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LEAST SQUARES SPECTRAL METHOD FOR VELOCITY-FLUX FORM OF THE COUPLED STOKES-DARCY EQUATIONS*

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Abstract. This paper develops least squares Legendre and Chebyshev spectral methods for the first order system of Stokes-Darcy equations. The least squares functional is based on the velocity-flux-pressure formulation with the enforcement of the Beavers-Joseph-Saffman interface conditions. Continuous and discrete homogeneous functionals are shown to be equivalent to the combination of weighted H^1 and $H(\mathrm{div})$-norm for the Stokes and Darcy equations. The spectral convergence for the Legendre and Chebyshev methods are derived and numerical experiments are also presented to illustrate the analysis.

Key words. Coupled Stokes-Darcy equation, first order system, least squares method, Legendre and Chebyshev pseudo-spectral method, Beavers-Joseph-Saffman law.

AMS subject classifications. 65N35, 65N12.

1. Introduction.

The most likely possible occurrences of the coupled Stokes-Darcy flow comprise groundwater flows, cross-flows and dead-end filtration processes, plasma separation from blood and heterogeneous catalytic reactions. The Stokes equation expresses the fluid dynamics in the free flow regime and the Darcy equation is used to express the fluid dynamics in porous medium. The mass conservation, balance of normal forces, and the Beavers-Joseph-Saffman law are used to model the connection between these two fluids.

To state the problem mathematically, let Ω be an open bounded domain in $\mathbf{R}^2$ divided into two sub-domains Ω_S and Ω_D with the curve (interface) Γ, such that $\overline{\Omega} = \overline{\Omega}_S \cup \overline{\Omega}_D \cup \Gamma$. The boundary of Ω is denoted by $\partial\Omega$ and $\partial\Omega_S = \overline{\Omega}_S \cap \partial\Omega$, $\partial\Omega_D = \overline{\Omega}_D \cap \partial\Omega$. The schematic of domain Ω with interface Γ is depicted in Figure 1.1. Suppose that the flow in Ω_S is governed

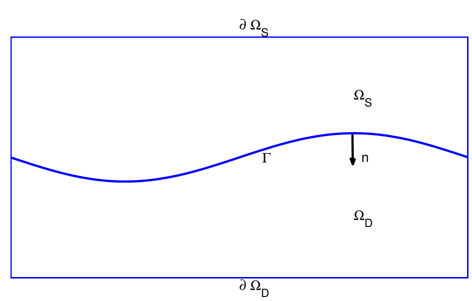

FIG. 1.1. *Schematic domain Ω for Stokes-Darcy problem with interface Γ.*

by the Stokes equation

$$\begin{cases} -\nabla \cdot \mathbf{T} = \mathbf{f}, & \text{in } \Omega_S, \\ \nabla \cdot \mathbf{u} = 0, & \text{in } \Omega_S, \end{cases} \tag{1.1}$$

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where $\mathbf{T} := -p\,\mathbf{I} + 2\nu\,\mathbf{E}(\mathbf{u})$ is the stress tensor, $\mathbf{u}$ is the velocity, and $\mathbf{f}$ is the external force function. Here $\mathbf{E}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ is the deformation rate tensor, $\nu > 0$ is the kinetic viscosity of the fluid, and p is the pressure with average zero. Suppose that the space averaged velocity and the pressure in the porous medium domain Ω_D are governed by the Darcy equation

$$\begin{cases} \mathbf{w} + K\nabla q = 0, & \text{in } \Omega_D, \\ \nabla\cdot\mathbf{w} = g, & \text{in } \Omega_D, \end{cases} \tag{1.2}$$

where $K > 0$ is the Darcy permeability, $\mathbf{w}$ is the velocity, q is the pressure and g is a given function. The following boundary conditions are considered

$$\begin{cases} \mathbf{u} = 0, & \text{on } \partial\Omega_S, \\ \mathbf{w}\cdot\mathbf{n} = 0, & \text{on } \partial\Omega_D, \end{cases} \tag{1.3}$$

where $\mathbf{n}$ is the outward unit normal vector on $\partial\Omega_S$ and $\partial\Omega_D$. On the interface Γ, the Beavers-Joseph-Saffman conditions are imposed

$$\begin{cases} \mathbf{u}\cdot\mathbf{n} - \mathbf{w}\cdot\mathbf{n} = 0, \\ \mathbf{n}\cdot(\mathbf{T}\cdot\mathbf{n}) + q = 0, \\ \beta\mathbf{n}\times(\mathbf{T}\cdot\mathbf{n}) + \mathbf{u}\times\mathbf{n} = 0, \end{cases} \tag{1.4}$$

where β is a positive constant and $\mathbf{n}$ is the unit normal vector pointing from Γ into Ω_D; for details on a proper choice of β, see [7]. In [21], the authors showed that the weak formulation of the Stokes-Darcy equation has a unique solution, i.e.,

$$(\mathbf{u}, p, \mathbf{w}, q) \in [H^1_{\partial\Omega_S}(\Omega_S)]^2 \times L^2(\Omega_S) \times H_{\partial\Omega_D}(\mathrm{div}, \Omega_D) \times L^2(\Omega_D).$$

The development of appropriate methods for the coupled Stokes-Darcy equations (1.1)–(1.4) has been investigated from the mathematical and numerical analysis viewpoints [7, 8, 10, 21, 26]. In the case of finite elements, the Stokes equation is analyzed using mixed formulation, while for the Darcy equation several approaches have been used, such as mixed formulation [21] and the standard variational formulation of the equivalent Poisson equation [9]. A discontinuous Galerkin method [25] and an edge stabilized method [4] have been proposed for coupled Stokes-Darcy problems. A survey for coupling Navier-Stokes-Darcy equations is given in [10]. Least squares methods of finite element type [23] and of pseudo-spectral type [14] have been used to approximate the solutions of the Stokes-Darcy equations. In the above least squares approach, the authors eliminated pressure in the Stokes domain and approximated the stress and velocity in the Stokes domain and velocity and pressure in the Darcy domain. In this work, we approximate all primitive variables as well as the gradient of velocity with spectral accuracy. Spectral methods of least-squares type have been the object of many recent studies, such as second order elliptic boundary value problem [19], the Stokes equation [20, 27], the Navier-Stokes equation [16], interface problem of Stokes [15], interface problem of the Navier-Stokes [17] and the Stokes-Darcy equation [14].

The motivation of the present work is to devise a pseudo-spectral approximation based on a first order system least squares method. Least squares methods have a great flexibility in the choice of solution spaces that is not restricted by the Ladyshenskaya-Babuška-Brezzi compatibility condition. Furthermore, least squares methods allow one to incorporate additional equations and impose additional boundary conditions, as long as the system is consistent. Additionally, pseudo-spectral methods have the benefit of simplicity and spectral accuracy.

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To apply the least squares principle, we reformulate the Stokes-Darcy equations as a first-order system derived in terms of an additional vector variable (the vector of gradients of the Stokes velocities). We then modify the Stokes-Darcy equations by extending the first order system with the curl and the gradient of the velocity flux variable for the Stokes domain, and the curl of the velocity in the Darcy domain. This enables us to prove fully H^1 ellipticity of the proposed method in the Legendre approximation. The least squares functional is defined as a combination of

a. the squared L^2_w-norm of the residuals in Stokes domain Ω_S scaled by viscosity constant ν,
b. the squared L^2_w-norm of the residuals in Darcy domain Ω_D,
c. the squared L^2_w-norm of the residuals of the interface conditions.

The Beavers-Joseph-Saffman interface conditions are treated as an extra least squares functional, while boundary conditions are imposed into solution spaces. The continuous and discrete Legendre least squares functional is established to have fully H^1 ellipticity, while the continuous and discrete Chebyshev least squares functional is shown to be equivalent to the product norm $\|\mathbf{U}\|^2_{\mathbf{V}_w,\Omega_S} + \|\mathbf{U}\cdot\mathbf{n}\|^2_{w,\Gamma} + \|\mathbf{u}\|^2_{1,w,\Omega_S} + \|p\|^2_{1,w,\Omega_S} + \|\mathbf{w}\|^2_{w,div,\Omega_D} + \|\mathbf{w}\cdot\mathbf{n}\|^2_{w,\Gamma} + \|q\|^2_{1,w,\Omega_D}$, under the H^2_w regularity assumption for the Stokes equations in the Stokes domain Ω_S. Spectral convergence of the proposed method for both Legendre and Chebyshev cases is presented.

The outline of the paper is as follows. Some preliminaries are prepared in Section 2. In Section 3, the Stokes-Darcy equation is recast into a first order system of equations. In Section 4, the Legendre and Chebyshev least squares functionals are defined and shown to be equivalent to an appropriate product norm. Spectral convergence of the proposed methods are also presented in this section. Numerical examples are given in Section 5 to demonstrate spectral convergence of our method. The paper is ended by concluding remarks in Section 6.

2. Preliminaries. The standard notations and definitions for the weighted Sobolev spaces for $\mathbf{D} = [-1,1]^2$, are given as follows. The weighted space $L^2_w(\mathbf{D})$ is defined as

$$L^2_w(\mathbf{D}) = \{v : \mathbf{D} \to \mathbf{R} \,\big|\, v \text{ is measurable and } \|v\|_{0,w,\mathbf{D}} < \infty\},$$

equipped with the norm and the associated scalar product

$$\|v\|_{0,w,\mathbf{D}} = \left(\int_{\mathbf{D}} |v(\mathbf{x})|^2 w(\mathbf{x})d\mathbf{x}\right)^{1/2}, \qquad (u,v)_{0,w,\mathbf{D}} = \int_{\mathbf{D}} u(\mathbf{x})v(\mathbf{x})w(\mathbf{x})d\mathbf{x}.$$

Define the weighted Sobolev space $H^s_w(\mathbf{D})$ for a non-negative integer s as

$$H^s_w(\mathbf{D}) = \{v \in L^2_w(\mathbf{D}) \,\big|\, v^{(\alpha)} \in L^2_w(\mathbf{D}),\, |\alpha| = 1,2,\ldots,s\},$$

where $\alpha = (\alpha_1,\alpha_2)$ with $\alpha_1,\ \alpha_2 \geq 0,\ |\alpha| = \alpha_1+\alpha_2$, and $v^{(\alpha)} = \frac{\partial^{|\alpha|}v}{\partial x^{\alpha_1}\partial y^{\alpha_2}}$, equipped with the norm and the associated scalar product

$$\|v\|_{s,w,\mathbf{D}} = \left(\sum_{|\alpha|\leq s} \|v^{(\alpha)}\|^2_{0,w,\mathbf{D}}\right)^{1/2}, \qquad (u,v)_{s,w,\mathbf{D}} = \sum_{|\alpha|\leq s} (u^{(\alpha)}, v^{(\alpha)})_{w,\mathbf{D}}.$$

We note that $w(\mathbf{x}) = \hat{w}(x)\hat{w}(y)$ is either the Legendre weight function with $\hat{w}(t) = 1$, or the Chebyshev weight function with $\hat{w}(t) = (1-t^2)^{-\frac{1}{2}}$. The space $H^0_w(\mathbf{D})$ denotes $L^2_w(\mathbf{D})$, in which the norm and inner product will be denoted by $\|\cdot\|_{w,\mathbf{D}}$ and $(\cdot,\cdot)_{w,\mathbf{D}}$, respectively. Let $H^1_{0,w}(\mathbf{D})$ be the subspace of $H^1_w(\mathbf{D})$, consisting of the functions which vanish at the boundary. Let $L^2_{w,0}(\mathbf{D})$ be the subspace of $L^2_w(\mathbf{D})$ whose functions have average zero, i.e.,

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$\int_{\mathbf{D}} p\, w\, dx = 0$. For the Legendre case, we will omit the subscript w, for example, $\|\cdot\|_{\mathbf{D}}$, $(\cdot,\cdot)_{\mathbf{D}}$. Denote by $H_w^{-1}(\mathbf{D})$ the dual space of the space $H_{0,w}^1(\mathbf{D})$ equipped with its norm [2]

$$\|u\|_{-1,w,\mathbf{D}} := \sup_{\phi \in H_{0,w}^1(\mathbf{D})} \frac{(u,\phi)_{w,\mathbf{D}}}{\|\phi\|_{1,w,\mathbf{D}}}.$$

Let

$$H_w(\mathrm{div},\mathbf{D}) = \{\mathbf{v} \in L_w^2(\mathbf{D})^2 \ : \ \nabla\cdot\mathbf{v} \in L_w^2(\mathbf{D})\}$$

and

$$H_w(\mathrm{curl},\mathbf{D}) = \{\mathbf{v} \in L_w^2(\mathbf{D})^2 \ : \ \nabla\times\mathbf{v} \in L_w^2(\mathbf{D})\},$$

which are Hilbert spaces under the respective norms

$$\|\mathbf{v}\|_{w,\mathrm{div},\mathbf{D}} = \left(\|\mathbf{v}\|_{w,\mathbf{D}}^2 + \|\nabla\cdot\mathbf{v}\|_{w,\mathbf{D}}^2\right)^{1/2}$$

and

$$\|\mathbf{v}\|_{w,\mathrm{curl},\mathbf{D}} = \left(\|\mathbf{v}\|_{w,\mathbf{D}}^2 + \|\nabla\times\mathbf{v}\|_{w,\mathbf{D}}^2\right)^{1/2}.$$

Define their subspaces

$$H_{0,w}(\mathrm{div},\mathbf{D}) = \{\mathbf{v} \in H_w(\mathrm{div},\mathbf{D}) \ : \ \mathbf{n}\cdot\mathbf{v} = 0 \ \text{ on } \ \partial\mathbf{D}\}$$

and

$$H_{0,w}(\mathrm{curl},\mathbf{D}) = \{\mathbf{v} \in H_w(\mathrm{curl},\mathbf{D}) \ : \ \mathbf{n}\times\mathbf{v} = 0 \ \text{ on } \ \partial\mathbf{D}\}.$$

Let $\mathcal{P}_N$ be the space of all polynomials of degree less than or equal to N. Let $\{\xi_i\}_{i=0}^N$ be the Legendre-Gauss-Lobatto (LGL) or Chebyshev-Gauss-Lobatto (CGL) points on $[-1,1]$ such that $-1 =: \xi_0 < \xi_1 < \cdots < \xi_{N-1} < \xi_N := 1$. For the Legendre case, $\{\xi_i\}_{i=0}^N$ are the zeros of $(1-t^2)L_N'(t)$ where L_N is the Nth Legendre polynomial and the corresponding quadrature weights $\{w_i\}_{i=0}^N$ are given by

$$w_0 = w_N = \frac{2}{N(N+1)}, \quad w_j = \frac{2}{N(N+1)}\frac{1}{[L_N(\xi_j)]^2}, \quad 1 \le j \le N-1.$$

For the Chebyshev case, $\{\xi_i\}_{i=0}^N$ are the zeros of $(1-t^2)T_N'(t)$ where T_N is the Nth Chebyshev polynomial and the corresponding quadrature weights $\{w_i\}_{i=0}^N$ are given by

$$w_0 = w_N = \frac{\pi}{2N}, \quad w_j = \frac{\pi}{N}, \quad 1 \le j \le N-1.$$

We have the following accuracy property for Gaussian quadrature rules,

$$\int_{-1}^1 v(t)\hat{w}(t)dt = \sum_{i=0}^N w_i\, v(\xi_i), \quad \forall\, v \in \mathcal{P}_{2N-1}. \tag{2.1}$$

Let $\{\phi_i\}_{i=0}^N$ be the set of Lagrange polynomials of degree N with respect to LGL or CGL points $\{\xi_i\}_{i=0}^N$ which satisfy

$$\phi_i(\xi_j) = \delta_{ij}, \qquad \forall i,j = 0,1,\ldots,N,$$

where δ_{ij} denotes the Kronecker delta function. For any continuous function v on $I = (-1, 1)$, denote by $I_N v \in \mathcal{P}_N$, its Lagrangian interpolant at the nodes $\{\xi_j\}_{j=0}^N$, i.e., $I_N v(\xi_j) = v(\xi_j)$. The interpolation error estimate [29] is given by

$$\|v - I_N v\|_{k,w,I} \le C\, N^{k-s} \|v\|_{s,w,I}, \quad k = 0, 1, \tag{2.2}$$

provided that $v \in H_w^s(I)$ for some $s \ge 1$. Define the discrete scalar product and norm as

$$\langle u, v\rangle_{w,I,N} = \sum_{j=0}^{N} u(\xi_j) v(\xi_j) w_j, \qquad \|v\|_{w,I,N} = \langle v,\, v\rangle_{w,I,N}^{1/2}.$$

By (2.1), we have

$$\langle u, v\rangle_{w,I,N} = (u, v)_{w,I}, \quad \text{ for } u, v \in \mathcal{P}_{2N-1}. \tag{2.3}$$

It is well-known that

$$\|v\|_{w,I} \le \|v\|_{w,I,N} \le \gamma^* \|v\|_{w,I}, \quad \forall\, v \in \mathcal{P}_N, \tag{2.4}$$

where $\gamma^* = \sqrt{2 + \frac{1}{N}}$ in the Legendre case, and $\gamma^* = \sqrt{2}$ in the Chebyshev case [29]. For $u \in H_w^s(I)$, $s \ge 1$, and $v_N \in \mathcal{P}_N$

$$|(u, v_N)_{w,I} - \langle u, v_N\rangle_{w,I,N}| \le C\, N^{-s}\, \|u\|_{s,w,I}\, \|v_N\|_{w,I}.$$

If the interval $[-1, 1]$ is replaced by $[a, b]$, we can use the following linear transformation

$$t = \frac{b-a}{2}(x+1) + a \,:[-1,1] \to [a,b] \tag{2.5}$$

to find the Gauss-points $\{\hat{\xi}_j\}_{j=0}^N$ and the quadrature weights $\{\hat{w}_j\}_{j=1}^N$

$$\hat{\xi}_j = \frac{b-a}{2}(\xi_j + 1) + a \quad \text{and} \quad \hat{w}_j = \frac{b-a}{2} w_j.$$

The two-dimensional LGL or CGL nodes $\{\mathbf{x}_{ij}\}$ and the corresponding weights $\{\mathbf{w}_{ij}\}$ are denoted by

$$\mathbf{x}_{ij} = (\xi_i, \xi_j), \quad \mathbf{w}_{ij} = w_i w_j, \quad i, j = 0, 1, \ldots, N.$$

Let $\mathcal{Q}_N$ be the space of all polynomials of degree less than or equal to N with respect to each single variable x and y. Define the basis for $\mathcal{Q}_N$ as

$$\psi_{ij}(x, y) = \phi_i(x)\phi_j(y), \qquad i, j = 0, 1, \ldots, N.$$

For any continuous functions u and v in $\overline{\mathbf{D}}$, the associated discrete scalar product and norm are given by

$$\langle u, v\rangle_{w,\mathbf{D},N} = \sum_{i,j=0}^{N} \mathbf{w}_{ij} u(\mathbf{x}_{ij}) v(\mathbf{x}_{ij}) \quad \text{and} \quad \|v\|_{w,\mathbf{D},N} = \langle v, v\rangle_{w,\mathbf{D},N}^{1/2}.$$

From (2.1), we have

$$\langle u, v\rangle_{w,\mathbf{D},N} = (u, v)_{w,\mathbf{D}}, \quad \text{for } u, v \in \mathcal{Q}_{2N-1}, \tag{2.6}$$

and it is well-known that

$$\|v\|_{w,\mathbf{D}} \leq \|v\|_{w,\mathbf{D},N} \leq \gamma^* \|v\|_{w,\mathbf{D}}, \quad \forall\, v \in \mathcal{Q}_N, \tag{2.7}$$

where $\gamma^* = \left(2 + \frac{1}{N}\right)$ for the Legendre case and $\gamma^* = 2$ for the Chebyshev case [29]. The interpolation error estimate is given by [2, 6]

$$\|v - I_N v\|_{k,w,\mathbf{D}} \leq C\, N^{k-s} \|v\|_{s,w,\mathbf{D}}, \quad k = 0, 1, \tag{2.8}$$

provided that $v \in H^s_w(\mathbf{D})$ for some $s \geq 2$. For $u \in H^s_w(\mathbf{D})$, $s \geq 2$, and $v_N \in \mathcal{Q}_N$

$$|(u, v_N)_{w,\mathbf{D}} - \langle u, v_N \rangle_{w,\mathbf{D},N}| \leq C\, N^{-s}\, \|u\|_{s,w,\mathbf{D}}\, \|v_N\|_{w,\mathbf{D}}. \tag{2.9}$$

LEMMA 2.1. *For any $\mathbf{v} \in [L^2_w(\mathbf{D})]^2$, we have*

$$\|\nabla \cdot \mathbf{v}\|_{-1,w,\mathbf{D}} \leq C \|\mathbf{v}\|_{w,\mathbf{D}}.$$

Proof. The proof is similar to Lemma 4.2 of [19]. □

LEMMA 2.2. *[11] For any $p \in L^2_0(\Omega_S)$ we have*

$$\|p\| \leq C \|\nabla p\|_{-1}.$$

We use the following bounds for traces from $H^1_w(\Omega_D)$ and $H^1_w(\Omega_S)$ [24]:

$$\|q\|^2_{1/2,w,\Gamma} \leq C_T(\|q\|^2_{0,w,D} + \|\nabla q\|^2_{0,w,D}), \tag{2.10}$$

$$\|\mathbf{v}\|^2_{1/2,w,\Gamma} \leq C_T(\|\mathbf{v}\|^2_{0,w,S} + \|\nabla \mathbf{v}\|^2_{0,w,S}), \tag{2.11}$$

$$\|\mathbf{v} \cdot \mathbf{n}\|_{-1/2,\Gamma} \leq C_T \|\mathbf{v}\|_{\text{div},D}, \tag{2.12}$$

with a constant C_T. We use the Poincaré-Friedrichs inequality [3] of the form

$$\|q\|_{0,D} \leq C_F \|K^{1/2} \nabla q\|_{0,D} \tag{2.13}$$

for all $q \in H^1(\Omega_D)$ which satisfies $\int_{\Omega_D} q = 0$.

REMARK 2.3. If the domain $\mathbf{D}$ is replaced by a simply connected domain, then the Gordon and Hall transformation [12, 13] can be used to map the simply connected domain into $\mathbf{D}$.

The following a priori estimate holds for the Stokes equation with homogeneous Dirichlet boundary condition on $\partial\mathbf{D}$

$$\|\nu \mathbf{u}\|_{1,\omega,\mathbf{D}} + \|p\|_{\omega,\mathbf{D}} \leq C\big(\| -\nu\Delta\mathbf{u} + \nabla p\|_{-1,\omega,\mathbf{D}} + \|\nu \nabla \cdot \mathbf{u}\|_{\omega,\mathbf{D}}\big). \tag{2.14}$$

Its proof can be found for the case $\nu = 1$ and $\omega = 1$ in [11] and for the Chebyshev weight ω in [2], the case for general ν is then immediate. A priori estimate for the Poisson equation $-\Delta q = g$ with the Neumann boundary condition $\frac{\partial q}{\partial \mathbf{n}} = 0$ on $\partial\mathbf{D}$, is [1]

$$\|q\|_{1,w,\mathbf{D}} \leq C \| -\Delta q\|_{-1,w,\mathbf{D}}, \tag{2.15}$$

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subject to solvability condition

$$\int_{\mathbf{D}} g = 0 \quad \text{or} \quad \int_{\mathbf{D}} q = 0.$$

THEOREM 2.4. *[11] Assume that the domain* $\mathbf{D}$ *is a bounded convex polyhedron or has* $C^{1,1}$ *boundary. Then for any* $\mathbf{v}$ *in either* $H_0(\text{div}, \mathbf{D}) \cap H(\text{curl}, \mathbf{D})$ *or* $H(\text{div}, \mathbf{D}) \cap H_0(\text{curl}, \mathbf{D})$ *we have*

$$\|\mathbf{v}\|_1^2 \le C(\|\mathbf{v}\|^2 + \|\nabla \cdot \mathbf{v}\|^2 + \|\nabla \times \mathbf{v}\|^2).$$

If the domain is simply connected then

$$\|\mathbf{v}\|_1^2 \le C(\|\nabla \cdot \mathbf{v}\|^2 + \|\nabla \times \mathbf{v}\|^2).$$

3. First order systems. In this section we transform the Stokes-Darcy equation into a system of first order equations by introducing the gradient of velocity in the Stokes domain as a new independent variable. To do so, for the velocity vector function $\mathbf{u} = (u_1, u_2)^t$, we introduce the gradient velocity variable $\mathbf{U} = \nabla \mathbf{u}^t = (\nabla u_1, \nabla u_2)$ which is a matrix with entries $\mathbf{U}_{ij} = \partial u_j / \partial x_i, \; 1 \le i, j \le 2$. Then the Stokes equation (1.1) can be recast as

$$\begin{cases} \mathbf{U} - \nabla \mathbf{u}^t = \mathbf{0}, & \text{in } \Omega_S, \\ -\nu(\nabla \cdot \mathbf{U})^t + \nabla p = \mathbf{f}, & \text{in } \Omega_S, \\ \nabla \cdot \mathbf{u} = 0, & \text{in } \Omega_S. \end{cases}$$

We extend the standard curl operator $\nabla \times \mathbf{u} = -\partial_y u_1 + \partial_x u_2$, divergence operator $\nabla \cdot \mathbf{u} = \partial_x u_1 + \partial_y u_2$ in $\mathbf{R}^2$ and tangential operator $\mathbf{n} \times U$ to $\mathbf{U} = (U_1, U_2)$, componentwise, i.e.,

$$\nabla \times \mathbf{U} = (\nabla \times U_1, \nabla \times U_2), \quad \nabla \cdot \mathbf{U} = (\nabla \cdot U_1, \nabla \cdot U_2)$$

and

$$\mathbf{n} \times \mathbf{U} = (\mathbf{n} \times U_1, \mathbf{n} \times U_2),$$

where $\mathbf{n}$ is the outward unit normal vector on $\partial\Omega_S$. Then it is easy to see that

$$\text{tr}\, \mathbf{U} = 0, \qquad \nabla \times \mathbf{U} = \mathbf{0} \;\text{ in }\; \Omega_S, \qquad \text{and} \qquad \mathbf{n} \times \mathbf{U} = \mathbf{0} \;\text{ on }\; \partial\Omega_S,$$

where $\text{tr}\, \mathbf{U} = \mathbf{U}_{11} + \mathbf{U}_{22}$. We also have

$$\mathbf{E}(\mathbf{u}) = \frac{1}{2}(\mathbf{U} + \mathbf{U}^t) \;\text{ and }\; \mathbf{n} \cdot \mathbf{T} \cdot \mathbf{n} + q = \nu \mathbf{n} \cdot (\mathbf{U} + \mathbf{U}^t) \cdot \mathbf{n} + q - p.$$

We consider the following extended first order system for the Stokes-Darcy equation

$$\begin{cases} -\nu(\nabla \cdot \mathbf{U})^t + \nabla p = \mathbf{f}, & \text{in } \Omega_S, \\ \nabla \cdot \mathbf{u} = 0, & \text{in } \Omega_S, \\ \mathbf{U} - \nabla \mathbf{u}^t = \mathbf{0}, & \text{in } \Omega_S, \\ \nabla(\text{tr}\, \mathbf{U}) = \mathbf{0}, & \text{in } \Omega_S, \\ \nabla \times \mathbf{U} = \mathbf{0}, & \text{in } \Omega_S, \\ \frac{1}{\sqrt{K}} \mathbf{w} + \sqrt{K} \nabla q = 0, & \text{in } \Omega_D, \\ \nabla \cdot \mathbf{w} = g, & \text{in } \Omega_D, \\ \nabla \times K^{-1} \mathbf{w} = 0 & \text{in } \Omega_D. \end{cases} \tag{3.1}$$

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along with boundary conditions

$$\begin{cases} \mathbf{u} = 0, & \text{on } \partial\Omega_S, \\ \mathbf{n} \times \mathbf{U} = \mathbf{0}, & \text{on } \partial\Omega_S, \\ \mathbf{w} \cdot \mathbf{n} = 0, & \text{on } \partial\Omega_D, \end{cases}$$

and interface conditions

$$\begin{cases} \mathbf{u} \cdot \mathbf{n} - \mathbf{w} \cdot \mathbf{n} = 0, & \text{on } \Gamma, \\ \nu \mathbf{n} \cdot (\mathbf{U} + \mathbf{U}^t) \cdot \mathbf{n} + q - p = 0, & \text{on } \Gamma, \\ \beta\nu\, \mathbf{n} \times (\mathbf{U} + \mathbf{U}^t) \cdot \mathbf{n} + \mathbf{u} \times \mathbf{n} = 0, & \text{on } \Gamma. \end{cases} \tag{3.2}$$

4. Least squares method. In this section we consider the Legendre and Chebyshev pseudo-spectral least squares methods for the first order system of equations (3.1)–(3.2) of the Stokes-Darcy equations. To this end, let

$$\begin{aligned} V_{w,S} &= [H^1_{w,\partial\Omega_S}(\Omega_S)]^2, \\ V_{w,D} &= \{\mathbf{v} \in H_w(\mathrm{div}, \Omega_\mathrm{D}) \,:\, \mathbf{v} \cdot \mathbf{n} = 0 \ \text{ on } \ \partial\Omega_\mathrm{D}, \ \text{ and } \ \mathbf{v} \cdot \mathbf{n} \in \mathrm{L}^2_\mathrm{w}(\Gamma)\}, \end{aligned}$$

where

$$H^1_{w,\partial\Omega_S}(\Omega_S) = \{\mathbf{v} \in H^1_w(\Omega_S) \,:\, \mathbf{v} = 0 \ \text{ on } \ \partial\Omega_S\}.$$

Let

$$\begin{aligned} V_1 &= \{\mathbf{V} \in [H^1(\Omega_S)]^4 \,:\, \mathbf{n} \times \mathbf{V} = \mathbf{0} \ \text{ on } \ \partial\Omega_S\}, \\ V_w &= \{\mathbf{V} \in [L^2_w(\Omega_S)]^4 \,:\, \mathbf{n} \times \mathbf{V} = \mathbf{0} \ \text{ on } \ \partial\Omega_S \ \text{ and } \ \|\mathbf{V}\|_{V_w} < \infty\}, \end{aligned}$$

equipped with the norm

$$\|\mathbf{V}\|_{V_w} = \left(\|\mathbf{V}\|^2_{w,S} + \|\nabla \cdot \mathbf{V}\|^2_{w,S} + \|\nabla \times \mathbf{V}\|^2_{w,S}\right)^{1/2}.$$

We note that V_1 is for Legendre and V_w is for the Chebyshev case. Define

$$\mathbf{W}_w = V_w \times V_{w,S} \times [H^1_w(\Omega_S) \cap L^2_{w,0}(\Omega_S)] \times V_{w,D} \times H^1_w(\Omega_D).$$

Let $\mathcal{U} = (\mathbf{U}, \mathbf{u}, p, \mathbf{w}, q)$ and $\mathcal{V} = (\mathbf{S}, \mathbf{v}, s, \mathbf{z}, r)$. Define the Legendre/Chebyshev least squares functional as

$$\mathcal{G}_w(\mathcal{U}; \mathbf{f}, g) = \mathcal{G}_{w,S}(\mathbf{U}, \mathbf{u}, p; \mathbf{f}) + \mathcal{G}_{w,D}(\mathbf{w}, q; g) + \mathcal{G}_{w,I}(\mathcal{U}) \tag{4.1}$$

over $\mathcal{U} \in \mathbf{W}_w$, where

$$\begin{aligned} \mathcal{G}_{w,S}(\mathbf{U}, \mathbf{u}, p; \mathbf{f}) &= \nu^2 \|\mathbf{U} - \nabla \mathbf{u}^t\|^2_{w,S} + \|\mathbf{f} + \nu(\nabla \cdot \mathbf{U})^t - \nabla p\|^2_{w,S} + \nu^2 \|\nabla \cdot \mathbf{u}\|^2_{w,S} \\ &\quad + \nu^2 \|\nabla(\mathrm{tr}\, \mathbf{U})\|^2_{w,S} + \nu^2 \|\nabla \times \mathbf{U}\|^2_{w,S}, \\ \mathcal{G}_{w,D}(\mathbf{w}, q; g) &= \|K^{-1/2}\mathbf{w} + K^{1/2}\nabla q\|^2_{w,D} + \|\nabla \cdot \mathbf{w} - g\|^2_{w,D} + \|\nabla \times K^{-1}\mathbf{w}\|^2_{w,D}, \\ \mathcal{G}_{w,I}(\mathcal{U}) &= \|\mathbf{u} \cdot \mathbf{n} - \mathbf{w} \cdot \mathbf{n}\|^2_{w,\Gamma} + \|\nu \mathbf{n} \cdot (\mathbf{U} + \mathbf{U}^t) \cdot \mathbf{n} + q - p\|^2_{w,\Gamma} \\ &\quad + \|\beta\nu \mathbf{n} \times (\mathbf{U} + \mathbf{U}^t) \cdot \mathbf{n} + \mathbf{u} \times \mathbf{n}\|^2_{w,\Gamma} \end{aligned}$$

for $\mathcal{U} \in \mathbf{W}_w$. The first order system least squares variational problem for (4.1) consists of minimizing the quadratic function $\mathcal{G}_w(\mathcal{U}; \mathbf{f}, g)$ over $\mathbf{W}_w$, that is: find $\mathcal{U} \in \mathbf{W}_w$ such that

$$\mathcal{G}_w(\mathcal{U}; \mathbf{f}, g) = \inf_{\mathcal{V} \in \mathbf{W}_w} \mathcal{G}_w(\mathcal{V}; \mathbf{f},\, g).$$

The corresponding variational problem is to find $\mathcal{U} \in \mathbf{W}_w$ such that

$$\mathcal{A}_w(\mathcal{U};\mathcal{V}) = \mathcal{F}_w(\mathcal{V}), \qquad \forall\, \mathcal{V} \in \mathbf{W}_w, \tag{4.2}$$

where

$$\begin{aligned}
\mathcal{A}_w(\mathcal{U};\mathcal{V}) = {} & (\nu(\nabla\cdot\mathbf{U})^t - \nabla p, \nu(\nabla\cdot\mathbf{S})^t - \nabla s)_{w,S} + \nu^2(\nabla\cdot\mathbf{u}, \nabla\cdot\mathbf{v})_{w,S} \\
& + \nu^2\big(\mathbf{U} - \nabla\mathbf{u}^t,\, \mathbf{S} - \nabla\mathbf{v}^t\big)_{w,S} + \nu^2\big(\nabla(\operatorname{tr}\mathbf{U}),\, \nabla(\operatorname{tr}\mathbf{S})\big)_{w,S} \\
& + \nu^2\big(\nabla\times\mathbf{U},\, \nabla\times\mathbf{S}\big)_{w,S} + (\nabla\cdot\mathbf{w}, \nabla\cdot\mathbf{z})_{w,D} \\
& + \left(\frac{1}{\sqrt{K}}\mathbf{w} + \sqrt{K}\nabla q, \frac{1}{\sqrt{K}}\mathbf{z} + \sqrt{K}\nabla r\right)_{w,D} \\
& + (\nabla\times K^{-1}\mathbf{w}, \nabla\times K^{-1}\mathbf{z})_{w,D} + (\mathbf{u}\cdot\mathbf{n} - \mathbf{w}\cdot\mathbf{n}, \mathbf{v}\cdot\mathbf{n} - \mathbf{z}\cdot\mathbf{n})_{w,\Gamma} \\
& + \big(\nu\mathbf{n}\cdot(\mathbf{U}+\mathbf{U}^t)\cdot\mathbf{n} + q - p,\, \nu\mathbf{n}\cdot(\mathbf{S}+\mathbf{S}^t)\cdot\mathbf{n} + r - s\big)_{w,\Gamma} \\
& + \big(\beta\nu\mathbf{n}\times(\mathbf{U}+\mathbf{U}^t)\cdot\mathbf{n} + \mathbf{u}\times\mathbf{n},\, \beta\nu\mathbf{n}\times(\mathbf{S}+\mathbf{S}^t)\cdot\mathbf{n} + \mathbf{v}\times\mathbf{n}\big)_{w,\Gamma}
\end{aligned}$$

and

$$\mathcal{F}_w(\mathbf{S},\mathbf{v},s,\mathbf{z},r) = (g,\, \nabla\cdot\mathbf{z})_{w,D} - (\mathbf{f}, \nu(\nabla\cdot\mathbf{S})^t - \nabla s)_{w,S}.$$

Let

$$\begin{aligned}
|||(\mathbf{U},\mathbf{u},p)||| &= \left(\nu^2\|\mathbf{U}\|_{1,S}^2 + \nu^2\|\mathbf{U}\cdot\mathbf{n}\|_{\Gamma}^2 + \nu^2\|\mathbf{u}\|_{1,S}^2 + \|p\|_{1,S}^2\right)^{1/2}, \\
|||(\mathbf{w},q)||| &= \left(\|q\|_{1,w,D}^2 + \|\mathbf{w}\|_{1,w,D}^2 + \|\mathbf{w}\cdot\mathbf{n}\|_{\Gamma}^2\right)^{1/2}, \\
|||(\mathbf{U},\mathbf{u},p)|||_w &= \left(\nu^2\|\mathbf{U}\|_{\mathbf{V}_w,S}^2 + \nu^2\|\mathbf{U}\cdot\mathbf{n}\|_{w,\Gamma}^2 + \nu^2\|\mathbf{u}\|_{1,w,S}^2 + \|p\|_{1,w,S}^2\right)^{1/2}, \\
|||(\mathbf{w},q)|||_w &= \left(\|q\|_{1,w,D}^2 + \|\mathbf{w}\|_{\operatorname{div},w,D}^2 + \|\mathbf{w}\cdot\mathbf{n}\|_{w,\Gamma}^2 + \|\nabla\times\mathbf{w}\|_{w,D}^2\right)^{1/2}
\end{aligned}$$

and

$$|||\mathcal{U}|||^2 = |||(\mathbf{U},\mathbf{u},p)|||^2 + |||(\mathbf{w},q)|||^2, \qquad |||\mathcal{U}|||_w^2 = |||(\mathbf{U},\mathbf{u},p)|||_w^2 + |||(\mathbf{w},q)|||_w^2.$$

LEMMA 4.1. *For* $(\mathbf{U},\mathbf{u},p) \in [\mathbf{V}_{w,S}]^2 \times [H^1_{w,S}(\Omega_S)]^2 \times H^1_{w,S}(\Omega_S)$, *we have*

$$\nu^2\|(\nabla\cdot\mathbf{U})^t\|_{w,S}^2 + \|\nabla p\|_{w,S}^2 \le C\big(\|\nu(\nabla\cdot\mathbf{U})^t - \nabla p\|_{w,S}^2 + \nu^2\|\nabla(\operatorname{tr}\mathbf{U})\|_{w,S}^2 + \nu^2\|\nabla\times\mathbf{U}\|_{w,S}^2\big).$$

Proof. The proof is similar to [15, Lemma 4.1] and [5, Theorem 3.2]. ∎

We now show the continuity and coercivity of the least squares functional for the Legendre approximation. We note that for the Legendre case, the subscript w is omitted.

THEOREM 4.2. *There exists a constant* C *such that*

$$\frac{1}{C}|||\mathcal{U}|||^2 \le \mathcal{G}(\mathcal{U};0,0) \le C|||\mathcal{U}|||^2, \qquad \forall\,\mathcal{U} \in \mathbf{W}. \tag{4.3}$$

Proof. By using (2.10), (2.11), the triangle and Cauchy-Schwarz inequalities, we have

$$\begin{aligned}\mathcal{G}_S(\mathbf{U},\mathbf{u},p;0) &\le C\big(\nu^2\|\mathbf{U}\|_{1,S}^2+\nu^2\|\mathbf{u}\|_{1,S}^2+\|p\|_{1,S}^2\big),\\ \mathcal{G}_D(\mathbf{w},q;0) &\le C\big(\|\mathbf{w}\|_{\mathrm{div},\,D}^2+\|q\|_{1,D}^2+\|\nabla\times\mathbf{w}\|_D^2\big),\\ \mathcal{G}_I(\mathbf{U},\mathbf{u},p,\mathbf{w},q) &\le C\|\mathbf{w}\cdot\mathbf{n}\|_\Gamma^2+C\nu^2\max\{1,\,\beta^2\}\|\mathbf{U}\cdot\mathbf{n}\|_\Gamma^2\\ &\quad+C_T\big(\|p\|_{1,S}^2+\|q\|_{1,D}^2+\|\mathbf{u}\|_{1,S}^2\big),\end{aligned}$$

which proves the upper bound of (4.3). To prove the lower bound, we have

$$\|\nabla\times K^{-1}\mathbf{w}\|_D^2\le\mathcal{G}_D(\mathbf{w},q;0).$$

By a similar idea of [28], using

$$2|(\mathbf{w}\cdot\mathbf{n},q)_\Gamma|\le\frac{1}{\delta\varepsilon}\|\mathbf{w}\cdot\mathbf{n}\|_{-1/2,\Gamma}^2+\varepsilon\delta\|q\|_{1/2,\Gamma}^2,$$

the ϵ-inequality with $\epsilon=1$ and the trace inequality (2.10), we have

$$\begin{aligned}&\|K^{-1/2}\mathbf{w}+K^{1/2}\nabla q\|_D^2\\ &\quad=\|K^{-1/2}\mathbf{w}\|_D^2+2(1-\delta)(\mathbf{w},\nabla q)_D+2\delta(\mathbf{w},\nabla q)_D+\|K^{1/2}\nabla q\|_D^2\\ &\quad=\|K^{-1/2}\mathbf{w}\|_D^2+2(1-\delta)(\mathbf{w},\nabla q)_D-2\delta(\nabla\cdot\mathbf{w},q)_D+\|K^{1/2}\nabla q\|_D^2\\ &\qquad\quad-2\delta(\mathbf{w}\cdot\mathbf{n},q)_\Gamma\\ &\quad\ge\delta\|K^{-1/2}\mathbf{w}\|_D^2-2\delta(\nabla\cdot\mathbf{w},q)_D+\delta\|K^{1/2}\nabla q\|_D^2-2\delta(\mathbf{w}\cdot\mathbf{n},q)_\Gamma\\ &\quad\ge\delta\|K^{-1/2}\mathbf{w}\|_D^2-2\delta(\nabla\cdot\mathbf{w},q)_D+\delta\|K^{1/2}\nabla q\|_D^2-\frac{1}{\varepsilon}\|\mathbf{w}\cdot\mathbf{n}\|_{-1/2,\Gamma}^2\\ &\qquad\quad-\delta^2\varepsilon\|q\|_{1/2,\Gamma}^2\\ &\quad\ge\delta\|K^{-1/2}\mathbf{w}\|_D^2-2\delta(\nabla\cdot\mathbf{w},q)_D+\delta\|K^{1/2}\nabla q\|_D^2\\ &\qquad\quad-\frac{1}{\varepsilon}\|\mathbf{w}\cdot\mathbf{n}\|_{-1/2,\Gamma}^2-\delta^2\varepsilon C_T^2\|q\|_{1,D}^2\end{aligned}$$

for $\delta\in(0,\,1)$. Using (2.13), we have

$$\begin{aligned}\mathcal{G}_D(\mathbf{w},q;0)&\ge\delta\|K^{-1/2}\mathbf{w}\|_D^2-2\delta(\nabla\cdot\mathbf{w},q)_D+\delta\|K^{1/2}\nabla q\|_D^2\\ &\qquad-\frac{1}{\varepsilon}\|\mathbf{w}\cdot\mathbf{n}\|_{-1/2,\Gamma}^2-\delta^2\varepsilon C_T^2\|q\|_{1,D}^2+\|\nabla\cdot\mathbf{w}\|_D^2\\ &\ge\delta\|K^{-1/2}\mathbf{w}\|_D^2-2\delta(\nabla\cdot\mathbf{w},q)_D-\frac{1}{\varepsilon}\|\mathbf{w}\cdot\mathbf{n}\|_{-1/2,\Gamma}^2\\ &\qquad+(\frac{\delta}{C_F^2}-\delta^2\varepsilon C_T^2)\|q\|_{1,D}^2+\|\nabla\cdot\mathbf{w}\|_D^2\\ &=\delta\|K^{-1/2}\mathbf{w}\|_D^2+\frac{1}{2}\|\nabla\cdot\mathbf{w}-2\delta q\|_D^2+\frac{1}{2}\|\nabla\cdot\mathbf{w}\|_D^2\\ &\qquad-2\delta^2\|q\|_D^2+(\frac{\delta}{C_F^2}-\delta^2\varepsilon C_T^2)\|q\|_{1,D}^2-\frac{1}{\varepsilon}\|\mathbf{w}\cdot\mathbf{n}\|_{-1/2,\Gamma}^2\\ &\ge\delta\|K^{-1/2}\mathbf{w}\|_D^2+\frac{1}{2}\|\nabla\cdot\mathbf{w}\|_D^2+(\frac{\delta}{C_F^2}-\delta^2\varepsilon C_T^2-2\delta^2)\|q\|_{1,D}^2\\ &\qquad\quad-\frac{1}{\varepsilon}\|\mathbf{w}\cdot\mathbf{n}\|_{-1/2,\Gamma}^2.\end{aligned}$$

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Choosing

$$\delta = \frac{1}{2C_F^2(2+\varepsilon C_T^2)}$$

and

$$C_E = \min\left\{\frac{1}{2},\, \delta,\, \frac{\delta}{2C_F^2}\right\},$$

we get

$$\mathcal{G}_D(\mathbf{w}, q; 0) \geq C_E\big(\|\mathbf{w}\|^2_{\mathrm{div},\,D} + \|q\|^2_{1,D}\big) - \frac{1}{\varepsilon}\|\mathbf{w}\cdot\mathbf{n}\|^2_{-1/2,\Gamma}.$$

Application of (2.12) implies

$$\mathcal{G}_D(\mathbf{w}, q; 0) \geq (C_E - \frac{C_T^2}{\varepsilon})\|\mathbf{w}\|^2_{\mathrm{div},\,D} + C_E\|q\|^2_{1,D}.$$

Therefore, for $\varepsilon \geq C_T^2/C_E$, there exists a constant C such that

$$\|\mathbf{w}\|^2_{\mathrm{div},\,D} + \|q\|^2_{1,D} \leq C\mathcal{G}_D(\mathbf{w}, q; 0). \tag{4.4}$$

To prove (4.3) in the Stokes domain Ω_S, let

$$\mathcal{F}_S(\mathbf{U}, \mathbf{u}, p) = \nu^2\|\mathbf{U} - \nabla\mathbf{u}^t\|^2_S + \|\nu(\nabla\cdot\mathbf{U})^t - \nabla p\|^2_{-1,S} + \nu^2\|\nabla\cdot\mathbf{u}\|^2_S$$

and

$$\mathbf{W}_1 = [H(\mathrm{div},\,\Omega_\mathrm{S})]^2 \times \mathrm{V_S} \times [\mathrm{H}^1(\Omega_\mathrm{S}) \cap \mathrm{L}^2_0(\Omega_\mathrm{S})].$$

Suppose that $(\mathbf{U}, \mathbf{u}, p) \in \mathbf{W}_1$ and let $\phi \in H^1_0(\Omega_S)$. We have [5]

$$\begin{aligned}(\nabla p, \phi)_S &= (-\nu(\nabla\cdot\mathbf{U})^t + \nabla p, \phi)_S - \nu(\mathbf{U}, \nabla\phi^t)_S \\ &\leq \|-\nu(\nabla\cdot\mathbf{U})^t + \nabla p\|_{-1,S}\|\phi\|_{1,S} + \nu\|\mathbf{U}\|_S\|\nabla\phi^t\|_S,\end{aligned}$$

from which Lemma 2.2 gives

$$\|p\|_S \leq C(\|-\nu(\nabla\cdot\mathbf{U})^t + \nabla p\|_{-1,S} + \nu\|\mathbf{U}\|_S). \tag{4.5}$$

From (4.5) and the Poincaré-Friedrichs inequality, we have

$$\begin{aligned}\nu^2\|\nabla\mathbf{u}^t\|^2_S &= \nu^2(\nabla\mathbf{u}^t - \mathbf{U}, \nabla\mathbf{u}^t)_S + \nu(-\nu(\nabla\cdot\mathbf{U})^t + \nabla p, \mathbf{u})_S + \nu(p, \nabla\cdot\mathbf{u})_S \\ &\quad - \nu(p\cdot\mathbf{n}, \mathbf{u})_\Gamma + \nu^2(\mathbf{U}\cdot\mathbf{n}, \mathbf{u})_\Gamma \\ &\leq \nu^2\|\nabla\mathbf{u}^t - \mathbf{U}\|_S\|\nabla\mathbf{u}^t\|_S + \nu\|-\nu(\nabla\cdot\mathbf{U})^t + \nabla p\|_{-1,S}\|\mathbf{u}\|_{1,S} \\ &\quad + \nu\|p\|_S\|\nabla\cdot\mathbf{u}\|_S + \nu(\nu\mathbf{U}\cdot\mathbf{n} - p\cdot\mathbf{n}, \mathbf{u})_\Gamma \\ &\leq \big(\nu\|\nabla\mathbf{u}^t - \mathbf{U}\|_S + \|-\nu(\nabla\cdot\mathbf{U})^t + \nabla p\|_{-1,S}\big)\nu\|\nabla\mathbf{u}^t\|_S \\ &\quad + C\nu\|\nabla\cdot\mathbf{u}\|_S\|-\nu(\nabla\cdot\mathbf{U})^t + \nabla p\|_{-1,S} + C\nu^2\|\nabla\cdot\mathbf{u}\|_S\|\mathbf{U}\|_S \\ &\quad + \nu(\nu\mathbf{U}\cdot\mathbf{n} - p\cdot\mathbf{n}, \mathbf{u})_\Gamma.\end{aligned}$$

Using the ϵ-inequality with $\epsilon = 1$ for the first two products yields

$$\nu^2\|\nabla\mathbf{u}^t\|^2_S \leq C\mathcal{F}_S(\mathbf{U}, \mathbf{u}, p) + C\nu^2\|\nabla\cdot\mathbf{u}\|_S\|\mathbf{U}\|_S + \nu(\nu\mathbf{U}\cdot\mathbf{n} - p\cdot\mathbf{n}, \mathbf{u})_\Gamma. \tag{4.6}$$

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From the interface condition (3.2) we have

$$\nu \mathbf{U}\cdot\mathbf{n} - p\cdot\mathbf{n} = -\nu\mathbf{U}^t\cdot\mathbf{n} - q\cdot\mathbf{n} \quad \text{and} \quad \mathbf{u}\cdot\mathbf{n} = \mathbf{w}\cdot\mathbf{n}. \tag{4.7}$$

By (4.4) and (4.7) we have

$$\begin{aligned}
\nu(\nu\mathbf{U}\cdot\mathbf{n} - p\cdot\mathbf{n},\mathbf{u})_\Gamma &= -\nu(\nu\mathbf{U}^t\cdot\mathbf{n} + q\cdot\mathbf{n},\mathbf{u})_\Gamma = -\nu(\nu\mathbf{n}\cdot\mathbf{U}^t\cdot\mathbf{n}+q,\mathbf{u}\cdot\mathbf{n})_\Gamma\\
&= -\nu(\nu\mathbf{n}\cdot\mathbf{U}^t\cdot\mathbf{n}+q,\mathbf{w}\cdot\mathbf{n})_\Gamma\\
&\le \nu\|\nu\mathbf{n}\cdot\mathbf{U}^t\cdot\mathbf{n}+q\|_{1/2,\Gamma}\|\mathbf{w}\cdot\mathbf{n}\|_{-1/2,\Gamma}\\
&\le \frac{1}{2\epsilon}\|\nu\mathbf{n}\cdot\mathbf{U}^t\cdot\mathbf{n}+q\|^2_{1/2,\Gamma} + \frac{\epsilon}{2}\nu^2\|\mathbf{w}\cdot\mathbf{n}\|^2_{-1/2,\Gamma}\\
&\le \frac{1}{\epsilon}\|\nu\mathbf{n}\cdot\mathbf{U}^t\cdot\mathbf{n}\|^2_{1/2,\Gamma} + \frac{1}{\epsilon}\|q\|^2_{1/2,\Gamma} + \frac{\epsilon}{2}\nu^2\|\mathbf{w}\cdot\mathbf{n}\|^2_{-1/2,\Gamma}\\
&\le \frac{\nu^2}{\epsilon}\|\mathbf{U}^t\cdot\mathbf{n}\|^2_\Gamma + \frac{1}{\epsilon}C_T^2\|q\|^2_{1,D} + \frac{\epsilon}{2}C_T^2\nu^2\|\mathbf{w}\|^2_{\mathrm{div},D}\\
&\le \frac{\nu^2}{\epsilon}\|\mathbf{U}^t\cdot\mathbf{n}\|^2_\Gamma + C\mathcal{G}_D(\mathbf{w},q;0).
\end{aligned} \tag{4.8}$$

Hence (4.6) becomes

$$\nu^2\|\nabla\mathbf{u}^t\|^2_S \le C\mathcal{F}_S(\mathbf{U},\mathbf{u},p) + C\mathcal{G}_D(\mathbf{w},q;0) + C\nu^2\|\nabla\cdot\mathbf{u}\|_S\|\mathbf{U}\|_S + \frac{\nu^2}{\epsilon}\|\mathbf{U}^t\cdot\mathbf{n}\|^2_\Gamma. \tag{4.9}$$

We have

$$\begin{aligned}
\nu^2\|\mathbf{U}\|^2_S &= \nu^2(\mathbf{U}-\nabla\mathbf{u}^t,\mathbf{U})_S + \nu(-\nu(\nabla\cdot\mathbf{U})^t+\nabla p,\mathbf{u})_S + \nu(p,\nabla\cdot\mathbf{u})_S\\
&\qquad -\nu(\mathbf{u},p\cdot\mathbf{n})_\Gamma + \nu^2(\mathbf{U}\cdot\mathbf{n},\mathbf{u})_\Gamma\\
&\le \nu^2\|\nabla\mathbf{u}^t-\mathbf{U}\|_S\|\mathbf{U}\|_S + \nu\|-\nu(\nabla\cdot\mathbf{U})^t+\nabla p\|_{-1,S}\|\mathbf{u}\|_{1,S}\\
&\qquad +\nu\|p\|_S\|\nabla\cdot\mathbf{u}\|_S + \nu(\nu\mathbf{U}\cdot\mathbf{n}-p\cdot\mathbf{n},\mathbf{u})_\Gamma\\
&\le (\nu\|\nabla\mathbf{u}^t-\mathbf{U}\|_S)(\nu\|\mathbf{U}\|_S) + C\nu\|-\nu(\nabla\cdot\mathbf{U})^t+\nabla p\|_{-1,S}\|\nabla\mathbf{u}^t\|_S\\
&\qquad +C\nu\|\nabla\cdot\mathbf{u}\|_S\|-\nu(\nabla\cdot\mathbf{U})^t+\nabla p\|_{-1,S} + C\nu\|\nabla\cdot\mathbf{u}\|_S(\nu\|\mathbf{U}\|_S)\\
&\qquad +\nu(\nu\mathbf{U}\cdot\mathbf{n}-p\cdot\mathbf{n},\mathbf{u})_\Gamma.
\end{aligned}$$

Using the ϵ-inequality with $\epsilon = 1$ twice, (4.8), and (4.9), we arrive at

$$\nu^2\|\mathbf{U}\|^2_S \le C\mathcal{F}_S(\mathbf{U},\mathbf{u},p) + C\mathcal{G}_D(\mathbf{w},q;0) + 4\frac{\nu^2}{\epsilon}\|\mathbf{U}^t\cdot\mathbf{n}\|^2_\Gamma. \tag{4.10}$$

By using (4.10) in (4.5) and (4.9), we get

$$\|p\|^2_S \le C\mathcal{F}_S(\mathbf{U},\mathbf{u},p) + C\mathcal{F}_D(\mathbf{w},q;0) + C\frac{\nu^2}{\epsilon}\|\mathbf{U}^t\cdot\mathbf{n}\|^2_\Gamma \tag{4.11}$$

and

$$\nu^2\|\nabla\mathbf{u}^t\|^2_S \le C\mathcal{F}_S(\mathbf{U},\mathbf{u},p) + C\mathcal{G}_D(\mathbf{w},q;0) + 5\frac{\nu^2}{\epsilon}\|\mathbf{U}^t\cdot\mathbf{n}\|^2_\Gamma. \tag{4.12}$$

Applying of (4.10), (4.11), and (4.12), we obtain

$$\begin{aligned}
C\big(\nu^2\|\mathbf{U}\|^2_S - \frac{\nu^2}{\epsilon}\|\mathbf{U}^t\cdot\mathbf{n}\|^2_\Gamma + \nu^2\|\mathbf{u}\|^2_{1,S} + \|p\|^2_S\big)&\\
\le \mathcal{F}_S(\mathbf{U},\mathbf{u},p) + \mathcal{G}_D(\mathbf{w},q;0).&
\end{aligned}$$

Since the H^{-1}-norm of a function is bounded by its L^2-norm and $\mathbf{W}_1 \subset \mathbf{W}$, then $\mathcal{F}_S(\mathbf{U},\mathbf{u},p)+\mathcal{G}_D(\mathbf{w},q;0) \le \mathcal{G}_S(\mathbf{U},\mathbf{u},p;0)+\mathcal{G}_D(\mathbf{w},q;0)$ on $\mathbf{W}$. Hence we have

$$C\big(\nu^2\|\mathbf{U}\|_S^2 - \frac{\nu^2}{\epsilon}\|\mathbf{U}^t\cdot\mathbf{n}\|_\Gamma^2 + \nu^2\|\mathbf{u}\|_{1,S}^2 + \|p\|_S^2\big) \le \mathcal{G}_S(\mathbf{U},\mathbf{u},p;0)+\mathcal{G}_D(\mathbf{w},q;0). \tag{4.13}$$

By Lemma 4.1, we have

$$\nu^2\|(\nabla\cdot\mathbf{U})^t\|_S^2 + \|\nabla p\|_S^2 \le C\mathcal{G}_S(\mathbf{U},\mathbf{u},p;0). \tag{4.14}$$

Hence by using Lemma 2.4, the Poincare inequality, (4.13), and (4.14) we get

$$C\big(\nu^2\|\mathbf{U}\|_{1,S}^2 - \frac{\nu^2}{\epsilon}\|\mathbf{U}^t\cdot\mathbf{n}\|_\Gamma^2 + \nu^2\|\mathbf{u}\|_{1,S}^2 + \|p\|_{1,S}^2\big) \le \mathcal{G}_S(\mathbf{U},\mathbf{u},p;0)+\mathcal{G}_D(\mathbf{w},q;0). \tag{4.15}$$

For the interface condition, by using $\|h_1+h_2\|^2 \ge \alpha\|h_1\|^2 - 2\alpha\|h_2\|^2$ for $\alpha \in (0, \frac{1}{2}]$, we have

$$\begin{aligned}\mathcal{G}_I(\mathbf{U},\mathbf{u},p,\mathbf{w},q) &\ge \alpha\big(\|\nu\mathbf{n}\cdot(\mathbf{U}+\mathbf{U}^t)\cdot\mathbf{n}\|_\Gamma^2 + \|\beta\nu\mathbf{n}\times(\mathbf{U}+\mathbf{U}^t)\cdot\mathbf{n}\|_\Gamma^2 \\ &\quad + \|\mathbf{w}\cdot\mathbf{n}\|_\Gamma^2\big) - 2\alpha\big(\|\mathbf{u}\cdot\mathbf{n}\|_\Gamma^2 + \|q-p\|_\Gamma^2 + \|\mathbf{u}\times\mathbf{n}\|_\Gamma^2\big) \\ &\ge \alpha\big(\|\nu\mathbf{n}\cdot(\mathbf{U}+\mathbf{U}^t)\cdot\mathbf{n}\|_\Gamma^2 + \|\beta\nu\mathbf{n}\times(\mathbf{U}+\mathbf{U}^t)\cdot\mathbf{n}\|_\Gamma^2 \\ &\quad + \|\mathbf{w}\cdot\mathbf{n}\|_\Gamma^2\big) - 2\alpha\big(\|\mathbf{u}\cdot\mathbf{n}\|_\Gamma^2 + \|q\|_\Gamma^2 + \|p\|_\Gamma^2 + \|\mathbf{u}\times\mathbf{n}\|_\Gamma^2\big) \\ &\ge \alpha\big(\|\mathbf{w}\cdot\mathbf{n}\|_\Gamma^2 + 4\nu^2\min\{1,\beta^2\}\|\mathbf{U}\cdot\mathbf{n}\|_\Gamma^2\big) \\ &\quad - 2\alpha\big(\|\mathbf{u}\|_\Gamma^2 + \|q\|_\Gamma^2 + \|p\|_\Gamma^2\big) \\ &\ge \alpha\big(\|\mathbf{w}\cdot\mathbf{n}\|_\Gamma^2 + 4\nu^2\min\{1,\beta^2\}\|\mathbf{U}\cdot\mathbf{n}\|_\Gamma^2\big) \\ &\quad - 2\alpha C_T\|\mathbf{u}\|_{1,S}^2 - C_T\|q\|_{1,D}^2 - C_T\|p\|_{1,S}^2.\end{aligned} \tag{4.16}$$

Hence, there exists a constant C such that

$$\|\mathbf{w}\cdot\mathbf{n}\|_\Gamma^2 + \nu^2\min\{1,\beta^2\}\|\mathbf{U}\cdot\mathbf{n}\|_\Gamma^2 \le C\big(\mathcal{G}_I(\mathcal{U}) + \|\mathbf{u}\|_{1,S}^2 + \|q\|_{1,D}^2 + \|p\|_{1,S}^2\big).$$

Application of the inequalities (4.4) and (4.15) gives

$$\|\mathbf{w}\cdot\mathbf{n}\|_\Gamma^2 + 4\nu^2\min\{1,\beta^2\}\|\mathbf{U}\cdot\mathbf{n}\|_\Gamma^2 \le C\mathcal{G}(\mathcal{U};0,0) + C\frac{\nu^2}{\epsilon}\|\mathbf{U}\cdot\mathbf{n}\|_\Gamma^2.$$

Therefore

$$\|\mathbf{w}\cdot\mathbf{n}\|_\Gamma^2 + \nu^2\|\mathbf{U}\cdot\mathbf{n}\|_\Gamma^2 \le C\mathcal{G}(\mathbf{U},\mathbf{u},p,\mathbf{w},q;0,0), \tag{4.17}$$

provided that $\epsilon > \frac{C}{4\min\{1,\beta^2\}}$. The coercivity (4.3) is now a consequence of (4.4), (4.15) and (4.17). □

We now wish to prove the continuity and coercivity of the least squares functionals for the Chebyshev approximation. First we assume that an a priori estimate (2.14) holds for the Stokes equation in Ω_S, that is

$$\|\nu\mathbf{u}\|_{1,\omega,S} + \|p\|_{\omega,S} \le C\big(\|-\nu\Delta\mathbf{u} + \nabla p\|_{-1,\omega,S} + \|\nu\nabla\cdot\mathbf{u}\|_{\omega,S}\big). \tag{4.18}$$

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We note that the elimination of $\mathbf{w}$ in (1.2) gives the Poisson equation $-K\Delta q = g$ with the Neumann boundary condition $\frac{\partial q}{\partial \mathbf{n}} = 0$ on $\partial\Omega_D$, and we also assume that an a priori estimate (2.15) holds for Darcy equation in Ω_D, that is,

$$\|q\|_{1,w,D} \leq C\| - \sqrt{K}\Delta q\|_{-1,w,D}. \tag{4.19}$$

THEOREM 4.3. *Assume that the inequalities (4.18) and (4.19) hold. Then there exists a positive constant C such that*

$$\tfrac{1}{C}\|\mathcal{U}\|_w^2 \leq \mathcal{G}_w(\mathcal{U};\mathbf{0},0) \leq C\|\mathcal{U}\|_w^2, \qquad \forall \mathcal{U} \in \mathbf{W}_w. \tag{4.20}$$

Proof. The upper bound of (4.20) is a consequence of the triangle inequality. To prove the lower bound of (4.20), (4.18) and Lemma 2.1 yield

$$\begin{aligned} \|\nu\mathbf{u}\|_{1,w,S}^2 + \|p\|_{w,S}^2 &\leq C(\| - \nu\Delta\mathbf{u} + \nabla p\|_{-1,w,S}^2 + \|\nu\nabla\cdot\mathbf{u}\|_{w,S}^2) \\ &= C\big(\| - \nabla\cdot(\mathbf{U} - \nabla\mathbf{u}^t)\|_{-1,w,S}^2 + \|\nu(\nabla\cdot\mathbf{U})^t \\ &\qquad + \nabla p\|_{-1,w,S}^2 + \|\nu\nabla\cdot\mathbf{u}\|_{w,S}^2\big) \\ &\leq C\big(\|\mathbf{U} - \nabla\mathbf{u}^t\|_{w,S}^2 + \|\nu(\nabla\cdot\mathbf{U})^t + \nabla p\|_{w,S}^2 \\ &\qquad + \|\nu\nabla\cdot\mathbf{u}\|_{w,S}^2\big) \\ &\leq C\mathcal{G}_{w,S}(\mathbf{U},\mathbf{u},p;\mathbf{0}). \end{aligned} \tag{4.21}$$

Using the triangle inequality and the above one we obtain

$$\nu^2\|\mathbf{U}\|_{w,S}^2 \leq 2\big(\|\mathbf{U} - \nabla\mathbf{u}^t\|_{w,S}^2 + \nu^2\|\nabla\mathbf{u}^t\|_{w,S}^2\big) \leq 2\mathcal{G}_{w,S}(\mathbf{U},\mathbf{u},p;\mathbf{0}). \tag{4.22}$$

By Lemma 4.1, we have

$$\nu^2\|(\nabla\cdot\mathbf{U})^t\|_{w,S}^2 + \|\nabla p\|_{w,S}^2 \leq C\mathcal{G}_{w,S}(\mathbf{U},\mathbf{u},p;0). \tag{4.23}$$

We obviously have

$$\|\nabla\times\mathbf{U}\|_{w,S}^2 \leq \mathcal{G}_{w,S}(\mathbf{U},\mathbf{u},p;0). \tag{4.24}$$

Hence, by (4.21), (4.22), (4.23), and (4.24) we have

$$\|\nu\mathbf{u}\|_{1,w,S}^2 + \nu^2\|\mathbf{U}\|_{\mathbf{V}_w}^2 + \|p\|_{1,w,S}^2 \leq C\mathcal{G}_{w,S}(\mathbf{U},\mathbf{u},p;0). \tag{4.25}$$

For the Darcy domain, by using (4.19), and Lemma 2.1, we have

$$\begin{aligned} \|q\|_{1,w,D}^2 &\leq C\| - \sqrt{K}\Delta q\|_{-1,w,D}^2 \\ &\leq C\| - \nabla\cdot(\sqrt{K}\nabla q + \frac{1}{\sqrt{K}}\mathbf{w})\|_{-1,w,D}^2 + C\|\frac{1}{\sqrt{K}}\nabla\cdot\mathbf{w}\|_{-1,w,D}^2 \\ &\leq C\|\sqrt{K}\nabla q + \frac{1}{\sqrt{K}}\mathbf{w}\|_{w,D}^2 + C\|\frac{1}{\sqrt{K}}\nabla\cdot\mathbf{w}\|_{w,D}^2 \\ &\leq C\mathcal{G}_{w,D}(\mathbf{w},q;0). \end{aligned} \tag{4.26}$$

The triangle inequality gives

$$\|\mathbf{w}\|_{w,D}^2 \leq C\|\frac{1}{\sqrt{K}}\mathbf{w} + \sqrt{K}\nabla q\|_{w,D}^2 + C\|q\|_{1,w,D}^2 \leq C\mathcal{G}_{w,D}(\mathbf{w},q;0).$$

Then, by using $\|\nabla \cdot \mathbf{w}\|_w \leq \mathcal{G}_{w,D}(\mathbf{w}, q; 0)$, we get

$$\|\mathbf{w}\|^2_{\mathrm{div},w,D} \leq C\mathcal{G}_{w,D}(\mathbf{w}, q; 0). \tag{4.27}$$

Obviously, it holds

$$\|\nabla \times \mathbf{w}\|^2_{w,D} \leq C\mathcal{G}_{w,D}(\mathbf{w}, q; 0). \tag{4.28}$$

Hence by (4.26), (4.27), and (4.28) we have

$$\|q\|^2_{1,w,D} + \|\mathbf{w}\|^2_{\mathrm{div},w,D} + \|\nabla \times \mathbf{w}\|^2_{w,D} \leq C\mathcal{G}_{w,D}(\mathbf{w}, q; 0). \tag{4.29}$$

Similarly to (4.16), there exists a constant C such that

$$\|\mathbf{w} \cdot \mathbf{n}\|^2_{w,\Gamma} + \|\mathbf{U} \cdot \mathbf{n}\|^2_{w,\Gamma} \leq C\mathcal{G}_w(\mathbf{U}, \mathbf{u}, p, \mathbf{w}, q; 0). \tag{4.30}$$

The lower bound of (4.20) follows from the inequalities (4.25), (4.29), and (4.30). □

We now define the discrete Legendre/Chebyshev pseudo-spectral least squares method for the Stokes/Darcy equations. First, define

$$V_{w,S,N} = V_{w,S} \cap \mathcal{Q}^2_N, \quad V_{w,N} = V_w \cap \mathcal{Q}^4_N, \quad V_{w,D,N} = V_{w,D} \cap \mathcal{Q}^2_N,$$

and

$$\mathbf{W}_{w,N} = V_{w,N} \times V_{w,S,N} \times [H^1_w(\Omega_S) \cap L^2_{w,0}(\Omega_S) \cap \mathcal{Q}_N] \times V_{w,D,N} \times [H^1_w(\Omega_D) \cap \mathcal{Q}_N].$$

Let us define the discrete Legendre/Chebyshev least squares functional as

$$\mathcal{G}_{w,N}(\mathcal{U}; \mathbf{f}, g) = \mathcal{G}_{w,S,N}(\mathbf{U}, \mathbf{u}, p; \mathbf{f}) + \mathcal{G}_{w,D,N}(\mathbf{w}, q; g) + \mathcal{G}_{w,I,N}(\mathcal{U}), \tag{4.31}$$

where

$$\begin{aligned}
\mathcal{G}_{w,S,N}(\mathbf{U}, \mathbf{u}, p; \mathbf{f}) &= \nu^2\|\mathbf{U} - \nabla \mathbf{u}^t\|^2_{w,S,N} + \|\mathbf{f} + \nu(\nabla \cdot \mathbf{U})^t - \nabla p\|^2_{w,S,N} \\
&\quad + \nu^2\|\nabla \cdot \mathbf{u}\|^2_{w,S,N} + \nu^2\|\nabla(\mathrm{tr}\,\mathbf{U})\|^2_{w,S,N} + \nu^2\|\nabla \times \mathbf{U}\|^2_{w,S,N}, \\
\mathcal{G}_{w,D,N}(\mathbf{w}, q; g) &= \|K^{-1/2}\mathbf{w} + K^{1/2}\nabla q\|^2_{w,D,N} + \|\nabla \cdot \mathbf{w} - g\|^2_{w,D,N} \\
&\quad + \|\nabla \times K^{-1}\mathbf{w}\|^2_{w,D,N}, \\
\mathcal{G}_{w,I,N}(\mathcal{U}) &= \|\mathbf{u} \cdot \mathbf{n} - \mathbf{w} \cdot \mathbf{n}\|^2_{w,\Gamma,N} + \|\nu\mathbf{n} \cdot (\mathbf{U} + \mathbf{U}^t) \cdot \mathbf{n} + q - p\|^2_{w,\Gamma,N} \\
&\quad + \|\beta\nu\mathbf{n} \times (\mathbf{U} + \mathbf{U}^t) \cdot \mathbf{n} + \mathbf{u} \times \mathbf{n}\|^2_{w,\Gamma,N}
\end{aligned}$$

for every $\mathcal{U} \in \mathbf{W}_{w,N}$. The first order system least squares variational problem for (4.31) consists of minimizing the quadratic function $\mathcal{G}_{w,N}(\mathcal{U}; \mathbf{f}, g)$ over $\mathbf{W}_{w,N}$, that is, find $\mathcal{U} \in \mathbf{W}_{w,N}$ such that

$$\mathcal{G}_{w,N}(\mathcal{U}; \mathbf{f}, g) = \inf_{\mathcal{V} \in \mathbf{W}_{w,N}} \mathcal{G}_{w,N}(\mathcal{V}; \mathbf{f}, g).$$

The corresponding variational problem is to find $\mathcal{U}_N \in \mathbf{W}_{w,N}$ such that

$$\mathcal{A}_{w,N}(\mathcal{U}_N; \mathcal{V}) = \mathcal{F}_{w,N}(\mathcal{V}), \qquad \forall\, \mathcal{V} \in \mathbf{W}_{w,N}, \tag{4.32}$$

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where

$$\begin{aligned}\mathcal{A}_{w,N}(\mathcal{U};\mathcal{V}) &= \big\langle \nu(\nabla\cdot\mathbf{U})^t - \nabla p, \nu(\nabla\cdot\mathbf{S})^t - \nabla s\big\rangle_{w,S,N} + \nu^2\big\langle \nabla\cdot\mathbf{u}, \nabla\cdot\mathbf{v}\big\rangle_{w,S,N} \\ &+ \nu^2\big\langle \mathbf{U} - \nabla\mathbf{u}^t,\, \mathbf{S} - \nabla\mathbf{v}^t\big\rangle_{w,S,N} + \nu^2\big\langle \nabla(\operatorname{tr}\mathbf{U}),\, \nabla(\operatorname{tr}\mathbf{S})\big\rangle_{w,S,N} \\ &+ \nu^2\big\langle \nabla\times\mathbf{U},\, \nabla\times\mathbf{S}\big\rangle_{w,S,N} + \big\langle \nabla\cdot\mathbf{w}, \nabla\cdot\mathbf{z}\big\rangle_{w,D,N} \\ &+ \big\langle \frac{1}{\sqrt{K}}\mathbf{w} + \sqrt{K}\nabla q,\, \frac{1}{\sqrt{K}}\mathbf{z} + \sqrt{K}\nabla r\big\rangle_{w,D,N} \\ &+ \big\langle \nabla\times K^{-1}\mathbf{w}, \nabla\times K^{-1}\mathbf{z}\big\rangle_{w,D,N} + \big\langle \mathbf{u}\cdot\mathbf{n} - \mathbf{w}\cdot\mathbf{n}, \mathbf{v}\cdot\mathbf{n} - \mathbf{z}\cdot\mathbf{n}\big\rangle_{w,\Gamma,N} \\ &+ \big\langle \nu\mathbf{n}\cdot(\mathbf{U}+\mathbf{U}^t)\cdot\mathbf{n} + q - p,\, \nu\mathbf{n}\cdot(\mathbf{S}+\mathbf{S}^t)\cdot\mathbf{n} + r - s\big\rangle_{w,\Gamma,N} \\ &+ \big\langle \beta\nu\mathbf{n}\times(\mathbf{U}+\mathbf{U}^t)\cdot\mathbf{n} + \mathbf{u}\times\mathbf{n},\, \beta\nu\mathbf{n}\times(\mathbf{S}+\mathbf{S}^t)\cdot\mathbf{n} + \mathbf{v}\times\mathbf{n}\big\rangle_{w,\Gamma,N}\end{aligned}$$

and

$$\mathcal{F}_{w,N}(\mathcal{V}) = \big\langle g,\, \nabla\cdot\mathbf{z}\big\rangle_{w,D,N} - \big\langle \mathbf{f}, \nu(\nabla\cdot\mathbf{S})^t - \nabla s\big\rangle_{w,S,N}.$$

The continuity and coercivity of $\mathcal{G}_{w,N}(\mathcal{U};\mathbf{f},g)$ is a consequence of that of $\mathcal{G}_w$. which is stated as follows.

THEOREM 4.4. *There exists a positive constant C such that*

$$\frac{1}{C}|\!|\!|\mathcal{U}|\!|\!|_w^2 \le \mathcal{G}_{w,N}(\mathcal{U};\mathbf{0},0) \le C|\!|\!|\mathcal{U}|\!|\!|_w^2, \quad \forall\mathcal{U}\in\mathbf{W}_{w,N}. \tag{4.33}$$

Proof. Since $\nu(\nabla\cdot\mathbf{U})^t - \nabla p \in \mathcal{Q}_{N-1}^2$, $\nabla\cdot\mathbf{u}, \nabla\cdot\mathbf{w} \in \mathcal{Q}_{N-1}$, $\mathbf{U} - \nabla\mathbf{u}^t \in \mathcal{Q}_N^4$, $\frac{1}{\sqrt{K}}\mathbf{w} + \sqrt{K}\nabla q \in \mathcal{Q}_N^2$, and $\mathbf{u}\cdot\mathbf{n} - \mathbf{w}\cdot\mathbf{n}$, $\nu\mathbf{n}\cdot(\mathbf{U}+\mathbf{U}^t)\cdot\mathbf{n} + q - p$, $\beta\nu\mathbf{n}\times(\mathbf{U}+\mathbf{U}^t)\cdot\mathbf{n} + \mathbf{u}\times\mathbf{n}$ are in $\mathcal{P}_N$. Hence, by applying (2.7) and (2.4), there is a constant C such that

$$\frac{1}{C}\,\mathcal{G}_w(\mathcal{U};\mathbf{0},0) \le \mathcal{G}_{w,N}(\mathcal{U};\mathbf{0},0) \le C\,\mathcal{G}_w(\mathcal{U};\mathbf{0},0).$$

Therefore the bounds on (4.33) are a consequence of Theorems 4.2 and 4.3 for the Legendre and Chebyshev approximations, respectively. □

Above we have proved the continuity and coercivity of the least squares functional. This proves, consequently, the existence and uniqueness of the solution of the Stokes-Darcy equation by the Lax-Milgram lemma. We are now going to prove spectral convergence of the proposed Legendre/Chebyshev pseudospectral method.

THEOREM 4.5. *Assume that the inequalities* (4.18) *and* (4.19) *hold for the Chebyshev approximation. Suppose that the solution $\mathcal{U}$ of* (4.2) *is in $\mathbf{W}_w \cap \big[H_w^s(\Omega_S)^4 \times H_w^s(\Omega_S)^2 \times H_w^s(\Omega_S) \times H_w^s(\Omega_D)^2 \times H_w^s(\Omega_D)\big]$ for some $s \ge 1$, and $g \in H_w^k(\Omega_D)$, $\mathbf{f} \in [H_w^\ell(\Omega_S)]^2$, for $\ell, k \ge 2$. Let $\mathcal{U}_N \in \mathbf{W}_{w,N}$ be the approximate solution of* (4.32). *Then there exists a constant C such that*

$$\begin{aligned}|\!|\!|\mathcal{U} - \mathcal{U}_N|\!|\!|_w \le C\Big(& N^{1-s}\big[\|\mathbf{U}\|_{s,w,S} + \|\mathbf{u}\|_{s,w,S} + \|p\|_{s,w,S} + \|\mathbf{w}\|_{s,w,D} \\ &+ \|q\|_{s,w,D}\big] + N^{-\ell}\|\mathbf{f}\|_{\ell,w,S} + N^{-k}\|g\|_{k,w,D}\Big).\end{aligned} \tag{4.34}$$

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Proof. Using Strang's Lemma [2], we get

$$\begin{aligned}\|\mathcal{U}-\mathcal{U}_N\|_w \le \inf_{\mathcal{V}\in\mathbf{W}_{w,N}} \left[\|\mathcal{U}-\mathcal{V}\|_w + \sup_{\mathcal{W}\in\mathbf{W}_{w,N}} \frac{|\mathcal{A}_w(\mathcal{V},\mathcal{W})-\mathcal{A}_{w,N}(\mathcal{V},\mathcal{W})|}{\|\mathcal{W}\|_w}\right] \\ + \sup_{\mathcal{W}\in\mathbf{W}_{w,N}} \frac{|\mathcal{F}_w(\mathcal{W})-\mathcal{F}_{w,N}(\mathcal{W})|}{\|\mathcal{W}\|_w}.\end{aligned} \tag{4.35}$$

The inequality (2.9) yields

$$\begin{aligned}|\mathcal{F}_w(\mathcal{W})-\mathcal{F}_{w,N}(\mathcal{W})| &= \big|(g,\,\nabla\cdot\mathbf{z})_{w,D} - (\mathbf{f},\nu(\nabla\cdot\mathbf{S})^t-\nabla s)_{w,S} \\ &\quad - \langle g,\,\nabla\cdot\mathbf{z}\rangle_{w,D,N} - \langle\mathbf{f},\nu(\nabla\cdot\mathbf{S})^t-\nabla s\rangle_{w,S,N}\big| \\ &\le CN^{-\ell}\|\mathbf{f}\|_{\ell,w,S}\|\nu(\nabla\cdot\mathbf{S})^t-\nabla s\|_{w,S} \\ &\quad + CN^{-k}\|g\|_{k,w,D}\|\nabla\cdot\mathbf{z}\|_{w,D} \\ &\le C(N^{-\ell}\|\mathbf{f}\|_{\ell,w,S}+N^{-k}\|g\|_{k,w,D})\|\mathcal{W}\|_w.\end{aligned} \tag{4.36}$$

If we take $\mathcal{V}\in\mathbf{W}_{w,N-1}$, then by using (2.3) for the interior inner products and (2.6) for the interface inner products, we have

$$\big|\mathcal{A}_w(\mathcal{V},\mathcal{W})-\mathcal{A}_{w,N}(\mathcal{V},\mathcal{W})\big| = 0, \qquad \mathcal{W}\in\mathbf{W}_{w,N}. \tag{4.37}$$

Hence, by (4.36) and (4.37), the inequality (4.35) becomes

$$\|\mathcal{U}-\mathcal{U}_N\|_w \le \inf_{\mathcal{V}\in\mathbf{W}_N}\|\mathcal{U}-\mathcal{V}\|_w + C(N^{-\ell}\|\mathbf{f}\|_{\ell,S}+N^{-k}\|g\|_{k,D}).$$

Applying (2.2) and (2.8), to the above inequality, results in (4.34) and the proof is complete. □

5. Implementation and numerical tests. This section provides a brief implementation of pseudospectral approximation (for details see [18, 19, 27]) and some numerical tests in order to confirm the spectral convergence of the presented least squares pseudospectral method for the Stokes-Darcy equations. We note that the numerical examples satisfy the non-homogeneous boundary condition (1.3). Let D_N be the one dimensional pseudo-spectral derivative matrix associated to the $N+1$ values of $\{\partial_N v(\xi_j)\}_{j=0}^N$ at LGL or CGL points [6]. The entries of D_N can be computed by differentiating the Lagrange polynomials ϕ_j. The LGL and CGL points are reordered from bottom to top, and then from left to right, such that $\mathbf{x}_{k(N+1)+l} := \mathbf{x}_{kl} = (\xi_k,\,\xi_l)$ for $k,l=1,\ldots,N$. The pseudo-spectral derivative matrix in the 2-dimensional space is defined via the Kronecker tensor product, that is,

$$D_x = D_N\otimes I_N \quad \text{and} \quad D_y = I_N\otimes D_N,$$

where I_N is identity matrix of the same order as D_N. For the continuous function r we use $\hat{\mathbf{r}}$ to denote the vector containing the nodal values, i.e.,

$$\hat{\mathbf{r}} = (r(x_0),\ldots,r(x_{(N+1)^2-1}))^T.$$

By the definition of discrete scalar inner product, we have

$$\langle v,z\rangle_{w,N} = \hat{\mathbf{z}}^T W\hat{\mathbf{v}} \quad \text{and} \quad \langle\partial_{t_1}v,\partial_{t_2}z\rangle_{w,N} = (D_{t_2}\hat{\mathbf{z}})^T W(D_{t_1}\hat{\mathbf{v}}),$$

where t_1 and t_2 are x or y, and $W=\operatorname{diag}\{w_i\}$ is the diagonal weight matrix. Then, the problem (4.32) can be assembled.

REMARK 5.1. In the implementation of the discrete least squares method, we use LGL or CGL points, which are defined in the interval $[-1,\,1]$. If the Stokes-Darcy equation is defined

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TABLE 5.1
L^2_w-discretization errors for Example 5.2.

N	$\|E_{\mathbf{U}}\|_w$	$\|E_{\mathbf{u}}\|_w$	$\|E_p\|_w$	$\|E_{\mathbf{w}}\|_w$	$\|E_q\|_w$
			Legendre approximation		
4	4.1080e-03	8.5709e-04	4.7171e-03	1.5724e-03	2.7597e-04
6	1.2657e-04	1.8891e-05	1.6569e-04	7.0949e-06	1.4258e-06
8	2.0361e-06	3.9323e-07	2.7640e-06	1.7992e-07	6.3527e-08
10	4.2830e-08	8.6139e-09	6.0595e-08	4.2996e-09	1.5640e-09
12	2.1704e-11	4.3976e-12	3.1159e-11	2.2377e-12	7.9158e-13
14	5.4480e-13	1.0967e-13	7.5334e-13	5.3601e-14	8.7046e-14
			Chebyshev approximation		
4	9.0684e-03	2.8718e-03	1.2252e-02	4.1880e-03	2.1751e-03
6	6.0338e-03	8.9482e-04	6.7867e-03	6.0708e-04	2.1063e-04
8	4.3721e-06	6.4370e-07	5.5952e-06	4.5264e-07	1.7633e-07
10	2.2318e-09	3.2886e-10	2.9373e-09	2.3635e-10	9.7432e-11
12	3.2112e-12	4.8785e-13	4.4752e-12	3.4076e-13	5.0783e-13
14	2.4990e-12	3.8468e-13	3.3192e-12	2.7540e-13	8.7372e-13

in a rectangular domain with a straight line interface, then transformation (2.5) can be used and, in the case of curved boundary and curve interface, the Gordon-Hall transformation [12, 13] can be used to transform the domain and equations into a rectangular domain. For a complete explanation and examples of Gordon-Hall transformation, see [18]. It is noteworthy that a major advantage of using the Gordon-Hall map and a pseudo-spectral approximation is that the collocation points always lie on the interface and two neighboring domains Ω_S and Ω_D share the same nodes on the interface, regardless of the interface shape. Owing to this property, the error discretization in our method does not include the mismatch parameter introduced in [22].

In the Examples 5.2–5.4, we take $\Omega = (0,1) \times (0,2)$ with $\Omega_D = (0,1) \times (0,1)$, $\Omega_S = (0,1) \times (1,2)$, and $\Gamma = (0,1) \times \{1\}$. The results are given for $K = 1,\ \beta = 1$ and $\nu = 1$. The functions $\mathbf{U}$ and $\mathbf{w}$ can be computed by the definition of $\mathbf{U}$ and equation (1.2), respectively. Let $(\mathbf{U}_N,\, \mathbf{u}_N,\, p_N,\, \mathbf{w}_N, q_N)$ be the approximate solution of the Stokes-Darcy equations by the Legendre or Chebyshev least squares method, and let $E_v = v - v_N$, for $v \in \{\mathbf{U},\, \mathbf{u},\, p,\, \mathbf{w},\, q\}$.

EXAMPLE 5.2. Let

$$\begin{cases} \mathbf{u}_1 = -\cos(\frac{\pi}{2}y)\sin(\frac{\pi}{2}x), \\ \mathbf{u}_2 = \sin(\frac{\pi}{2}y)\cos(\frac{\pi}{2}x) - 1 + x, \\ p = \frac{1}{2} - x, \\ q = \frac{2}{\pi}\cos(\frac{\pi}{2}x)\cos(\frac{\pi}{2}y) - y(x-1), \end{cases}$$

be the exact solution of the Stokes-Darcy equations. The discretization errors for the Legendre and Chebyshev approximations are given in Tables 5.1 and 5.2, which show that the spectral errors decay exponentially with respect to N.

EXAMPLE 5.3. We consider the following velocity and pressure:

$$\begin{cases} \mathbf{u}_1 = \exp(x+y) + y, \\ \mathbf{u}_2 = -\exp(x+y) - x, \\ p = \cos(\pi x)\cos(\pi y) + x - \frac{1}{2}, \\ q = \exp(x+y) - \cos(\pi x) + yx, \end{cases}$$

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TABLE 5.2
H^1_w-*discretization errors for Example 5.2.*

N	$\|E_{\mathbf{U}}\|_{1,w}$	$\|E_{\mathbf{u}}\|_{1,w}$	$\|E_p\|_{1,w}$	$\|E_{\mathbf{w}}\|_{1,w}$	$\|E_q\|_{1,w}$
Legendre approximation					
4	3.2723e-02	5.3085e-03	2.7545e-02	1.0885e-02	1.6032e-03
6	8.2747e-04	8.9677e-05	7.6937e-04	7.0137e-05	9.6244e-06
8	1.2690e-05	1.6098e-06	1.2373e-05	6.8969e-07	1.9330e-07
10	2.6403e-07	3.4711e-08	2.6057e-07	1.5299e-08	4.5983e-09
12	1.3406e-10	1.7682e-11	1.3257e-10	7.9145e-12	2.3791e-12
14	3.3360e-12	4.4185e-13	3.3019e-12	1.9394e-13	1.1203e-13
Chebyshev approximation					
4	6.0344e-02	1.9460e-02	4.5903e-02	2.4181e-02	7.4832e-03
6	4.8013e-02	4.9008e-03	3.9871e-02	3.2964e-03	6.8104e-04
8	4.5170e-05	3.5574e-06	3.9833e-05	2.5476e-06	5.0788e-07
10	2.6740e-08	1.8907e-09	2.3863e-08	1.7968e-09	4.2169e-10
12	3.4081e-11	2.7486e-12	3.0534e-11	2.5228e-12	1.0089e-12
14	2.2993e-11	2.1063e-12	2.0430e-11	1.5314e-12	1.2557e-12

TABLE 5.3
L^2_w-*discretization errors for Example 5.3.*

N	$\|E_{\mathbf{U}}\|_{w}$	$\|E_{\mathbf{u}}\|_{w}$	$\|E_p\|_{w}$	$\|E_{\mathbf{w}}\|_{w}$	$\|E_q\|_{w}$
Legendre approximation					
4	2.6404e-02	2.2206e-03	3.1493e-02	1.3263e-02	1.4158e-02
6	1.9205e-04	2.7308e-05	2.9518e-04	7.4952e-05	1.9430e-04
8	5.8800e-06	1.1377e-06	8.1192e-06	6.6253e-07	1.6026e-06
10	2.1014e-07	4.2258e-08	2.9737e-07	2.1158e-08	1.1566e-08
12	2.3002e-10	4.6610e-11	3.3174e-10	2.4346e-11	3.4577e-11
14	3.6359e-12	7.2209e-13	5.2272e-12	3.4598e-13	2.3270e-13
Chebyshev approximation					
4	9.2662e-02	8.3589e-03	1.9296e-01	1.6701e-01	1.2027e-01
6	1.7560e-02	2.5955e-03	1.9743e-02	2.7258e-03	1.3267e-03
8	1.0442e-05	1.5550e-06	1.3321e-05	1.7008e-05	6.2780e-06
10	6.4516e-09	5.8487e-10	2.9639e-08	9.5675e-08	3.4423e-08
12	2.3514e-11	1.3861e-12	1.1828e-10	3.8769e-10	1.3310e-10
14	1.8150e-12	2.5433e-13	2.0914e-12	1.1649e-12	7.0745e-13

as an exact solution of the Stokes-Darcy equations. The discretization errors for Example 5.3, corresponding to both Legendre and Chebyshev approximations, are displayed in Tables 5.3 and 5.4, which show spectral convergence of the errors.

EXAMPLE 5.4. In this example, let the exact solutions of the Stokes-Darcy equation be

$$\begin{cases} \mathbf{u}_1 = -\cos(\pi x)\sin(\pi y), \\ \mathbf{u}_2 = \sin(\pi x)\cos(\pi y), \\ p = \sin(\pi x) - \frac{2}{\pi}, \\ q = y\sin(\pi x). \end{cases}$$

The discretization errors for Example 5.4, for Legendre and Chebyshev approximations, are presented in Tables 5.5 and 5.6, which show the spectral convergence of the proposed method.

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TABLE 5.4
H^1_w-discretization errors for Example 5.3.

N	$\|E_{\mathbf{U}}\|_{1,w}$	$\|E_{\mathbf{u}}\|_{1,w}$	$\|E_p\|_{1,w}$	$\|E_{\mathbf{w}}\|_{1,w}$	$\|E_q\|_{1,w}$
Legendre approximation					
4	1.3215e-01	1.7397e-02	1.3532e-01	4.5072e-02	8.5626e-02
6	1.3939e-03	1.3215e-04	2.1471e-03	7.0086e-04	2.0271e-03
8	3.6775e-05	4.6533e-06	4.2075e-05	6.5112e-06	2.3921e-05
10	1.2957e-06	1.7029e-07	1.2889e-06	8.3257e-08	1.7150e-07
12	1.4231e-09	1.8742e-10	1.6128e-09	1.6877e-10	8.1357e-10
14	2.2358e-11	2.9318e-12	2.2384e-11	1.3511e-12	2.8244e-12
Chebyshev approximation					
4	5.9432e-01	6.7927e-02	5.6767e-01	9.3997e-01	2.6007e-01
6	1.3963e-01	1.4179e-02	1.1610e-01	3.3371e-02	5.3978e-03
8	1.0941e-04	8.7607e-06	1.0793e-04	4.6644e-04	5.4721e-05
10	7.6404e-08	4.9536e-09	3.7317e-07	3.8397e-06	3.7447e-07
12	2.4672e-10	1.6459e-11	1.7773e-09	2.0613e-08	1.7421e-09
14	1.5384e-11	1.4745e-12	1.4958e-11	7.8129e-11	5.9324e-12

TABLE 5.5
L^2_w-discretization errors for Example 5.4.

N	$\|E_{\mathbf{U}}\|_{w}$	$\|E_{\mathbf{u}}\|_{w}$	$\|E_p\|_{w}$	$\|E_{\mathbf{w}}\|_{w}$	$\|E_q\|_{w}$
Legendre approximation					
4	2.0737e-01	4.0619e-02	2.5063e-01	5.1443e-02	1.2925e-02
6	3.7222e-03	7.7173e-04	5.1587e-03	7.6340e-04	1.8120e-04
8	1.4605e-04	2.9422e-05	2.0442e-04	1.5607e-05	5.4847e-06
10	8.1525e-06	1.6335e-06	1.1492e-05	8.0811e-07	2.9278e-07
12	1.0081e-08	2.0173e-09	1.4291e-08	9.9919e-10	3.6071e-10
14	3.1499e-11	6.3004e-12	4.4846e-11	3.1132e-12	1.1138e-12
16	4.3518e-12	8.7959e-13	6.2548e-12	4.4551e-13	1.2082e-13
Chebyshev approximation					
4	1.4916e-01	3.4011e-02	6.5356e-02	7.8975e-02	2.3771e-02
6	1.8656e-02	2.9435e-03	2.1295e-02	2.2268e-03	7.7432e-04
8	6.2222e-05	1.0750e-05	7.3509e-05	1.1832e-05	3.1119e-06
10	5.2815e-07	8.4915e-08	6.6412e-07	7.8161e-08	2.3058e-08
12	3.5054e-09	5.4157e-10	4.5602e-09	4.2674e-10	1.4341e-10
14	1.8411e-11	2.7520e-12	2.4454e-11	1.9514e-12	6.3886e-13
16	5.0970e-13	4.2134e-14	8.1079e-13	1.6911e-14	8.1472e-15

EXAMPLE 5.5. This example is taken from a real engineering application on a domain of 100 by 100 meters, with material parameters $K = 8.25\text{e-}5$, $\nu = 1\text{e-}4$ and $\beta = 2\frac{\sqrt{K}}{\nu}$ [23]. The boundary conditions used for this example are the following:

$$\begin{cases} \mathbf{w}\cdot\mathbf{n} = \frac{\pi}{2}\sin(\frac{\pi}{100}x), & \text{on } \Gamma_1, \\ \mathbf{u} = (0,-1), & \text{on } \Gamma_2, \\ \mathbf{w}\cdot\mathbf{n} = 0, & \text{on } \Gamma_3, \\ \mathbf{T}\cdot\mathbf{n} = 0, & \text{on } \Gamma_4; \end{cases}$$

the schematic domain is depicted in Figure 5.1.

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TABLE 5.6
H^1_w-discretization errors for Example 5.4.

N	$\|E_{\mathbf{U}}\|_{1,w}$	$\|E_{\mathbf{u}}\|_{1,w}$	$\|E_p\|_{1,w}$	$\|E_{\mathbf{w}}\|_{1,w}$	$\|E_q\|_{1,w}$
Legendre approximation					
4	1.3669e+00	1.5956e-01	9.8057e-01	2.8906e-01	4.0020e-02
6	2.8122e-02	3.5718e-03	2.1866e-02	7.4118e-03	5.4998e-04
8	9.1746e-04	1.2056e-04	8.8322e-04	1.0136e-04	1.6028e-05
10	5.0324e-05	6.5931e-06	4.9613e-05	2.9522e-06	8.6341e-07
12	6.2704e-08	8.1849e-09	6.1605e-08	4.7224e-09	1.0613e-09
14	1.9662e-10	2.5598e-11	1.9320e-10	1.5633e-11	3.2990e-12
16	2.7066e-11	3.5403e-12	2.6773e-11	1.5720e-12	4.5047e-13
Chebyshev approximation					
4	1.2642e+00	2.5036e-01	3.4214e-01	6.0876e-01	1.4714e-01
6	1.5470e-01	1.7923e-02	1.2343e-01	1.9292e-02	4.7021e-03
8	7.0277e-04	1.2235e-04	4.5845e-04	2.1130e-04	6.1141e-05
10	6.3597e-06	1.0156e-06	4.5517e-06	1.6659e-06	4.9462e-07
12	4.2259e-08	5.7641e-09	3.2941e-08	8.8438e-09	2.6285e-09
14	2.1936e-10	2.4510e-11	1.8144e-10	3.3905e-11	9.9318e-12
16	6.0879e-12	3.8101e-13	5.3604e-12	1.7201e-13	3.1977e-14

Since the exact solution is not known, we are not able to compute the errors as we did in previous examples. However the least squares functionals establish an effective and reliable error estimator. To demonstrate the efficiency of our least squares method we compute the least squares functional. The numerical results are given in Table 5.7, which shows the spectral convergence of the proposed method.

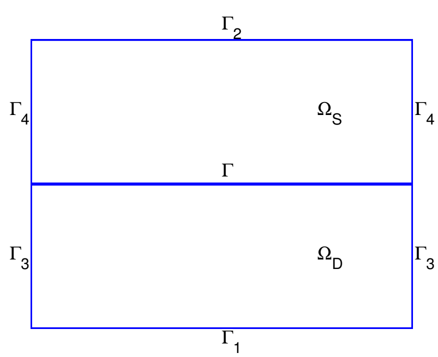

FIG. 5.1. *Schematic domain Ω for the Stokes-Darcy equation of Example 5.5.*

6. Conclusion. This paper combined least squares technique and pseudospectral method to approximate the solution of Stokes-Darcy equations. The gradient of velocity is introduced as a new independent variable in the Stokes subsystem, and the Stokes-Darcy equation is recast into a first order system of equations. The first order system is then extended by gradient and curl operator in the Stokes subsystem and curl operator in the Darcy subsystem. The least squares functional is defined as a combination of

a. the squared L^2_w-norm of residuals in a Stokes subsystem scaled by the viscosity constant ν,

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TABLE 5.7
Discretization errors for Example 5.5.

N	$\mathcal{G}_{w,S}(\mathbf{U},\mathbf{u},p;\mathbf{f})$	$\mathcal{G}_{w,D}(\mathbf{w},q;g)$	$\mathcal{G}_{w,I}(\mathbf{U},\mathbf{u},p,\mathbf{w},q)$
Legendre approximation			
4	1.2665e-01	2.0889e+00	2.7733e-03
6	1.3629e-04	1.8208e-03	4.4932e-05
8	3.3024e-08	4.5319e-07	1.2915e-07
10	4.2467e-12	2.1770e-10	5.0023e-09
12	6.3643e-15	2.2204e-14	2.3257e-12
Chebyshev approximation			
4	3.6097e-01	4.1110e+00	2.4055e-02
6	4.0788e-04	5.8283e-03	9.5122e-05
8	9.2369e-08	9.3731e-07	4.9744e-07
10	6.7283e-12	7.0756e-11	3.0396e-09
12	3.0911e-13	3.6242e-12	2.0451e-11

b. the squared L^2_w-norm of residuals in a Darcy subsystem,
c. the squared L^2_w-norm of residuals of interface conditions.

Continuous and discrete homogeneous least squares functionals for Legendre approximation is shown to be fully H^1 elliptic, and for Chebyshev approximation are shown to be equivalent to $\|\mathbf{U}\|^2_{\mathbf{V}_w,S}+\|\mathbf{U}\cdot\mathbf{n}\|^2_{w,\Gamma}+\|\mathbf{u}\|^2_{1,w,S}+\|p\|^2_{1,w,S}+\|\mathbf{w}\|^2_{w,div,D}+\|\mathbf{w}\cdot\mathbf{n}\|^2_{w,\Gamma}+\|q\|^2_{1,w,D}$, that is, the Chebyshev least squares functional is equivalent to the weighted *div-curl* product norm. The spectral convergence for both Legendre and Chebyshev pseudospectral methods are derived. To illustrate the analysis, several numerical tests are given. The proposed method can be applied to the Stokes-Darcy equations in three-dimensional space with no essential changes. It can also be applied to the Navier-Stokes-Darcy equation in two and three dimensions requiring, however, more analysis.

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