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Differential superordination defined by Sălăgean operator

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Abstract

By using the Sălăgean operator $D^n f$, we introduce a class of holomorphic functions denoted by $S(\alpha)$, and we obtain some superordinations results related to this class.

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1 Introduction

Let Ω be any set in the complex plane $\mathbb{C}$, let p be analytic in the unit disk U and let $\psi(\gamma, s, t; z) : \mathbb{C}^3 \times U \to \mathbb{C}$. In a series of articles the authors and many others [1] have determined properties of functions p that satisfy the differential subordination

$$\{\psi(p(z), zp'(z), z^2p'(z); z)|\ z \in U\} \subset \Omega.$$

In this article we consider the dual problem of determining properties of function p that satisfy the differential superordination

$$\Omega \subset \{\psi(p(z), zp'(z), z^2p'(z); z)|\ z \in U\}.$$

This problem was introduced in [2].

We let $\mathcal{H}(U)$ denote the class of holomorphic functions in the unit disk $U = \{z \in \mathbb{C} : \ |z| < 1\}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$ we let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(U), \ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots, \ z \in U\}$$

and

$$A = \{f \in \mathcal{H}(U), f(z) = z + a_2 z^2 + \dots, z \in U\}.$$

For $0 < r < 1$, we let $U_r = \{z, \ |z| < r\}$.

Definition 1 (see [2]). *Let $\varphi : \mathbb{C}^2 \times U \to \mathbb{C}$ and let h be analytic in U. If p and $\varphi(p(z), zp'(z); z)$ are univalent in U and satisfy the (first-order) differential superordination*

$$(1) \qquad h(z) \prec \varphi(p(z), zp'(z); z)$$

then p is called a solution of the differential superordination. An analytic function q is called a subordinant of the solutions of the differential superordination, or more simply a subordinant if $q \prec p$ for all p satisfying (1). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (1) is said to be the best subordinant. Note that the best subordinant is unique up to a rotation of U.

For Ω a set in $\mathbb{C}$, with φ and p as given in Definition 1, suppose (1) is replaced by

$$(1') \qquad \Omega \subset \{\varphi(p(z), zp'(z); z) | \ z \in U\}.$$

Although this more general situation is a "differential containment", the condition in (1 will also be referred to as a differential superordination, and the definitions of solution, subordinant and best dominant as given above can be extended to this generalization.

Definition 2 (see [2]). *We denote by Q the set of functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where*

$$E(f) = \{\zeta \in \partial U : \ \lim_{z \to \zeta} f(z) = \infty\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

The subclass of Q for which $f(0) = a$ is denoted by $Q(a)$.

Definition 3 (see [2]). *Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H}[a, n]$ with $q'(z) \neq 0$. The class of admissible functions $\phi_n[\Omega, q]$, consist of those functions $\varphi : \mathbb{C}^2 \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\varphi\left(q(z), \frac{zq'(z)}{m}; \zeta\right) \in \Omega \tag{2}$$

where $z \in U$, $\zeta \in \partial U$ and $m \geq n \geq 1$.

In order to prove the new results we shall use the following lemma:

Lemma A (see [2]). *Let h be convex in U, with $h(0) = a$, $\gamma \neq 0$ with* Re $\gamma \geq 0$, *and $p \in \mathcal{H}[a, 1] \cap Q$. If $p(x) + \dfrac{zp'(z)}{\gamma}$ is univalent in U,*

$$h(z) \prec p(z) + \frac{zp'(z)}{\gamma}$$

then

$$q(z) \prec p(z),$$

where

$$q(z) = \frac{\gamma}{z^\gamma} \int_0^z h(t) t^{\gamma - 1} dt, \quad z \in U.$$

The function q is convex and is the best subordinant.

Lemma B (see [2]). *Let q be convex in U and let h be defined by*

$$h(z) = q(z) + \frac{zq'(z)}{\gamma}, \quad z \in U,$$

with Re $\gamma \geq 0$. *If $p \in \mathcal{H}[a, 1] \cap Q$, $p(z) + \dfrac{zp'(z)}{\gamma}$ is univalent in U, and*

$$q(z) + \frac{zq'(z)}{\gamma} \prec p(z) + \frac{zp'(z)}{\gamma}, \quad z \in U$$

then

$$q(z) \prec p(z),$$

where

$$q(z) = \frac{\gamma}{z^\gamma} \int_0^z h(t) t^{\gamma - 1} dt.$$

The function q is the best subordinant.

Definition 4. [G. S. Sălăgean 3] *For $f \in A_n$ and $n \geq 0$, $n \in \mathbb{N}$, the operator $D^n f$ is defined by*

$$D^0 f(z) = f(z)$$

$$D^{n+1} f(z) = z[D^n f(z)]', \quad z \in U.$$

2 Main results

If $0 \leq \alpha < 1$ and $n \in \mathbb{N}$, let $S(\alpha)$ denote the class of functions $f \in A$ which satisfy the inequality

$$\mathrm{Re}\ [D^n f(z)]' > \alpha.$$

Theorem 1. *Let*

$$h(z) = \frac{1 + (2\alpha - 1)z}{1 + z}$$

be convex in U, with $h(0) = 1$.

Let $f \in S(\alpha)$, and suppose that $[D^{n+1} f(z)]'$ is univalent and $[D^n f(z)]' \in \mathcal{H}[1, 1] \cap Q$.

If

(3) $$h(z) \prec [D^{n+1} f(z)]', \quad z \in U,$$

then

$$q(z) \prec [D^n f(z)]', \quad z \in U,$$

where

(4) $$q(z) = \frac{1}{z} \int_0^z \frac{1 + (2\alpha - 1)t}{1 + t} dt = 2\alpha - 1 + (2 - 2\alpha) \frac{\ln(1 + z)}{z}.$$

The function q is convex and is the best subordinant.

Proof. Let $f \in S(\alpha)$. By using the properties of the operator $D^n f(z)$ we have

(5) $$D^{n+1} f(z) = z[D^n f(z)]', \quad z \in U.$$

Differentiating (5), we obtain

$$[D^{n+1}]'f(z) = [D^n f(z)]' + z[D^n f(z)]', \quad z \in U. \tag{6}$$

If we let $p(z) = [D^n f(z)]'$ then (6) becomes

$$[D^{n+1} f(z)]' = p(z) + zp'(z), \quad z \in U.$$

Then (3) becomes

$$h(z) \prec p(z) + zp'(z), \quad z \in U.$$

By using Lemma A, we have

$$q(z) \prec p(z) = [D^n f(z)]', \quad z \in U,$$

where

$$q(z) = \frac{1}{z}\int_0^z \frac{1+(2\alpha-1)t}{1+t}dt = 2\alpha - 1 + (2-2\alpha)\frac{\ln(1+z)}{z}, \quad z \in U.$$

The function q is the best subordinant.

Theorem 2. *Let*

$$h(z) = \frac{1+(2\alpha-1)z}{1+z}$$

be convex in U, with $h(0) = 1$. Let $f \in S(\alpha)$ and suppose that $[D^n f(z)]'$ is univalent and $\dfrac{D^n f(z)}{z} \in \mathcal{H}[1,1] \cap Q$.

If

$$h(z) \prec [D^n f(z)]', \quad z \in U, \tag{7}$$

then

$$q(z) \prec \frac{D^n f(z)}{z}, \quad z \in U,\ z \neq 0,$$

where

$$q(z) = \frac{1}{z}\int_0^z \frac{1+(2\alpha-1)t}{1+t}dt = 2\alpha - 1 + (2-2\alpha)\frac{\ln(1+z)}{z}.$$

The function q is convex and is the best subordinant.

Proof. We let

$$p(z)=\frac{D^n f(z)}{z}, \quad z\in U,\ z\neq 0$$

and we obtain

$$D^n f(z)=zp(z), \quad z\in U,\ z\neq 0. \tag{8}$$

By differentiating (8) we obtain

$$[D^n f(z)]'=p(z)+zp'(z), \quad z\in U,\ z\neq 0.$$

Then (7) becomes

$$h(z)\prec p(z)+zp'(z), \quad z\in U,\ z\neq 0.$$

By using Lemma A we have

$$q(z)\prec p(z)=\frac{D^n f(z)}{z}, \quad z\in U,\ z\neq 0,$$

where

$$q(z)=2\alpha-1+(2-2\alpha)\frac{\ln(1+z)}{z}.$$

The function q is convex and is the best subordinant.

Theorem 3. *Let q be convex in U and let h be defined by*

$$h(z)=q(z)+zq'(z), \quad z\in U.$$

Let $f\in S(\alpha)$ and suppose that $[D^{n+1}f(z)]'$ is univalent in U, $[D^n f(z)]'\in\mathcal{H}[1,1]\cap Q$ and

$$h(z)=q(z)+zq'(z)\prec[D^{n+1}f(z)]', \quad z\in U, \tag{9}$$

then

$$q(z)\prec[D^n f(z)]', \quad z\in U$$

where

$$q(z)=\frac{1}{z}\int_0^z h(t)dt, \quad z\in U.$$

The function q is the best subordinant.

Proof. Let $f \in S(\alpha)$. By using the properties of the operator $D^n f(z)$, we have

(10) $$D^{n+1}f(z) = z[D^n f(z)]', \quad z \in U.$$

Differentiating (10), we obtain

(11) $$[D^{n+1}f(z)]' = [D^n f(z)]' + z[D^n f(z)]', \quad z \in U.$$

If we let $p(z) = [D^n f(z)]'$ then (11) becomes

$$[D^{n+1}f(z)]' = p(z) + zp'(z), \quad z \in U.$$

By using Lemma B, we have

$$q(z) \prec p(z) = [D^n f(z)]', \quad z \in U,$$

where

$$q(z) = \frac{1}{z}\int_0^z h(t)dt.$$

The function q is best subordinant.

Theorem 4. *Let q be convex in U and let h be defined by*

$$h(z) = q(z) + zq'(z), \quad z \in U.$$

Let $f \in S(\alpha)$ and suppose that $[D^n f(z)]'$ is univalent in U, $\frac{D^n f(z)}{z} \in \mathcal{H}[1,1] \cap Q$ and

(12) $$h(z) = q(z) + zq'(z) \prec [D^n f(z)]', \quad z \in U$$

then

$$q(z) \prec \frac{D^n f(z)}{z}, \quad z \in U, \ z \neq 0,$$

where

$$q(z) = \frac{1}{z}\int_0^z h(t)dt.$$

The function q is the best subordinant.

Proof. We let

$$p(z) = \frac{D^n f(z)}{z}, \quad z \in U, \ z \neq 0$$

and we obtain

$$D^n f(z) = zp(z), \quad z \in U, \ z \neq 0. \tag{13}$$

By differentiating (13), we obtain

$$[D^n f(z)]' = p(z) + zp'(z), \quad z \in U, \ z \neq 0.$$

Then (12) becomes

$$q(z) + zq'(z) \prec p(z) + zp'(z), \quad z \in U, \ z \neq 0.$$

By using Lemma B we have

$$q(z) \prec p(z) = \frac{D^n f(z)}{z}, \quad z \in U, \ z \neq 0,$$

where

$$q(z) = \frac{1}{z}\int_0^z h(t)dt.$$

The function q is the best subordinant.

References

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