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OSCILLATION OF SECOND ORDER HALF-LINEAR DIFFERENTIAL EQUATIONS WITH DAMPING

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Abstract. This paper is concerned with a class of second order half-linear damped differential equations. Using the generalized Riccati transformation and the averaging technique, new oscillation criteria are obtained which are either extensions of or complementary to a number of the existing results.

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1. Introduction

Qualitative properties of half-linear differential equations of the form

$$\left[r(t)\,|x'(t)|^{\alpha-1}x'(t)\right]' + q(t)\,|x(t)|^{\alpha-1}x(t) = 0$$

have been investigated by many authors. In particular, Wang in [1] obtained a number of oscillation theorems by means of the generalized Riccati transformation and the averaging technique. On the other hand, Kong in [2] (see also Li and Agarwal [3]) also obtained a number of oscillation criteria based on Wirtinger type inequalities when $\alpha = 1$. It turns out that with extra effort, their results can be extended to an associated nonlinear differential equation with damping term

$$\left[r(t)\,|x'(t)|^{\alpha-1}x'(t)\right]' + p(t)\,|x'(t)|^{\alpha-1}x'(t) + q(t)\,|x(t)|^{\alpha-1}x(t) = 0, \quad t \ge t_0, \quad (1)$$

where $t_0 \ge 0$, $r \in C^1([t_0,\infty);(0,\infty))$, $p, q \in C([t_0,\infty);R)$ and $\alpha > 0$.

Our results are also extensions of a number of the existing results for linear equations with damping, and therefore will be of interest to researchers working in this area.

As is well known, the following inequality [4] is useful in working with half-linear equations: Let X and Y be nonnegative numbers and $q > 1$, then $X^q + (q-1)Y^q - qXY^{q-1} \ge 0$, where the equality holds if and only if $X = Y$.

2. Oscillation Criteria

As in [1], we set $D_0 = \{(t,s) : t > s \ge t_0\}$ and $D = \{(t,s) : t \ge s \ge t_0\}$. We first establish oscillation criteria that involve the behavior of the functions p, q and r on the entire half-line $[t_0,\infty)$.

Theorem 1. *Let functions $H \in C(D;R)$, $h \in C(D_0;R)$, $k, \rho \in C^1([t_0,\infty);(0,\infty))$ satisfy the following conditions:*

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(I) $H(t,t) = 0$ *for* $t \geq t_0$, $H(t,s) > 0$ *on* D_0;
(II) $H(t,s)$ *has a continuous and nonpositive partial derivative* $\partial H/\partial s$ *on* D_0;
(III) *for* $(t,s) \in D_0$,

$$-\frac{\partial}{\partial s}\big(H(t,s)\,k(s)\big) + H(t,s)\,k(s)\left[\frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)}\right] = h(t,s). \tag{2}$$

If

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \Bigg[H(t,s)\,k(s)\,\rho(s)\,q(s) - \frac{\rho(s)\,r(s)\,|h(t,s)|^{\alpha+1}}{(\alpha+1)^{\alpha+1}\,(H(t,s)\,k(s))^{\alpha}}\Bigg]ds = \infty, \tag{3}$$

then equation (1) *is oscillatory.*

Proof. By comparing Theorem 1 with that in [1], we see that the only difference is the additional term $p(s)/r(s)$ in (2). Therefore we expect a similar proof. Indeed, we can give a sketch below. Let $x(t)$ be a nonoscillatory solution of equation (1). Without loss of generality, we may assume that $x(t) \neq 0$ for $t \geq T_0 \geq t_0$. Define

$$w(t) = \rho(t)\frac{r(t)|x'(t)|^{\alpha-1}x'(t)}{|x(t)|^{\alpha-1}x(t)}\,. \tag{4}$$

Differentiating (4) and making use of (1) as well as the assumptions of our theorem, we may see that for $t \geq T_0$,

$$w'(t) = -\rho(t)\,q(t) - \left[\frac{p(t)}{r(t)} - \frac{\rho'(t)}{\rho(t)}\right]w(t) - \alpha\,\frac{|w(t)|^{(\alpha+1)/\alpha}}{(\rho(t)\,r(t))^{1/\alpha}}\,. \tag{5}$$

If we replace t in (5) by s, multiply the resulting equation by $H(t,s)\,k(s)$ and then integrate from T to t, where $t \geq T \geq T_0$, we have

$$\begin{aligned}
\int_T^t H(t,s)\,k(s)\,\rho(s)\,q(s)ds &= H(t,T)\,k(T)\,w(T) \\
&\quad - \int_T^t \Big[-\frac{\partial}{\partial s}\big(H(t,s)\,k(s)\big) + H(t,s)\,k(s)\Big(\frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)}\Big)\Big]w(s)\,ds \\
&\quad - \int_T^t \alpha H(t,s)\,k(s)\frac{|w(s)|^{(\alpha+1)/\alpha}}{(\rho(s)\,r(s))^{1/\alpha}}\,ds = H(t,T)\,k(T)\,w(T) \\
&\quad - \int_T^t h(t,s)\,w(s)\,ds - \int_T^t \alpha\,H(t,s)\,k(s)\,\frac{|w(s)|^{(\alpha+1)/\alpha}}{(\rho(s)\,r(s))^{1/\alpha}}\,ds
\end{aligned}$$

$$\le H(t,T)\,k(T)\,w(T) + \int_T^t |h(t,s)\,w(s)|\,ds$$

$$- \int_T^t \alpha\,H(t,s)\,k(s)\frac{|w(s)|^{(\alpha+1)/\alpha}}{(\rho(s)\,r(s))^{1/\alpha}}\,ds. \tag{6}$$

If we let $q = (\alpha+1)/\alpha$,

$$X = \big(\alpha\,H(t,s)\,k(s)\big)^{\alpha/(\alpha+1)}\,\frac{|w(s)|}{(\rho(s)\,r(s))^{1/(\alpha+1)}},$$

and

$$Y = \left[\frac{\alpha}{\alpha+1}\right]^{\alpha}\left[\frac{\rho(s)\,r(s)}{(\alpha H(t,s)\,k(s))^{\alpha}}\right]^{\alpha/(\alpha+1)}|h(t,s)|^{\alpha},$$

then, in view of the inequality mentioned before, we obtain for $t > s \ge t_0$

$$\big|(t,s)w(s)\big| - \alpha H(t,s)\,k(s)\frac{|w(s)|^{(\alpha+1)/\alpha}}{(\rho(s)\,r(s))^{1/\alpha}}$$

$$\le \frac{\rho(s)\,r(s)}{(\alpha+1)^{\alpha+1}(H(t,s)\,k(s))^{\alpha}}|h(t,s)|^{\alpha+1}. \tag{7}$$

It follows from (6) and (7) that

$$\int_T^t \left[H(t,s)\,k(s)\,\rho(s)\,q(s) - \frac{\rho(s)\,r(s)}{(\alpha+1)^{\alpha+1}(H(t,s)\,k(s))^{\alpha}}\,|h(t,s)|^{\alpha+1}\right]ds$$

$$\le H(t,t_0)\,k(T)\,|w(T)| \tag{8}$$

for every $t \ge T \ge T_0 \ge t_0$. Thus we obtain

$$\int_{t_0}^t \left[H(t,s)\,k(s)\,\rho(s)\,q(s) - \frac{\rho(s)\,r(s)}{(\alpha+1)^{\alpha+1}(H(t,s)\,k(s))^{\alpha}}|h(t,s)|^{\alpha+1}\right]ds$$

$$\le \left\{\int_{t_0}^{T_0} + \int_{T_0}^{t}\right\}\Bigg[H(t,s)\,k(s)\,\rho(s)\,q(s)$$

$$- \frac{\rho(s)\,r(s)}{(\alpha+1)^{\alpha+1}(H(t,s)\,k(s))^{\alpha}}|h(t,s)|^{\alpha+1}\Bigg]\,ds$$

$$\le H(t,t_0)\left[\int_{t_0}^{T_0} k(s)\,\rho(s)|q(s)|\,ds + k(T_0)|w(T_0)|\right]. \tag{9}$$

Dividing both sides of (9) by $H(t,t_0)$ and taking the superior limit as $t \to \infty$, we have

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\,k(s)\,\rho(s)\,q(s) - \frac{\rho(s)\,r(s)|h(t,s)|^{\alpha+1}}{(\alpha+1)^{\alpha+1}(H(t,s)\,k(s))^{\alpha}} \right] ds$$

$$\le \int_{t_0}^{T_0} |k(s)\,\rho(s)\,q(s)|\,ds + k(T_0)|w(T_0)| < \infty,$$

which is contrary to (3). □

In view of the previous presentation, it is not difficult to follow the details in [1] and check the following statement.

Theorem 2. *Theorems* 2 *and* 3 *in* [1] *are formally valid for our equation* (1) *if we assume that condition* (III) *in* [1] *is replaced by condition* (III) *in our Theorem* 1.

Theorem 3. *Theorems* 4, 5 *and* 6 *in* [1] *are formally valid for our equation* (1) *if the condition*

$$-\frac{\partial}{\partial s}(H(t,s)\,k(s)) - H(t,s)\,k(s)\frac{\rho'(s)}{\rho(s)} = h(t,s)\sqrt{H(t,s)\,k(s)}, \quad (t,s) \in D,$$

in [1, *Theorem* 4] *is replaced by*

$$-\frac{\partial}{\partial s}(H(t,s)\,k(s)) + H(t,s)\,k(s)\left[\frac{p(s)}{r(s)} - \frac{\rho'(s)}{\rho(s)}\right] = h(t,s)\sqrt{H(t,s)\,k(s)}, \quad (t,s) \in D.$$

We now turn to oscillation criteria which involve the behavior of the functions p, q, r over the sequences of subintervals of $[t_0, \infty)$.

Theorem 4. *Let functions* $H \in C(D;R)$, $h_1, h_2 \in C(D_0;R)$, $k, \rho \in C^1([t_0,\infty);(0,\infty))$ *satisfy the following conditions:*

(I) $H(t,t) = 0$ *for* $t \ge t_0$, $H(t,s) > 0$ *on* D_0;

(II) *for* $(t,s) \in D_0$,

$$\frac{\partial}{\partial t}(H(t,s)\,k(t)) + H(t,s)\,k(t)\left[\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r(t)}\right] = h_1(t,s);$$

(III) *for* $(t,s) \in D_0$,

$$\frac{\partial}{\partial s}(H(t,s)\,k(s)) + H(t,s)\,k(s)\left[\frac{\rho'(s)}{\rho(s)} - \frac{p(s)}{r(s)}\right] = -h_2(t,s).$$

If for each $T_0 \ge t_0$, *there exist* $a, b, c \in R$ *with* $T_0 \le a < c < b$ *such that*

$$\frac{1}{H(c,a)} \int_{a}^{c} H(s,a)\,k(s)\,\rho(s)\,q(s)\,ds + \frac{1}{H(b,c)} \int_{c}^{b} H(b,s)\,k(s)\,\rho(s)\,q(s)\,ds$$

$$> \frac{1}{(\alpha+1)^{\alpha+1}} \frac{1}{H(c,a)} \int_a^c \frac{\rho(s)\,r(s)}{(H(s,a)\,k(s))^\alpha} |h_1(s,a)|^{\alpha+1}\,ds$$

$$+ \frac{1}{(\alpha+1)^{\alpha+1}} \frac{1}{H(b,c)} \int_c^b \frac{\rho(s)\,r(s)}{(H(b,s)\,k(s))^\alpha} |h_2(b,s)|^{\alpha+1}\,ds, \tag{10}$$

then equation (1) *is oscillatory.*

Proof. Without loss of generality, we may assume that there exists a nonoscillatory solution $x(t)$ of (1) such that $x(t) > 0$ for $t \geq t_0$. Define $w(t)$ as in (4). As in the proof of Theorem 2.1, we can obtain (8). If we replace $T, h(t,s)$ by c and $h_2(t,s)$ respectively, then

$$\int_c^t \left[H(t,s)\,k(s)\,\rho(s)\,q(s) - \frac{1}{(\alpha+1)^{\alpha+1}} \frac{\rho(s)\,r(s)}{(H(t,s)\,k(s))^\alpha} |h_2(t,s)|^{\alpha+1} \right] ds \leq H(t,c)\,k(c)\,w(c), \tag{11}$$

where $t \in [c,b)$. Letting $t \to b^-$ in (11) and then dividing both sides by $H(b,c)$, we have

$$\frac{1}{H(b,c)} \int_c^b \left[H(b,s)\,k(s)\,\rho(s)\,q(s) - \frac{1}{(\alpha+1)^{\alpha+1}} \frac{\rho(s)\,r(s)}{(H(b,s)\,k(s))^\alpha} |h_2(b,s)|^{\alpha+1} \right] ds \leq k(c)\,w(c). \tag{12}$$

By symmetry considerations, we may also show that

$$\frac{1}{H(c,a)} \int_c^a \left[H(s,a)\,k(s)\,\rho(s)\,q(s) - \frac{\rho(s)\,r(s)}{(\alpha+1)^{\alpha+1}(H(s,a)\,k(s))^\alpha} |h_1(s,a)|^{\alpha+1} \right] ds \leq -k(c)\,w(c). \tag{13}$$

Now we assert that x has at least one zero in (a,b). For otherwise adding (12) and (13) would yield an inequality which contradicts assumption (10). Finally, the proof is completed by choosing $\{T_i\} \subset [t_0,\infty)$ such that $T_i \to \infty$ as $i \to \infty$, and then applying what we have just shown to conclude x has zero in each (T_i,∞).

If $h_1(t,s)$ and $h_2(t,s)$ are replaced by $h_1(t,s)\sqrt{H(t,s)\,k(s)}$ and $h_2(t,s) \times \sqrt{H(t,s)\,k(s)}$ in Theorem 4, respectively, we can obtain the following result. The proof is similar and therefore omitted. □

Theorem 5. *Let functions* $H \in C(D;R)$, $h_1, h_2 \in C(D_0;R)$, $k, \rho \in C^1([t_0,\infty);(0,\infty))$ *satisfy the following conditions:*

(I) $H(t,t) = 0$ *for* $t \geq t_0$, $H(t,s) > 0$ *on* D_0;

(II) *for* $(t,s) \in D_0$,

$$\frac{\partial}{\partial t}\big(H(t,s)\,k(t)\big) + H(t,s)\,k(t)\left[\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r(t)}\right] = h_1(t,s)\sqrt{H(t,s)\,k(t)},$$

(III) *for* $(t,s)\in D_0$,

$$\frac{\partial}{\partial s}\big(H(t,s)\,k(s)\big)+H(t,s)\,k(s)\left[\frac{\rho'(s)}{\rho(s)}-\frac{p(s)}{r(s)}\right]=-h_2(t,s)\sqrt{H(t,s)\,k(s)}.$$

If for each $T_0\ge t_0$, *there exist* $a,b,c\in R$ *with* $T_0\le a<c<b$ *such that*

$$\frac{1}{H(c,a)}\int_a^c H(s,a)\,k(s)\rho(s)\,q(s)\,ds+\frac{1}{H(b,c)}\int_c^b H(b,s)\,k(s)\,\rho(s)\,q(s)\,ds$$
$$>\frac{1}{(\alpha+1)^{\alpha}}\,\frac{1}{H(c,a)}\int_a^c \rho(s)\,r(s)\big[H(s,a)\,k(s)\big]^{\frac{1-\alpha}{2}}|h_1(s,a)|^{\alpha+1}\,ds$$
$$+\frac{1}{(\alpha+1)^{\alpha}}\,\frac{1}{H(b,c)}\int_c^b \rho(s)\,r(s)\big[H(b,s)\,k(s)\big]^{\frac{1-\alpha}{2}}|h_2(b,s)|^{\alpha+1}ds, \quad (14)$$

then equation (1) *is oscillatory.*

As an immediate consequence of Theorems 4 and 5, we get the following corollaries: (i) Under the assumptions of Theorem 4, if (10) is replaced by requiring that

$$\limsup_{t\to\infty}\int_l^t\left[H(s,l)\,k(s)\,\rho(s)\,q(s)-\frac{\rho(s)\,r(s)|h_1(s,l)|^{\alpha+1}}{(\alpha+1)^{\alpha+1}(H(s,l)\,k(s))^{\alpha}}\right]ds>0 \qquad (15)$$

and

$$\limsup_{t\to\infty}\int_l^t\left[H(t,s)\,k(s)\,\rho(s)\,q(s)-\frac{\rho(s)\,r(s)|h_2(t,s)|^{\alpha+1}}{(\alpha+1)^{\alpha+1}(H(t,s)\,k(s))^{\alpha}}\right]ds>0 \qquad (16)$$

for each sufficiently large $l\ge t_0$, then equation (1) is oscillatory; (ii) under the assumptions of Theorem 5, if condition (14) is replaced by requiring that

$$\limsup_{t\to\infty}\int_l^t\Big[H(s,l)\,k(s)\,\rho(s)\,q(s)$$
$$-\frac{\rho(s)\,r(s)|h_1(s,l)|^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\big[H(s,l)\,k(s)\big]^{(1-\alpha)/2}ds\Big]ds>0 \qquad (17)$$

and

$$\limsup_{t\to\infty}\int_l^t\Big[H(t,s)k(s)\rho(s)q(s)$$
$$-\frac{\rho(s)r(s)|h_2(t,s)|^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\big[H(t,s)k(s)\big]^{(1-\alpha)/2}ds\Big]ds>0 \qquad (18)$$

for each sufficiently large $l\ge t_0$, then equation (1) is oscillatory.

Indeed, for any $T \geq T_0 \geq t_0$, let $a = T$. In (15) we choose $l = a$. Then there exists $c > a$ such that

$$\int_a^c \left[H(s,a)\,k(s)\,\rho(s)\,q(s) - \frac{\rho(s)\,r(s)|h_1(s,a)|^{\alpha+1}}{(\alpha+1)^{\alpha+1}(H(s,a)\,k(s))^{\alpha}} \right] ds > 0. \tag{19}$$

In (16) we choose $l = c$. Then there exists $b > c$ such that

$$\int_c^b \left[H(b,s)\,k(s)\,\rho(s)\,q(s) - \frac{\rho(s)\,r(s)|h_1(b,s)|^{\alpha+1}}{(\alpha+1)^{\alpha+1}(H(b,s)k(s))^{\alpha}} \right] ds > 0. \tag{20}$$

Combining (19) and (20), we obtain (10). The conclusion (i) thus follows from Theorem 4. The conclusion (ii) is similarly proved.

By choosing different $H(t,s)$, $k(s)$ and $\rho(s)$, a number of specific oscillation criteria can be obtained. For instance, let

$$R(t) = \int_l^t \frac{1}{r(s)}\,ds, \quad t \geq l \geq t_0,$$

and let

$$H(t,s) = [R(t) - R(s)]^{\lambda}, \quad t \geq t_0,$$

where $\lambda > \alpha$ is a constant.

Corollary 1. *Let $\rho(t) = \exp \int_{t_0}^t (p(s)/r(s))ds$ for $t \geq t_0$. Then equation (1) is oscillatory provided that there is some $\lambda > \alpha$ such that one of the following conditions is satisfied:*

(i) *for any $l \geq t_0$,*

$$\limsup_{t\to\infty} \frac{1}{R^{\lambda-\alpha}(t)} \int_l^t \rho(s)[R(s) - R(l)]^{\lambda} \times \left\{ q(s) - \frac{\lambda^{\alpha+1}[R(s)-R(l)]^{-\alpha-1}}{(\alpha+1)^{\alpha+1}} \frac{1}{r^{\alpha}(s)} \right\} ds > 0 \tag{21}$$

and

$$\limsup_{t\to\infty} \frac{1}{R^{\lambda-\alpha}(t)} \int_l^t \rho(s)[R(t) - R(s)]^{\lambda} \times \left\{ q(s) - \frac{\lambda^{\alpha+1}[R(t)-R(s)]^{-\alpha-1}}{(\alpha+1)^{\alpha+1}} \frac{1}{r^{\alpha}(s)} \right\} ds > 0; \tag{22}$$

(ii) *$r(t) \equiv 1$, $p(t) \equiv 0$ and for any $l \geq t_0$,*

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda-\alpha}} \int_l^t (s-l)^{\lambda} q(s)\,ds > \frac{\lambda^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \frac{1}{\lambda-\alpha} \tag{23}$$

and

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda-\alpha}} \int_l^t (t-s)^\lambda q(s)\,ds > \frac{\lambda^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \frac{1}{\lambda-\alpha}\,. \tag{24}$$

Proof. (i) Pick $k(t) \equiv 1$ and $\rho(t) = \exp \int_{t_0}^t (p(s)/r(s))ds$. Then $\rho'(t)/\rho(t) - p(t)/r(t) = 0$ for $t > t_0$. It is easy to see that

$$h_1(t,s) = \lambda\big[R(t) - R(s)\big]^{\frac{\lambda}{2}-1} \frac{1}{r(t)}$$

and

$$h_2(t,s) = \lambda\big[R(t) - R(s)\big]^{\frac{\lambda}{2}-1} \frac{1}{r(s)}\,.$$

Noting that

$$\int_l^t \frac{\rho(s)r(s)}{(\alpha+1)^{\alpha+1}} [H(s,l)]^{(1-\alpha)/2} |h_1(s,l)|^{\alpha+1}\,ds = \int_l^t \frac{\lambda^{\alpha+1}\,[R(s)-R(l)]^{\lambda-\alpha-1}}{(\alpha+1)^{\alpha+1}} \frac{1}{r^\alpha(s)}\,ds \tag{25}$$

and

$$\int_l^t \frac{\rho(s)r(s)}{(\alpha+1)^{\alpha+1}} [H(t,s)]^{(1-\alpha)/2} |h_2(t,s)|^{\alpha+1}\,ds = \int_l^t \frac{\lambda^{\alpha+1}[R(t)-R(s)]^{\lambda-\alpha-1}}{(\alpha+1)^{\alpha+1}} \frac{1}{r^\alpha(s)}\,ds, \tag{26}$$

from (21) and (25) we have

$$\limsup_{t\to\infty} \frac{1}{R^{\lambda-1}(t)} \int_l^t \bigg[H(s,l)\,\rho(s)\,q(s) - \frac{\rho(s)r(s)}{(\alpha+1)^{\alpha+1}}[H(s,l)]^{(1-\alpha)/2} \times |h_1(s,l)|^{\alpha+1} ds\bigg] ds = \limsup_{t\to\infty} \frac{1}{R^{\lambda-\alpha}(t)} \int_l^t \rho(s)[R(s)-R(l)]^\lambda \times \bigg\{q(s) - \frac{\lambda^{\alpha+1}[R(s)-R(l)]^{-\alpha-1}}{(\alpha+1)^{\alpha+1}} \frac{1}{r^\alpha(s)}\bigg\} ds > 0.$$

It follows that

$$\limsup_{t\to\infty}\frac{1}{R^{\lambda-1}(t)}\int_l^t\Big[H(s,l)\rho(s)q(s) - \frac{\rho(s)r(s)}{(\alpha+1)^{\alpha+1}}[H(s,l)]^{(1-\alpha)/2}|h_1(s,l)|^{\alpha+1}\Big]ds>0,$$

i.e., (17) holds. Similarly, (22) implies that (18) holds. From the above Corollary, equation (1) is oscillatory.

(ii) Because $r(t)=1$ and $p(t)=0$, we have $\rho(t)=1$ and $H(t,s)=(t-s)^\alpha$; it follows that

$$\lim_{t\to\infty}\frac{\lambda^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\frac{1}{t^{\lambda-\alpha}}\int_l^t(s-l)^{\lambda-\alpha-1}ds=\frac{\lambda^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\frac{1}{\lambda-\alpha} \tag{27}$$

so that

$$\limsup_{t\to\infty}\frac{1}{t^{\lambda-\alpha}}\int_l^t\Big\{(s-l)^\lambda q(s)-\frac{\lambda^{\alpha+1}}{(\alpha+1)^{\alpha+1}}(s-l)^{\lambda-\alpha-1}\Big\}ds$$

$$=\limsup_{t\to\infty}\frac{1}{t^{\lambda-\alpha}}\int_l^t[s-l]^\lambda q(s)\,ds-\frac{\lambda^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\frac{1}{\lambda-\alpha}>0.$$

Similarly, (24) implies that (22) holds. In view of (i), equation (1) is oscillatory. □

We remark that other choices of $k(s),\rho(s)$ include 1, s, etc.; while other choices of H include $H(t,s)=(t-s)^\lambda$, $H(t,s)=[R(t)-R(s)]^\lambda$, $H(t,s)=[\log Q(t)/Q(s)]^\lambda$, or $H(t,s)=\big[\int_s^t dz/w(z)\big]^\lambda$, etc., for $t\ge s\ge t_0$, where $\lambda>1$ is a constant, $R(t)=\int_{t_0}^t ds/u(s)$, $Q(t)=\int_t^\infty ds/u(s)<\infty$, for $t\ge t_0$, and $w\in C([t_0,\infty),R_+)$ satisfying $\int_{t_0}^\infty ds/w(s)=\infty$.

3. Examples

As our first example, let $q(t)$ be decreasing for $t\ge t_0=1$ and $\lim\limits_{t\to\infty}t^{\alpha+1}q(t)=\mu>0$. For example,

$$q(t)=\frac{\mu}{t^{\alpha+1}}\left(1+\frac{n}{(\log t)^m}\right),\quad m,n\ge 0,$$

or

$$q(t)=\frac{\mu}{\sqrt[m]{t^{m\alpha+2}(t-1)^{m-2}}},\quad m\ge 1.$$

Consider the half-linear equation

$$\big[|x'(t)|^{\alpha-1}x'(t)\big]'+q(t)|x(t)|^{\alpha-1}x(t)=0,\ ;\quad t\ge 1. \tag{28}$$

It is well known that equation (28) with $\alpha=1$ and $q(t)=\mu t^{-2}$ is oscillatory if and only if $\mu>1/4$. However, most of the standard oscillation criteria cannot be used to show that equation (28) is oscillatory for general α. In spite of this, note that for $\lambda > \alpha$,

$$\lim_{t\to\infty} \frac{1}{t^{\lambda-\alpha}} \int_l^t (s-l)^\lambda q(t)ds = \frac{1}{\lambda-\alpha} \lim_{t\to\infty} \frac{(t-l)^\lambda}{t^{\lambda-\alpha-1}t^{\alpha+1}} \lim_{t\to\infty} t^{\alpha+1}q(t) = \frac{\mu}{\lambda-\alpha}, \quad (29)$$

and

$$\int_l^t (t-l)^\lambda\, q(s)\, ds \geq \int_l^t (s-l)^\lambda\, q(s)\, ds. \quad (30)$$

Let

$$F(t) = \int_l^t \big\{(t-l)^\lambda - (s-l)^\lambda\big\} q(s)\, ds.$$

Then $F(l) = 0$, and for $t \geq l$

$$F'(t) = \int_l^t \lambda(t-l)^{\lambda-1} q(s)\, ds - (t-l)^\lambda q(t) \geq q(t) \int_l^t \lambda(t-l)^{\lambda-1} ds - (t-l)^\lambda q(t) = 0.$$

Hence $F(t) \geq F(l) = 0$ for $t \geq l$, i.e., (23) holds. By (29) and (30), for any $\mu > \alpha^{\alpha+1}/(\alpha+1)^{\alpha+1}$, there exists $\lambda > \alpha$ such that $\mu/(\lambda-\alpha) > \alpha^{\alpha+1}/\big((\alpha+1)^{\alpha+1}(\lambda-\alpha)\big)$. This means that (23) and (24) hold for the same λ. Applying Corollary 1, we find that (28) is oscillatory for $\mu > \alpha^{\alpha+1}/(\alpha+1)^{\alpha+1}$.

As another example, consider the half-linear equation

$$\begin{aligned} &\big[t|x'(t)|^{\alpha-1}x'(t)\big]'(\alpha-1)|x'(t)|^{\alpha-1}x'(t) \\ &+ \frac{\mu}{t^\alpha (\ln t)^{\alpha+1}} |x(t)|^{\alpha-1} x(t) = 0, \quad t \geq 1, \end{aligned} \quad (31)$$

where $\mu > 0$. Let $R(t) = \ln t$, $\rho(t) = t^{\alpha-1}$ and $H(t,s) = [R(t) - R(s)]^\lambda = (t-s)^\lambda$. Then for $\lambda > \alpha$, we have

$$\begin{aligned} &\lim_{t\to\infty} \frac{1}{R^{\lambda-\alpha}(t)} \int_l^t \rho(s)[R(s)-R(l)]^\lambda \left\{ q(s) - \frac{\lambda^{\alpha+1}[R(s)-R(l)]^{-\alpha-1}}{(\alpha+1)^{\alpha+1}} \frac{1}{r^\alpha(s)} \right\} ds \\ &= \lim_{t\to\infty} \frac{1}{(\ln t)^{\lambda-\alpha}} \int_l^t s^{\alpha-1} (\ln s - \ln l)^\lambda \left\{ \frac{\mu}{s^\alpha (\ln s)^{\alpha+1}} - \frac{\lambda^{\alpha+1}(\ln s - \ln l)^{-\alpha-1}}{(\alpha+1)^{\alpha+1}} \frac{1}{s^\alpha} \right\} ds \\ &= \lim_{t\to\infty} \frac{1}{(\ln t)^{\lambda-\alpha}} \left\{ \int_l^t \frac{\mu(\ln s - \ln l)^\lambda}{s(\ln s)^{\alpha+1}} ds - \frac{\lambda^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \int_l^t (\ln s - \ln l)^{\lambda-\alpha-1} \frac{ds}{s} \right\} \\ &= \lim_{t\to\infty} \frac{t}{(\lambda-\alpha)(\ln t)^{\lambda-\alpha-1}} \frac{\mu(\ln t - \ln l)^\lambda}{t(\ln t)^{\alpha+1}} - \frac{\lambda^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \lim_{t\to\infty} \frac{(\ln t - \ln l)^{\lambda-\alpha}}{(\lambda-\alpha)(\ln t)^{\lambda-\alpha}} \\ &= \frac{\mu}{\lambda-\alpha} - \frac{\lambda^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \frac{1}{\lambda-\alpha}. \end{aligned} \quad (32)$$

Next, we prove that

$$\int_l^t s^{\alpha-1}(\ln t-\ln l)^\lambda \frac{\mu}{s^\alpha(\ln s)^{\alpha+1}}ds \geq \int_l^t s^{\alpha-1}(\ln s-\ln l)^\lambda \frac{\mu}{s^\alpha(\ln s)^{\alpha+1}}ds. \tag{33}$$

Let

$$G(t)=\int_l^t \left\{(\ln t-\ln l)^\lambda-(\ln s-\ln l)^\lambda\right\}\frac{\mu}{s(\ln s)^{\alpha+1}}ds.$$

Then $G(l)=0$, and for $t\geq l$

$$\begin{aligned} G'(t) &= \int_l^t \lambda(\ln t-\ln l)^{\lambda-1}\frac{1}{s}\frac{\mu}{s(\ln s)^{\alpha+1}}ds-(\ln t-\ln l)^\lambda\frac{\mu}{t(\ln t)^{\alpha+1}} \\ &\geq \frac{\mu}{t(\ln t)^{\alpha+1}}\int_l^t \lambda(\ln t-\ln l)^{\lambda-1}\frac{1}{s}ds-\frac{\mu(\ln t-\ln l)^\lambda}{t(\ln t)^{\alpha+1}}=0. \end{aligned}$$

Thus $G(t)\geq G(l)=0$ for $t\geq l$, i.e., (33) holds. By (32) and (33), for any $\mu>\alpha^{\alpha+1}/(\alpha+1)^{\alpha+1}$, there exists $\lambda>\alpha$ such that $\mu/(\lambda-\alpha)>\alpha^{\alpha+1}/\big((\alpha+1)^{\alpha+1}(\lambda-\alpha)\big)$. This means that (21) and (22) hold for the same λ. Applying Corollary 1, equation (31) is oscillatory for $\mu>\alpha^{\alpha+1}/(\alpha+1)^{\alpha+1}$.

As our final example, let us consider the following nonhomogeneous equation

$$\left[r(t)|x'(t)|^{\alpha-1}x'(t)\right]'+p(t)|x'(t)|^{\alpha-1}x'(t)+q(t)|x(t)|^{\alpha-1}x(t)=e(t),\quad t\geq t_0, \tag{34}$$

where $t_0\geq 0$, $\alpha>0$, $r(t)\in C^1(I;(0,\infty))$, $p(t),q(t)$ and $e(t)\in C(I;R)$.

Let $x(t)$ be a solution of equation (34) and assume the conditions of Theorem 1 hold. Suppose $\liminf_{t\to\infty}|x(t)|=L>0$. Then $x(t)$ is nonoscillatory. Without loss of generality, we may assume that $x(t)\geq L>0$ for some $T_0\geq t_0$. Define $w(t)$ as in (4); then differentiating (4), making use of (1) and the assumptions of the theorem, it follows that for all $t\geq T_0$

$$\begin{aligned} w'(t) &= -\rho(t)q(t)-\left[\frac{p(s)}{r(s)}-\frac{\rho'(s)}{\rho(s)}\right]w(t)-\alpha\frac{|w(t)|^{(\alpha+1)/\alpha}}{(\rho(t)r(t))^{1/\alpha}}+\frac{\rho(t)e(t)}{|x(t)|^{\alpha-1}x(t)} \\ &\leq -\rho(t)q(t)-\left[\frac{p(s)}{r(s)}-\frac{\rho'(s)}{\rho(s)}\right]w(t)-\alpha\frac{|w(t)|^{(\alpha+1)/\alpha}}{(\rho(t)r(t))^{1/\alpha}}+\frac{\rho(t)|e(t)|}{L^\alpha}. \end{aligned}$$

As in the proof of Theorem 1, we may show that for all $t\geq T_0$,

$$\begin{aligned} &\int_{T_0}^t \left[H(t,s)\,k(s)\,\rho(s)\,q(s)-\frac{\rho(s)\,r(s)|h(t,s)|^{\alpha+1}}{(\alpha+1)^{\alpha+1}(H(t,s)\,k(s))^\alpha}\right]ds \\ &\qquad\leq H(t,t_0)k(T_0)|w(T_0)|+\frac{1}{L^\alpha}\int_{t_0}^t H(t,s)k(s)\rho(s)|e(s)|\,ds. \end{aligned}$$

It follows that

$$\int_{t_0}^{t}\left[H(t,s)\,k(s)\rho(s)\,q(s)-\frac{\rho(s)\,r(s)|h(t,s)|^{\alpha+1}}{(\alpha+1)^{\alpha+1}(H(t,s)\,k(s))^{\alpha}}\right]ds$$

$$=\int_{t_0}^{T_0}\left[H(t,s)\,k(s)\,\rho(s)\,q(s)-\frac{\rho(s)\,r(s)|h(t,s)|^{\alpha+1}}{(\alpha+1)^{\alpha+1}(H(t,s)\,k(s))^{\alpha}}\right]ds$$

$$+\int_{T_0}^{t}\left[H(t,s)\,k(s)\,\rho(s)\,q(s)-\frac{\rho(s)\,r(s)|h(t,s)|^{\alpha+1}}{(\alpha+1)^{\alpha+1}(H(t,s)\,k(s))^{\alpha}}\right]ds$$

$$\leq H(t,t_0)\left[\int_{t_0}^{T_0}k(s)\,\rho(s)|q(s)|ds+k(T_0)|w(T_0)|\right]$$

$$+\frac{1}{L^{\alpha}}\int_{t_0}^{t}H(t,s)\,k(s)\,\rho(s)|e(s)|\,ds. \tag{35}$$

Hence

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\left[H(t,s)\,k(s)\rho(s)\,q(s)-\frac{\rho(s)\,r(s)|h(t,s)|^{\alpha+1}}{(\alpha+1)^{\alpha+1}(H(t,s)\,k(s))^{\alpha}}\right]ds$$

$$\leq\int_{t_0}^{T_0}k(s)\,\rho(s)|q(s)|\,ds+k(T_0)|w(T_0)|$$

$$+\frac{1}{L^{\alpha}}\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\,k(s)\rho(s)|e(s)|\,ds<\infty.$$

In other words, under the conditions of Theorem 1, if in addition

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\,k(s)\rho(s)|e(s)|\,ds<\infty, \tag{36}$$

then every solution $x(t)$ of equation (34) satisfies $\liminf_{t\to\infty}|x(t)|=0$.

By similar arguments, we may also, based on Theorems 1 and 2, establish several sufficient conditions for every solution $x(t)$ of equation (34) to satisfy $\liminf_{t\to\infty}|x(t)|=0$. No new principles, however, are involved.

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