundary conditions:

$$\mathcal{L}u = \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(p(x,t)\frac{\partial u}{\partial x}\right) - \frac{\partial^2}{\partial t \partial x}\left(q(x,t)\frac{\partial u}{\partial x}\right) + r(x,t)u = f\,(x,t)\,, \qquad (x,t) \in Q, \tag{2.1}$$

$$\ell u = u(x,0) = u_0(x), \qquad x \in (a,b)\,, \tag{2.2}$$

$$\frac{\partial u(0,t)}{\partial x} = 0, \qquad t \in (0,T), \tag{2.3}$$

$$\int_a^b u(x,t)\,dx = 0, \qquad t \in (0,T), \tag{2.4}$$

with

$$u_0'(a) = 0, \qquad \int_a^b u_0(x)dx = 0, \tag{2.5}$$

where

$$f(x,t) = \mathrm{f}(x,t) - \mathcal{L}U$$

and

$$u_0(x) = \theta_0(x) - U(x,0).$$

Assumption A1: For all $(x,t) \in \overline{Q}$, we assume

$$0 < c_0 \le p(x,t) \le c_1, \;\; 0 < c_2 \le q(x,t) \le c_3, \;\; r(x,t) \le c_4,$$
$$\left|\frac{\partial p}{\partial t}\right| \le c_5, \;\; \left|\frac{\partial p}{\partial x}\right| \le c_6, \;\; \left|\frac{\partial q}{\partial t}\right| \le c_7, \;\; \left|\frac{\partial^2 q}{\partial x \partial t}\right| \le c_8, \;\; \frac{\partial^2 q}{\partial x^2} \le 0.$$

Assumption A2: For all $(x,t) \in \overline{Q}$, we assume

$$\frac{\partial p}{\partial t} \ge 0, \;\; \frac{\partial q}{\partial t} \ge 0, \;\; r \ge 0, \;\; \left|\frac{\partial^2 p}{\partial t^2}\right| \le c_9, \;\; \left|\frac{\partial^2 q}{\partial t^2}\right| \le c_{10}, \;\; \left|\frac{\partial r}{\partial t}\right| \le c_{11}.$$

In Assumptions A1–A2, and in the rest of the paper, c_i, $i = 0, \ldots, 16$ are positive constants.

Let us introduce function spaces needed for our investigation. Let $L^2(a,b)$, $H^1(a,b)$ be the usual Lebesgue and Sobolev spaces. Let H be a Hilbert space with norm $\|\cdot\|_H$, and let $u(\cdot,t) : (0,T) \to H$ be an abstract function. We denote by $L^2(0,T;H)$ the set of all measurable abstract functions $u(\cdot,t)$ from $(0,T)$ to H with the finite norm

$$\|u\|_{L^2(0,T;H)} = \left(\int_0^T \|u(\cdot,t)\|_H^2\,dt\right)^{1/2}$$

and the associated scalar product

$$(u,v)_{L^2(0,T;H)} = \int_0^T (u(\cdot,t), v(\cdot,t))_H\,dt.$$

We denote by $C([0,T];H)$ the set of all continuous abstract functions with

$$\|u\|_{C([0,T];H)} = \max_{t\in[0,T]} \|u(\cdot,t)\|_H < \infty.$$

Let $\widetilde{H}^{1,1}(0,T;H)$ be the Hilbert space obtained by endowing $C^{1,1}([0,T];H)$ with the norm

$$\|u\|_{1,1}^2 = \|u\|_{L^2(0,T;H)}^2 + \left\|\frac{\partial u}{\partial x}\right\|_{L^2(0,T;H)}^2 + \left\|\frac{\partial u}{\partial t}\right\|_{L^2(0,T;H)}^2 + \left\|\frac{\partial^2 u}{\partial x \partial t}\right\|_{L^2(0,T;H)}^2.$$

The scalar product in this space is defined by the equality

$$(u,v)_{1,1} = (u,v)_{L^2(0,T;H)} + \left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}\right)_{L^2(0,T;H)} + \left(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}\right)_{L^2(0,T;H)} + \left(\frac{\partial^2 u}{\partial x \partial t}, \frac{\partial^2 v}{\partial x \partial t}\right)_{L^2(0,T;H)}.$$

By $C_0(a,b)$ we denote the space of continuous functions with compact support in (a,b). We denote by $B_2^1(a,b)$ the so-called Bouziani space [1, 4], the completion of $C_0(a,b)$ for the norm

$$\|u\|_{B_2^1(a,b)} = \|\Im_x u\|_{L^2(a,b)},$$

where $\Im_x u = \int_a^x u(\xi,.)\,d\xi$. The associated scalar product is

$$(u,w)_{B_2^1(a,b)} = \int_a^b \Im_x u \Im_x w dx.$$

It is easy to get

$$\|u\|^2_{B_2^1(a,b)} \leq \frac{(b-a)^2}{2} \|u\|^2_{L^2(a,b)}. \tag{2.6}$$

More generally, we denote by $B_2^{m(t)}(a,b)$ the space of square summable primitive functions of order $m(t)$. It can be considered as a completion of the space $C_0(a,b)$ for the norm

$$\|u\|_{B_2^{m(t)}(a,b)} = \left(\int_a^b \left(\Im_x^{m(t)} u\right)^2 dx\right)^{\frac{1}{2}},$$

with the associated scalar product

$$(u,w)_{B_2^{m(t)}(a,b)} = \int_a^b \Im_x^{m(t)} u \cdot \Im_x^{m(t)} w dx,$$

where

$$\Im_x^{m(t)} u = \frac{1}{(m(t)-1)!} \int_a^x (x-\xi)^{m(t)-1} u(\xi,.)\,d\xi,$$

the function $m(t)$ assumes nonnegative integer values and is bounded. For $m(t) = m$, we have the space $B_2^m(a,b)$ defined in [3], for $m = 0$ we have $B_2^0(a,b) = L^2(a,b)$.

We reformulate problem (2.1)–(2.4) as a problem of solving the operator equation

$$Lu = (f, u_0), \tag{2.7}$$

where L is the operator which maps $u(x,t)$ to the pair of elements $\mathcal{L}u$ and ℓu so that

$$Lu = (\mathcal{L}u, \ell u).$$

Let $D(L)$ be the set of all functions $u \in L^2\left(0,T;B_2^1\left(a,b\right)\right)$ for which $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial t \partial x}$, $\frac{\partial^3 u}{\partial t \partial x^2} \in L^2\left(0,T;B_2^1\left(a,b\right)\right)$, and u verifies conditions (2.3)-(2.4). Let B be the Banach space which is the completion of $D(L)$ in the norm

$$\|u\|_B^2 = \|u\|_{1,1}^2 + \|u\|_{C([0,T];H^1(a,b))}^2 .$$

The elements $u \in B$ are continuous functions on $[0,T]$ which have values in $H^1(a,b)$. Hence the mapping

$$\ell : B \ni u \rightarrow \ell u = u_{|t=0} \in H^1(a,b)$$

is defined and continuous on B. F denotes the Hilbert space $L^2\left(0,T;B_2^1\left(a,b\right)\right) \times H^1(a,b)$ equipped with the scalar product

$$((f,u_0),(f',u_0'))_F := (f,f')_{L^2\left(0,T;B_2^1(a,b)\right)} + (u_0,u_0')_{H^1(a,b)}$$

and the associated norm

$$\|(f,u_0)\|_F := \left(\|f\|_{L^2\left(0,T;B_2^1(a,b)\right)}^2 + \|u_0\|_{H^1(a,b)}^2\right)^{1/2}.$$

The operator L is considered from B to F with the domain $D(L)$, for which we establish the following a priori estimates.

3. A Priori Estimates

3.1. **Continuity of the operator L^{-1}.** We take the scalar product, in $B_2^1\left(a,b\right)$, of equation (2.1) and $\frac{\partial u}{\partial t}$:

$$\begin{aligned}
&\left\|\frac{\partial u\left(\cdot,t\right)}{\partial t}\right\|_{B_2^1(a,b)}^2 - \left(\frac{\partial}{\partial x}\left(p\frac{\partial u\left(\cdot,t\right)}{\partial x}\right), \frac{\partial u\left(\cdot,t\right)}{\partial t}\right)_{B_2^1(a,b)} \\
&- \left(\frac{\partial^2}{\partial t \partial x}\left(q\frac{\partial u\left(\cdot,t\right)}{\partial x}\right), \frac{\partial u\left(\cdot,t\right)}{\partial t}\right)_{B_2^1(a,b)} \\
&+ \left(ru\left(\cdot,t\right), \frac{\partial u\left(\cdot,t\right)}{\partial t}\right)_{B_2^1(a,b)} = \left(f\left(\cdot,t\right), \frac{\partial u\left(\cdot,t\right)}{\partial t}\right)_{B_2^1(a,b)}. \qquad (3.1)
\end{aligned}$$

Integrating by parts the second and the third member on the left-hand side of (3.1), we obtain

$$\begin{aligned}
&- \left(\frac{\partial}{\partial x}\left(p\frac{\partial u\left(\cdot,t\right)}{\partial x}\right), \frac{\partial u\left(\cdot,t\right)}{\partial t}\right)_{B_2^1(a,b)} \\
&= - \left(p\frac{\partial u\left(\cdot,t\right)}{\partial x}, \Im_x \frac{\partial u\left(\cdot,t\right)}{\partial t}\right)_{L^2(a,b)} \\
&= \frac{1}{2}\frac{\partial}{\partial t}\int_a^b pu^2 dx - \frac{1}{2}\int_a^b \frac{\partial p}{\partial t}u^2 dx + \int_a^b \frac{\partial p}{\partial x} u \Im_x \frac{\partial u}{\partial t} dx,
\end{aligned}$$

$$-\left(\frac{\partial^2}{\partial t\partial x}\left(q\frac{\partial u(\cdot,t)}{\partial x}\right),\frac{\partial u(\cdot,t)}{\partial t}\right)_{B_2^1(a,b)}=\int_a^b q\left(\frac{\partial u}{\partial t}\right)^2dx$$
$$-\frac{1}{2}\int_a^b\frac{\partial^2 q}{\partial x^2}\left(\Im_x\frac{\partial u}{\partial t}\right)^2dx+\int_a^b\frac{\partial q}{\partial t}u\frac{\partial u}{\partial t}dx+\int_a^b\frac{\partial^2 q}{\partial t\partial x}u\Im_x\frac{\partial u}{\partial t}dx.$$

Thus equality (3.1) becomes

$$\left\|\frac{\partial u(\cdot,t)}{\partial t}\right\|^2_{B_2^1(a,b)}+\int_a^b q\left(\frac{\partial u}{\partial t}\right)^2dx+\frac{1}{2}\frac{\partial}{\partial t}\int_a^b pu^2dx$$
$$=\left(f(\cdot,t),\frac{\partial u(\cdot,t)}{\partial t}\right)_{B_2^1(a,b)}+\frac{1}{2}\int_a^b\frac{\partial p}{\partial t}u^2dx$$
$$+\frac{1}{2}\int_a^b\frac{\partial^2 q}{\partial x^2}\left(\Im_x\frac{\partial u}{\partial t}\right)^2dx-\int_a^b\frac{\partial p}{\partial x}u\Im_x\frac{\partial u}{\partial t}dx-\int_a^b\frac{\partial q}{\partial t}u\frac{\partial u}{\partial t}dx$$
$$-\int_a^b\frac{\partial^2 q}{\partial t\partial x}u\Im_x\frac{\partial u}{\partial t}dx-\int_a^b\Im_x(ru)\Im_x\frac{\partial u}{\partial t}dx,$$

from which, by integration over $(0,\tau)$, with $0\le\tau\le T$, we have

$$\int_0^\tau\left\|\frac{\partial u(\cdot,t)}{\partial t}\right\|^2_{B_2^1(a,b)}dt+\int_0^\tau\int_a^b q\left(\frac{\partial u}{\partial t}\right)^2dxdt$$
$$+\frac{1}{2}\int_a^b p(x,\tau)u^2(x,\tau)dx$$
$$=\int_0^\tau\left(f(\cdot,t),\frac{\partial u(\cdot,t)}{\partial t}\right)_{B_2^1(a,b)}dt+\frac{1}{2}\int_a^b p(x,0)u_0^2(x)dx$$
$$+\frac{1}{2}\int_0^\tau\int_a^b\frac{\partial p}{\partial t}u^2dxdt+\frac{1}{2}\int_0^\tau\int_a^b\frac{\partial^2 q}{\partial x^2}\left(\Im_x\frac{\partial u}{\partial t}\right)^2dxdt$$
$$-\int_0^\tau\int_a^b\frac{\partial p}{\partial x}u\Im_x\frac{\partial u}{\partial t}dxdt-\int_0^\tau\int_a^b\frac{\partial q}{\partial t}u\frac{\partial u}{\partial t}dxdt$$
$$-\int_0^\tau\int_a^b\frac{\partial^2 q}{\partial t\partial x}u\Im_x\frac{\partial u}{\partial t}dxdt-\int_0^\tau\int_a^b\Im_x(ru)\Im_x\frac{\partial u}{\partial t}dxdt.\qquad(3.2)$$

In view of the Cauchy inequality, the first and the last four terms on the right-hand side of (3.2) are bounded by

$$\frac{\varepsilon_1}{2}\int_0^\tau\|f(\cdot,t)\|^2_{B_2^1(a,b)}dt+\frac{1}{2\varepsilon_1}\int_0^\tau\left\|\frac{\partial u(\cdot,t)}{\partial t}\right\|^2_{B_2^1(a,b)}dt,$$
$$\frac{\varepsilon_2}{2}\int_0^\tau\int_a^b\left(\frac{\partial p}{\partial x}\right)^2u^2dxdt+\frac{1}{2\varepsilon_2}\int_0^\tau\left\|\frac{\partial u(\cdot,t)}{\partial t}\right\|^2_{B_2^1(a,b)}dt,$$
$$\frac{\varepsilon_3}{2}\int_0^\tau\int_a^b\left(\frac{\partial q}{\partial t}\right)^2u^2dxdt+\frac{1}{2\varepsilon_3}\int_0^\tau\left\|\frac{\partial u(\cdot,t)}{\partial t}\right\|^2_{L^2(a,b)}dt,$$

$$\frac{\varepsilon_4}{2}\int_0^\tau\int_a^b\left(\frac{\partial^2 q}{\partial t\partial x}\right)^2 u^2dxdt+\frac{1}{2\varepsilon_4}\int_0^\tau\left\|\frac{\partial u(\cdot,t)}{\partial t}\right\|^2_{B_2^1(a,b)}dt,$$

$$\frac{\varepsilon_5}{2}\int_0^\tau\int_a^b\left(\Im_x(ru)\right)^2 u^2dxdt+\frac{1}{2\varepsilon_5}\int_0^\tau\left\|\frac{\partial u(\cdot,t)}{\partial t}\right\|^2_{B_2^1(a,b)}dt.$$

According to Assumption A1 and inequality (2.7), we obtain by choosing $\varepsilon_i = 2$ $(i = 1, 2, 4, 5)$ and $\varepsilon_3 = 1/c_2$

$$\begin{aligned}
&c_2\int_0^\tau\left\|\frac{\partial u(\cdot,t)}{\partial t}\right\|^2_{L^2(a,b)}dt+c_0\left\|u(\cdot,\tau)\right\|^2_{L^2(a,b)}\\
&\leq 2\int_0^\tau\left\|f(\cdot,t)\right\|^2_{B_2^1(a,b)}dt+c_1\left\|u_0\right\|^2_{L^2(a,b)}\\
&\quad+\left((b-a)^2c_4^2+c_5+2c_6^2+c_7^2/\,c_2+2c_8^2\right)\int_0^\tau\left\|u(\cdot,t)\right\|^2_{L^2(a,b)}dt. \qquad (3.3)
\end{aligned}$$

On the other hand, applying the operator $\Im_x$ to equation (2.1), we get

$$-p\frac{\partial u}{\partial x}-q\frac{\partial^2 u}{\partial x\partial t}=\Im_x f-\Im_x\frac{\partial u}{\partial t}+\frac{\partial q}{\partial t}\frac{\partial u}{\partial x}-\Im_x(ru),$$

which implies

$$\begin{aligned}
&c_0^2\int_0^\tau\left\|\frac{\partial u(\cdot,t)}{\partial x}\right\|^2_{L^2(a,b)}dt+c_2^2\int_0^\tau\left\|\frac{\partial^2 u(\cdot,t)}{\partial x\partial t}\right\|^2_{L^2(a,b)}dt\\
&+c_0c_2\left\|\frac{\partial u(\cdot,\tau)}{\partial x}\right\|^2_{L^2(a,b)}\leq 4\int_0^\tau\left\|f(\cdot,t)\right\|^2_{B_2^1(a,b)}dt\\
&+c_1c_3\left\|u_0'\right\|^2_{L^2(a,b)}+2(b-a)^2c_4^2\int_0^\tau\left\|u(\cdot,t)\right\|^2_{L^2(a,b)}dt\\
&+2(b-a)^2\int_0^\tau\left\|\frac{\partial u(\cdot,t)}{\partial t}\right\|^2_{L^2(a,b)}dt+4c_7^2\int_0^\tau\left\|\frac{\partial u(\cdot,t)}{\partial x}\right\|^2_{L^2(a,b)}dt.
\end{aligned}$$

Multiplying the last inequality by $c_2\,/4\,(b-a)^2$ and adding the result to inequality (3.3),

$$\begin{aligned}
&\frac{c_2}{2}\int_0^\tau\left\|\frac{\partial u(\cdot,t)}{\partial t}\right\|^2_{L^2(a,b)}dt+\frac{c_2c_0^2}{4(b-a)^2}\int_0^\tau\left\|\frac{\partial u(\cdot,t)}{\partial x}\right\|^2_{L^2(a,b)}dt\\
&+\frac{c_2^3}{4(b-a)^2}\int_0^\tau\left\|\frac{\partial^2 u(\cdot,t)}{\partial x\partial t}\right\|^2_{L^2(a,b)}dt+c_0\left\|u(\cdot,\tau)\right\|^2_{L^2(a,b)}\\
&+\frac{c_0c_2^2}{4(b-a)^2}\left\|\frac{\partial u(\cdot,\tau)}{\partial x}\right\|^2_{L^2(a,b)}\\
&\leq\left(2+\frac{c_2}{(b-a)^2}\right)\int_0^\tau\left\|f(\cdot,t)\right\|^2_{B_2^1(a,b)}dt
\end{aligned}$$

$$+ c_1 \|u_0\|^2_{L^2(a,b)} + \frac{c_1 c_2 c_3}{4(b-a)^2} \|u_0'\|^2_{L^2(a,b)}$$
$$+ \left(\frac{c_2 c_4^2}{2} + (b-a)^2 c_4^2 + c_5 + 2c_6^2 + \frac{c_7^2}{c_2} + 2c_8^2 \right) \int_0^\tau \|u(\cdot,t)\|^2_{L^2(a,b)} dt$$
$$+ \frac{c_2 c_7^2}{(b-a)^2} \int_0^\tau \left\| \frac{\partial u(\cdot,t)}{\partial x} \right\|^2_{L^2(a,b)} dt. \tag{3.4}$$

From Poincaré's inequality we have

$$\frac{c_2 c_0^2}{4(b-a)^4} \int_0^\tau \|u(\cdot,t)\|^2_{L^2(a,b)} dt \le \frac{c_2 c_0^2}{8(b-a)^2} \int_0^\tau \left\| \frac{\partial u(\cdot,t)}{\partial x} \right\|^2_{L^2(a,b)} dt,$$

by which inequality (3.4) can be rewritten as

$$\int_0^\tau \left(\|u(\cdot,t)\|^2_{L^2(a,b)} + \left\| \frac{\partial u(\cdot,t)}{\partial t} \right\|^2_{L^2(a,b)} + \left\| \frac{\partial u(\cdot,t)}{\partial x} \right\|^2_{L^2(a,b)} \right.$$
$$\left. + \left\| \frac{\partial^2 u(\cdot,t)}{\partial x \partial t} \right\|^2_{L^2(a,b)} \right) dt + \|u(\cdot,\tau)\|^2_{H^1(a,b)}$$
$$\le c_{12} \left(\int_0^\tau \|f(\cdot,t)\|^2_{B_2^1(a,b)} dt + \|u_0\|^2_{H^1(a,b)} \right) + c_{13} \int_0^\tau \|u(\cdot,t)\|^2_{H^1(a,b)} dt,$$

where

$$c_{12} = \frac{\max\left(c_1, 2 + \frac{c_2}{(b-a)^2}, \frac{c_1 c_2 c_3}{4(b-a)^2}\right)}{\min\left(c_0, \frac{c_2}{2}, \frac{c_2 c_0^2}{4(b-a)^4}, \frac{c_2 c_0^2}{8(b-a)^2}, \frac{c_2^3}{4(b-a)^2}, \frac{c_0 c_2^2}{4(b-a)^4}\right)}$$

and

$$c_{13} = \frac{\max\left(\frac{c_2 c_4^2}{2} + (b-a)^2 c_4^2 + c_5 + 2c_6^2 + \frac{c_7^2}{c_2} + 2c_8^2, \frac{c_2 c_7^2}{(b-a)^2}\right)}{\min\left(c_0, \frac{c_2}{2}, \frac{c_2 c_0^2}{4(b-a)^4}, \frac{c_2 c_0^2}{8(b-a)^2}, \frac{c_2^3}{4(b-a)^2}, \frac{c_0 c_2^2}{4(b-a)^4}\right)}.$$

From Gronwall's Lemma we deduce

$$\int_0^\tau \left(\|u(\cdot,t)\|^2_{L^2(a,b)} + \left\| \frac{\partial u(\cdot,t)}{\partial t} \right\|^2_{L^2(a,b)} + \left\| \frac{\partial u(\cdot,t)}{\partial x} \right\|^2_{L^2(a,b)} \right.$$
$$\left. + \left\| \frac{\partial^2 u(\cdot,t)}{\partial x \partial t} \right\|^2_{L^2(a,b)} \right) dt + \|u(\cdot,\tau)\|^2_{H^1(a,b)}$$
$$\le c_{12} \exp(c_{13}\tau) \left(\int_0^\tau \|f(\cdot,t)\|^2_{B_2^1(a,b)} dt + \|u_0\|^2_{H^1(a,b)} \right).$$

As the right-hand side of this inequality can be taken to be independent of τ, we replace the left-hand side by its upper bound with respect to τ from 0 to T, and as a result we have

$$\|u\|^2_{1,1} + \|u\|^2_{C([0,T];H^1(a,b))} \le c_{12} \exp(c_{13}T) \left(\int_0^T \|f(\cdot,t)\|^2_{B_2^1(a,b)} dt + \|u_0\|^2_{H^1(a,b)} \right).$$

We have therefore obtained

Theorem 1. *Let Assumption* A1 *be fulfilled. The operator L has, on $R(L)$, the continuous inverse L^{-1}, i.e., the following estimate*

$$\|u\|_B \leq c_{14} \|Lu\|_F , \tag{3.5}$$

holds for any function $u \in D(L)$, where $c_{14} = c_{12}^{1/2} \exp(c_{13}T/2)$.

Corollary 1. *Under the conditions of Theorem* 1*, if u is the solution of problem* (2.1)–(2.4) *corresponding to the given data f, u_0, and u^* is the solution of problem* (2.1)–(2.4) *corresponding to the given data f^*, u_0^*, then we have*

$$\|u-u^*\|_B \leq c_{14} \left(\|f-f^*\|^2_{L^2\left(0,T;B_2^1(a,b)\right)} + \|u_0-u_0^*\|^2_{H^1(a,b)} \right)^{1/2} .$$

Corollary 2. *Under the conditions of Theorem* 1*, the solution of problem* (2.1)–(2.4) *is unique.*

3.2. **Continuity of the operator** L**.** If we take the square of the norm of $\mathcal{L}u$ in the space $L^2(0,T;B_2^1(a,b))$, then we obtain

$$\begin{aligned} \|\mathcal{L}u\|^2_{L^2\left(0,T;B_2^1(a,b)\right)} &\leq 2(b-a)^2 c_4^2 \|u\|^2_{L^2(0,T;L^2(a,b))} \\ &+ 2(b-a)^2 \left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(0,T;L^2(a,b))} + 8\left(c_1^2+c_7^2\right) \left\|\frac{\partial u}{\partial x}\right\|^2_{L^2(0,T;L^2(a,b))} \\ &+ 4c_3^2 \left\|\frac{\partial^2 u}{\partial x \partial t}\right\|^2_{L^2(0,T;L^2(a,b))} . \end{aligned} \tag{3.6}$$

Observe that

$$\|\ell u\|^2_{H^1(a,b)} \leq \|u\|^2_{C([0,T];H^1(a,b))} . \tag{3.7}$$

Then from (3.6)–(3.7) follows

Theorem 2. *Under Assumption* A1 *the following estimate*

$$\|Lu\|_F \leq c_{15} \|u\|_B \tag{3.8}$$

holds for all $u \in D(L)$, where

$$c_{15} = \sqrt{\max\left(2(b-a)^2 c_4^2, 2(b-a)^2, 8\left(c_1^2+c_7^2\right), 4c_3^2, 1\right)}.$$

By virtue of Theorems 1 and 2 we have

Corollary 3. *The operator L maps B homeomorphically onto the closed set $R(L) \subset F$.*

4. Existence of a Solution

Theorem 3. *Suppose that Assumptions A1 and A2 are satisfied. Then problem* (2.1)–(2.4) *possesses a unique solution $u = L^{-1}(f, u_0)$ for any function $f \in L^2(0,T;B_2^1(a,b))$ and $u_0 \in H^1(a,b)$.*

Proof. To establish the existence of a solution, it suffices to show that $R(L) = F$. To this end, we first prove that $R(L)$ is dense in F for a special case in which u belongs to $D_0(L) = \{u/u \in D(L) : \ell u = 0\}$:

Proposition 1. *Let the assumptions of Theorem* 3 *hold. If*
$$(\mathcal{L}u,\omega)_{L^2\left(0,T;B_2^1(a,b)\right)}=0 \tag{4.1}$$
for all $u\in D_0(L)$ *and some* $\omega\in L^2\left(0,T;B_2^1\left(a,b\right)\right)$, *then* ω *vanishes almost everywhere in* Q.

Proof of Proposition 1. As equality (4.1) is fulfilled for any function $u\in D_0\left(L\right)$, we express it in the following way:

Let s be any arbitrary fixed number belonging to $[0,T]$; we set
$$v=\begin{cases}0, & \text{if } 0\le t\le s,\\ \int_s^t \frac{\partial u}{\partial\tau}d\tau, & \text{if } s\le t\le T,\end{cases} \tag{4.2}$$
and let $\frac{\partial v}{\partial t}$ be a solution of the equation
$$-q\left(\sigma,t\right)\frac{\partial^2 v}{\partial x\partial t}=\Im_t^*\left(\Im_x\omega\right)=\int_t^T\Im_x\omega d\tau, \tag{4.3}$$
where σ is a fixed number in $\left[a,b\right]$.

It is easy to see from relations (4.2) and (4.3) that $v\in D_s(L)=\{v\mid v\in D(L): v=0 \text{ for } t\le s\}\subseteq D_0\left(L\right), \frac{\partial v(x,T)}{\partial t}=0$, and possesses a high order of smoothness.

Differentiating (4.3) with respect to t and x, we get
$$\omega=\frac{\partial}{\partial t}\left(q\left(\sigma,t\right)\frac{\partial^3 v}{\partial x^2\partial t}\right). \tag{4.4}$$

We use the following assertion the proof of which is the same as that in [2].

Lemma 1. *If the assumptions of Theorem* 3 *are satisfied, then the function* ω *defined by* (4.4) *belongs to* $L^2\left(s,T;B_2^1\left(a,b\right)\right)$.

Substituting (2.1) and (4.4) into (4.1) and by taking into account the particular form of v, we obtain
$$\begin{aligned}&\left(\frac{\partial v}{\partial t},\frac{\partial}{\partial t}\left(q\left(\sigma,t\right)\frac{\partial^3 v}{\partial x^2\partial t}\right)\right)_{L^2\left(s,T;B_2^1(a,b)\right)}\\ &-\left(\frac{\partial}{\partial x}\left(p\frac{\partial v}{\partial x}\right),\frac{\partial}{\partial t}\left(q\left(\sigma,t\right)\frac{\partial^3 v}{\partial x^2\partial t}\right)\right)_{L^2\left(s,T;B_2^1(a,b)\right)}\\ &-\left(\frac{\partial^2}{\partial t\partial x}\left(q\frac{\partial v}{\partial x}\right),\frac{\partial}{\partial t}\left(q\left(\sigma,t\right)\frac{\partial^3 v}{\partial x^2\partial t}\right)\right)_{L^2\left(s,T;B_2^1(a,b)\right)}\\ &+\left(rv,\frac{\partial}{\partial t}\left(q\left(\sigma,t\right)\frac{\partial^3 v}{\partial x^2\partial t}\right)\right)_{L^2\left(s,T;B_2^1(a,b)\right)}=0.\end{aligned} \tag{4.5}$$

The standard integration by parts of each term of (4.5) leads to
$$\left(\frac{\partial v}{\partial t},\frac{\partial}{\partial t}\left(q\left(\sigma,t\right)\frac{\partial^3 v}{\partial x^2\partial t}\right)\right)_{L^2\left(s,T;B_2^1(a,b)\right)}$$

$$= \frac{1}{2}\int_a^b q(\sigma,s)\left(\frac{\partial v(x,s)}{\partial t}\right)^2 dx - \frac{1}{2}\int_s^T \int_a^b q'(\sigma,t)\left(\frac{\partial v}{\partial t}\right)^2 dxdt, \tag{4.6}$$

$$-\left(\frac{\partial}{\partial x}\left(p(x,t)\frac{\partial v}{\partial x}\right), \frac{\partial}{\partial t}\left(q(\sigma,t)\frac{\partial^3 v}{\partial x^2 \partial t}\right)\right)_{L^2\left(s,T;B_2^1(a,b)\right)}$$
$$= \int_s^T \int_a^b p(x,t)\,q(\sigma,t)\left(\frac{\partial^2 v}{\partial x \partial t}\right)^2 dxdt$$
$$+ \frac{1}{2}\int_a^b \frac{\partial p(x,T)}{\partial t} q(\sigma,T)\left(\frac{\partial v(x,T)}{\partial x}\right)^2 dx$$
$$- \frac{1}{2}\int_s^T \int_a^b \left(\frac{\partial^2 p(x,t)}{\partial t^2} q(\sigma,t) + \frac{\partial p(x,t)}{\partial t} q'(\sigma,t)\right)\left(\frac{\partial v}{\partial x}\right)^2 dxdt, \tag{4.7}$$

$$-\left(\frac{\partial^2}{\partial t \partial x}\left(q\frac{\partial v}{\partial x}\right), \frac{\partial}{\partial t}\left(q(\sigma,t)\frac{\partial^3 v}{\partial x^2 \partial t}\right)\right)_{L^2\left(s,T;B_2^1(a,b)\right)}$$
$$= \frac{1}{2}\int_a^b q(x,s)\,q(\sigma,s)\left(\frac{\partial^2 v(x,s)}{\partial x \partial t}\right)^2 dx$$
$$+ \frac{3}{2}\int_s^T \int_a^b \frac{\partial q(x,t)}{\partial t} q(\sigma,t)\left(\frac{\partial^2 v}{\partial x \partial t}\right)^2 dxdt$$
$$- \frac{1}{2}\int_s^T \int_a^b q(x,t)\,q'(\sigma,t)\left(\frac{\partial^2 v}{\partial x \partial t}\right)^2 dxdt$$
$$+ \int_s^T \int_a^b \frac{\partial^2 q(x,t)}{\partial t^2} q(\sigma,t)\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x \partial t} dxdt, \tag{4.8}$$

$$\left(rv, \frac{\partial}{\partial t}\left(q(\sigma,t)\frac{\partial^3 v}{\partial x^2 \partial t}\right)\right)_{L^2\left(s,T;B_2^1(a,b)\right)}$$
$$= -\int_s^T \int_a^b \Im_x\left(r\frac{\partial v}{\partial t}\right) q(\sigma,t)\frac{\partial^2 v}{\partial x \partial t} dxdt$$
$$- \int_s^T \int_a^b \Im_x\left(\frac{\partial r}{\partial t} v\right) q(\sigma,t)\frac{\partial^2 v}{\partial x \partial t} dxdt. \tag{4.9}$$

It then follows from (4.5)-(4.9) that

$$\int_a^b \frac{\partial p(x,T)}{\partial t} q(\sigma,T)\left(\frac{\partial v(x,T)}{\partial x}\right)^2 dx + \int_a^b q(\sigma,s)\left(\frac{\partial v(x,s)}{\partial t}\right)^2 dx$$
$$\int_a^b q(x,s)\,q(\sigma,s)\left(\frac{\partial^2 v(x,s)}{\partial x \partial t}\right)^2 dx$$
$$+ \int_s^T \int_a^b \left(3\frac{\partial q(x,t)}{\partial t} q(\sigma,t) + 2p(x,t)\,q(\sigma,t)\right)\left(\frac{\partial^2 v}{\partial x \partial t}\right)^2 dxdt$$

$$
\begin{aligned}
&= \int_s^T \int_a^b \left(\frac{\partial^2 p(x,t)}{\partial t^2} q(\sigma,t) + \frac{\partial p(x,t)}{\partial t} q'(\sigma,t) \right) \left(\frac{\partial v}{\partial x} \right)^2 dxdt \\
&+ \int_s^T \int_a^b q'(\sigma,t) \left(\frac{\partial v}{\partial t} \right)^2 dxdt + \int_s^T \int_a^b qq'(\sigma,t) \left(\frac{\partial^2 v}{\partial x \partial t} \right)^2 dxdt \\
&- 2 \int_s^T \int_a^b \frac{\partial^2 q(x,t)}{\partial t^2} q(\sigma,t) \frac{\partial v}{\partial x} \frac{\partial^2 v}{\partial x \partial t} dxdt \\
&+ 2 \int_s^T \int_a^b \Im_x \left(r \frac{\partial v}{\partial t} \right) q(\sigma,t) \frac{\partial^2 v}{\partial x \partial t} dxdt \\
&+ 2 \int_s^T \int_a^b \Im_x \left(\frac{\partial r}{\partial t} v \right) q(\sigma,t) \frac{\partial^2 v}{\partial x \partial t} dxdt. \qquad (4.10)
\end{aligned}
$$

According to the Cauchy inequality and inequality (2.7), the right-hand side of the above inequality is then dominated by

$$
\begin{aligned}
&\int_s^T \int_a^b \Bigg(\frac{\partial^2 p(x,t)}{\partial t^2} q(\sigma,t) + \frac{\partial p(x,t)}{\partial t} q'(\sigma,t) \\
&+ \left(\frac{\partial^2 q(x,t)}{\partial t^2} \right)^2 \Bigg) \left(\frac{\partial v}{\partial x} \right)^2 dxdt + \frac{(b-a)^2}{2} \int_s^T \int_a^b \left(\frac{\partial r(x,t)}{\partial t} \right)^2 v^2 dxdt \\
&+ \int_s^T \int_a^b \left(q'(\sigma,t) + \frac{(b-a)^2}{2} r^2 \right)(x,t) \left(\frac{\partial v}{\partial t} \right)^2 dxdt \\
&+ \int_s^T \int_a^b \left(3q^2(\sigma,t) + q(x,t) q'(\sigma,t) \right) \left(\frac{\partial^2 v}{\partial x \partial t} \right)^2 dxdt.
\end{aligned}
$$

Substituting it into (4.10) and using Assumptions A1–A2, we obtain

$$
\begin{aligned}
&c_2 \left\| \frac{\partial v(\cdot,s)}{\partial t} \right\|^2_{L^2(a,b)} + c_2^2 \left\| \frac{\partial^2 v(\cdot,s)}{\partial x \partial t} \right\|^2_{L^2(a,b)} \\
&\leq \frac{(b-a)^2 c_{11}^2}{2} \int_s^T \|v(\cdot,t)\|^2_{L^2(a,b)} dt \\
&+ \left(c_3 c_9 + c_5 c_7 + c_{10}^2 \right) \int_s^T \left\| \frac{\partial v(\cdot,t)}{\partial x} \right\|^2_{L^2(a,b)} dt \\
&+ \left(c_7 + \frac{(b-a)^2 c_4^2}{2} \right) \int_s^T \left\| \frac{\partial v(\cdot,t)}{\partial t} \right\|^2_{L^2(a,b)} dt \\
&+ \left(3c_3^2 + c_3 c_7 \right) \int_s^T \left\| \frac{\partial^2 v(\cdot,t)}{\partial x \partial t} \right\|^2_{L^2(a,b)} dt. \qquad (4.11)
\end{aligned}
$$

In view of the elementary inequalities

$$
\|v(\cdot,T)\|^2_{L^2(a,b)} \leq \int_s^T \|v(\cdot,t)\|^2_{L^2(a,b)} dt + \int_s^T \left\| \frac{\partial v(\cdot,t)}{\partial t} \right\|^2_{L^2(a,b)} dt,
$$

$$\left\|\frac{\partial v(\cdot,T)}{\partial x}\right\|^2_{L^2(a,b)} \leq \int_s^T \left\|\frac{\partial v(\cdot,t)}{\partial x}\right\|^2_{L^2(a,b)} + \int_s^T \left\|\frac{\partial^2 v(\cdot,t)}{\partial x \partial t}\right\|^2_{L^2(a,b)} dt,$$

estimate (4.11) becomes

$$\|v(\cdot,T)\|^2_{H^1(a,b)} + \left\|\frac{\partial v(\cdot,s)}{\partial t}\right\|^2_{H^1(a,b)} \leq c_{16}\int_s^T \left(\|v(\cdot,t)\|^2_{H^1(a,b)} + \left\|\frac{\partial v(\cdot,t)}{\partial t}\right\|^2_{H^1(a,b)}\right) dt, \tag{4.12}$$

where

$$c_{16} = \frac{1+\max\left(\frac{(b-a)^2 c_{11}^2}{2}, c_7 + \frac{(b-a)^2 c_4^2}{2}, c_3 c_9 + c_5 c_7 + c_{10}^2, 3c_3^2 + c_3 c_7\right)}{\min\left(1, c_2, c_2^2\right)}.$$

Now, if we set

$$\eta(x,t) = \int_t^T \frac{\partial v(x,\tau)}{\partial \tau} d\tau,$$

then we have

$$\eta(x,t) = v(x,T) - v(x,t) \text{ and } \eta(x,s) = v(x,T),$$

therefore

$$v(x,t) = \eta(x,s) - \eta(x,t).$$

Let us substitute this expression for v into (4.12) and then majorize the right-hand side of the obtained inequality. As a result,

$$\begin{aligned}
&\|\eta(\cdot,s)\|^2_{H^1(a,b)} + \left\|\frac{\partial v(\cdot,s)}{\partial t}\right\|^2_{H^1(a,b)} \\
&\leq c_{16}\int_s^T \left(\|\eta(\cdot,s) - \eta(\cdot,t)\|^2_{H^1(a,b)} dt + \left\|\frac{\partial v(\cdot,t)}{\partial t}\right\|^2_{H^1(a,b)}\right) dt \\
&\leq 2c_{16}(T-s)\|\eta(\cdot,s)\|^2_{H^1(a,b)} + 2c_{16}\int_s^T \|\eta(\cdot,t)\|^2_{H^1(a,b)} dt \\
&\quad + c_{16}\int_s^T \left\|\frac{\partial v(\cdot,t)}{\partial t}\right\|^2_{H^1(a,b)} dt.
\end{aligned}$$

For $0 \leq 2c_{16}(T-s) \leq 1/2$, we obtain from the above inequality

$$\|\eta(\cdot,s)\|^2_{H^1(a,b)} + \left\|\frac{\partial v(\cdot,s)}{\partial t}\right\|^2_{H^1(a,b)} \leq 4c_{16}\int_s^T \left(\|\eta(\cdot,t)\|^2_{H^1(a,b)} + \left\|\frac{\partial v(\cdot,t)}{\partial t}\right\|^2_{H^1(a,b)}\right) dt,$$

which holds for all $s \in [T-s_0, T]$, where $s_0 = 1/(4c_{16})$.

Applying Gronwall's Lemma for the case of the reversed interval to the last inequality, we conclude that

$$\|\eta(\cdot,s)\|^2_{H^1(a,b)} + \left\|\frac{\partial v(\cdot,s)}{\partial t}\right\|^2_{H^1(a,b)} = 0, \qquad T - s_0 \le s \le T,$$

and hence $\omega \equiv 0$ almost everywhere in $(a,b) \times (T - s_0, T)$. If we repeat this argument step by step along rectangles of length s_0, and so on, we shall exhaust the entire rectangle Q, thus ω vanishes almost everywhere in Q. □

After proving Proposition 1, we turn to the proof of Theorem 3. Let $V = (f, u_0)$ of F be orthogonal to every element of $R(L)$, i.e., be such that

$$\begin{aligned}(Lu, V)_F &= ((\mathcal{L}u, \ell u), (f, u_0))_F \\ &= (\mathcal{L}u, f)_{L^2\left(0,T;B_2^1(a,b)\right)} + (\ell u, u_0)_{H^1(a,b)} = 0\end{aligned} \tag{4.13}$$

for all $u \in D(L)$. We have to show that $V = 0$. Suppose, in (4.13), that u is replaced by an element of $D_0(L)$; then (4.13) becomes

$$(\mathcal{L}u, f)_{L^2\left(0,T;B_2^1(a,b)\right)} = 0.$$

It then follows from Proposition 3 that $f = 0$. Because of this (4.13) takes the form

$$(\ell u, u_0)_{H^1(a,b)} = 0, \qquad u \in D(L).$$

Since the range $R(\ell)$ of the trace operator ℓ is dense in $H^1(a,b)$, the last equality implies that $u_0 = 0$ (we recall that u_0 satisfies condition (2.5)) [7]. Hence $V = 0$ and thus $\overline{R(L)} = F$. Since $R(L)$ is closed, we have proved that $R(L) = F$. This completes the proof of Theorem 3. □

5. Generalization

In this section, we state some results concerning the generalization of problem (1.6)-(1.9), together with some indications about their proofs.

1. Our results can be extended to a mixed problem for a pluripseudoparabolic equation with Neumann-integral boundary conditions [or, more generally, for a pluripseudoparabolic equation with nonlocal initial conditions and Neumann-integral boundary conditions, respectively]:

$$\begin{aligned}\mathcal{L}\theta &= \sum_{i=1}^{n} \frac{\partial\theta}{\partial t_i} - \frac{\partial}{\partial x}\left(p(x,t)\frac{\partial\theta}{\partial x}\right) \\ &\quad - \sum_{i=1}^{n} \frac{\partial^2}{\partial t_i \partial x}\left(q(x,t)\frac{\partial\theta}{\partial x}\right) + r(x,t)\theta \\ &= \mathrm{f}(x,t), \quad x \in (a,b), \quad t = (t_1,\dots,t_n) \in \prod_{i=1}^{n}(0,T_i),\end{aligned} \tag{5.1}$$

$$\left[\mathcal{L}\theta = \sum_{i=1}^{n} \frac{\partial\theta}{\partial t_i} - \sin\mathrm{g}\prod_{i=1}^{n}\left(1 - |\lambda_i|^2\right)\left\{\frac{\partial}{\partial x}\left(p(x,t)\frac{\partial\theta}{\partial x}\right)\right.\right.$$

$$- \sum_{i=1}^{n} \frac{\partial^2}{\partial t_i \partial x} \left(q(x,t) \frac{\partial \theta}{\partial x} \right) \Bigg\} + r(x,t)\theta$$

$$= \mathrm{f}\,(x,t)\,,\, x \in (a,b)\,,\; t = (t_1, \ldots, t_n) \in \prod_{i=1}^{n} (0,T_i)\,,\ \text{respectively}\Bigg] \tag{5.1$'$}$$

$$\begin{aligned} \ell\theta &= \theta(x, t_1, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_n) \\ &= \theta_0(x, t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n), \\ &\quad (x, t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n) \in (a,b) \times \prod_{\substack{j=1 \\ j \neq i}}^{n} (0, T_j)\,, \end{aligned} \tag{5.2}$$

$$\begin{aligned} \Big[\ell\theta &= \theta(x, t_1, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_n) \\ &\qquad - \lambda_i \theta(x, t_1, \ldots, t_{i-1}, T_i, t_{i+1}, \ldots, t_n) \\ &= \theta_0(x, t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n), \\ &\quad (x, t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n) \in (a,b) \times \prod_{\substack{j=1 \\ j \neq i}}^{n} (0, T_j)\,,\ \text{respectively}\Big] \end{aligned} \tag{5.2$'$}$$

$$\frac{\partial \theta(a, t_1, \ldots, t_n)}{\partial x} = \mu(t_1, \ldots, t_n), \qquad (t_1, \ldots, t_n) \in \prod_{i=1}^{n} (0, T_i)\,, \tag{5.3}$$

$$\int_a^b \theta\,(x, t_1, \ldots, t_n)\, dx = E(t_1, \ldots, t_n), \quad (t_1, \ldots, t_n) \in \prod_{i=1}^{n} (0, T_i)\,. \tag{5.4}$$

For problems (5.1)–(5.4) [(5.1′)–(5.2′), (5.3)–(5.4), respectively] we show that the operator L^{-1} is bounded by taking the scalar product in $B_2^1\,(a,b)$ of the considered equation and $\sum_{i=1}^{n} \frac{\partial u}{\partial t_i}$; then we proceed exactly as in the proof of Theorem 1. Concerning the continuity of the operator L, the proof is analogous to that of Theorem 2. As for the existence of a solution, we first construct an appropriate ω

$$\omega = \sum_{i=1}^{n} \omega_i = \sum_{i=1}^{n} \frac{\partial}{\partial t_i} \left(q\,(\sigma, t)\, \frac{\partial^3 v}{\partial x^2 \partial t_i} \right),$$

then take the scalar product of ω and the considered equation in the space $L^2\left(\prod_{i=1}^{n}(0,T_i)\,; B_2^1\,(a,b)\right)$, and, using the same reasoning as in the proof of Theorem 3, get the results.

2. We can prove that our results still hold for a $2m$-pseudoparabolic equation [or, more generally, for a pseudoparabolic equation with variable order, respectively], with Neumann and integral conditions; i.e.,

$$\mathcal{L}\theta = \frac{\partial \theta}{\partial t} + (-1)^m \frac{\partial^{2m-1}}{\partial x^{2m-1}} \left(p \frac{\partial \theta}{\partial x} \right) + (-1)^m \frac{\partial^{2m}}{\partial t \partial x^{2m-1}} \left(q \frac{\partial \theta}{\partial x} \right) + P_m \theta = \mathrm{f},$$

$$\left[\mathcal{L}\theta = \frac{\partial\theta}{\partial t} + (-1)^{m(t)} \frac{\partial^{2m(t)-1}}{\partial x^{2m(t)-1}} \left(p\frac{\partial\theta}{\partial x}\right) + (-1)^{m(t)} \frac{\partial^{2m(t)}}{\partial t \partial x^{2m(t)-1}} \left(q\frac{\partial\theta}{\partial x}\right) \right.$$
$$\left. + P_{m(t)}\theta = \text{f, respectively}\right].$$

Here P_m [$P_{m(t)}$, respectively] denotes a differential operator of order m [$m(t)$, respectively].

In this case, it suffices to replace in the proofs of the corresponding theorems the space $B_2^1(a,b)$ by $B_2^m(a,b)$ [$B_2^{m(t)}(a,b)$, respectively].

3. Of course, we can combine these problems to obtain a more general problem, the solvability of which is the same as that of the studied problem.

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