The Electronic Journal of Linear Algebra.
A publication of the International Linear Algebra Society.
Volume 4, pp. 39-47, October 1998.
ISSN 1081-3810. http://math.technion.ac.il/iic/ela

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P^{α}-MATRICES AND LYAPUNOV SCALAR STABILITY*

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Abstract. For partitions α of $\{1,\ldots,n\}$, the classes of P^{α}-matrices are defined, unifying the classes of the real P-matrices and of the real positive definite matrices. Lyapunov scalar stability of matrices is defined and characterized, and it is shown also that every real Lyapunov α-scalar stable matrix is a P^{α}-matrix. Implication relations between Lyapunov scalar stability and H-stability are discussed.

Key words. P-matrices, P^{α}-matrices, stability, scalar stability, positive definite matrices, H-stability

AMS subject classifications. 15A15, 15A18, 15A57

1. Introduction.

In this paper we relate two well known classes of matrices, that is, the class of P-matrices consisting of the matrices with all principal minors positive, and the class of positive definite matrices. While a matrix A is positive definite if and only if for every nonzero vector v in $\mathbb{C}^n$ the product v^*Av is positive, real P-matrices are known to be characterized by the fact that they do not reverse the sign of all components of a nonzero vector, that is, for every nonzero vector v in $\mathbb{R}^n$ there exists k such that $v_k(Av)_k > 0$. Motivated by this, for partitions α of $\{1,\ldots,n\}$ we define the class of P^{α}-matrices of all real matrices A such that for every nonzero vector v in $\mathbb{R}^n$ there exists k such that $v[\alpha_k]^T(Av)[\alpha_k] > 0$. Our definition thus unifies the classes of the real P-matrices and of the real positive definite matrices.

In Section 2 we discuss some properties of P^{α}-matrices. In Section 3 we define Lyapunov scalar stability of matrices and we prove that every real Lyapunov α-scalar stable matrix is a P^{α}-matrix, generalizing the known result that every real Lyapunov diagonally stable matrix is a P-matrix. We also prove two different theorems that characterize Lyapunov α-scalar stability. One of these theorems involves P^{α}-matrices. Both theorems generalize known characterizations of Lyapunov diagonal stability. In Section 4 we show that Lyapunov α-scalar stability implies $H(\alpha)$-stability, another new specific type of matrix stability which generalizes the notion of H-stability.

2. P^{α}-matrices .

DEFINITION 2.1. A real $n\times n$ matrix A is said to be *positive definite [semidefinite]* if for every nonzero vector v in $\mathbb{R}^n$ the product v^TAv is positive [nonnegative].

Note that in many references positive (semi)definite matrices are assumed to be Hermitian. This is not the case in this article.

DEFINITION 2.2. An $n \times n$ complex matrix A is said to be a P-matrix if all the principal minors of A are positive.

The following is a well known characterization of real P-matrices, e.g. [7].

*Received by the editors on 26 July 1998. Accepted for publication on 20 October 1998. Handling Editor: Volker Mehrmann.

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Proposition 2.3. *A real $n \times n$ matrix A is a P-matrix if and only if for every nonzero vector v in $\mathbb{R}^n$ there exists $k \in \{1, \dots, n\}$ such that $v_k(Av)_k > 0$.*

Motivated by Proposition 2.3 we now define a generalization of the real P-matrices as well as of the real positive definite matrices.

Notation 2.4. Let v be a vector in $\mathbb{R}^n$ and let γ be a subset of $\{1, \dots, n\}$. We denote by $v[\gamma]$ the subvector of v indexed by γ.

Definition 2.5. Let $\alpha = \{\alpha_1, \dots, \alpha_p\}$ be a partition of $\{1, \dots, n\}$. A real $n \times n$ matrix A is said to be a *P^α-matrix* if for every nonzero vector v in $\mathbb{R}^n$ there exists $k \in \{1, \dots, p\}$ such that $v[\alpha_k]^T(Av)[\alpha_k] > 0$.

Note that the real P-matrices are exactly the $P^{\{\{1\},\dots,\{n\}\}}$-matrices, while the real positive definite matrices are exactly the $P^{\{\{1,\dots,n\}\}}$-matrices.

Definition 2.6. Let $\alpha = \{\alpha_1, \dots, \alpha_p\}$ and $\beta = \{\beta_1, \dots, \beta_q\}$ be partitions of $\{1, \dots, n\}$. We say that $\beta \subseteq \alpha$ if every set β_i of β is contained in some set α_j of α.

The following proposition follows immediately from Definitions 2.5 and 2.6.

Proposition 2.7. *Let α and β be partitions of $\{1, \dots, n\}$ such that $\beta \subseteq \alpha$. Then every P^α-matrix is a P^β-matrix.*

Remark 2.8. It follows from Proposition 2.7 that for every partition α of $\{1, \dots, n\}$, every P^α-matrix is a P-matrix.

The converse of the statement in Remark 2.8 does not hold in general, as is demonstrated by the following example.

Example 2.9. The matrix

$$A = \begin{bmatrix} 4 & 4 \\ 9 & 10 \end{bmatrix}$$

is a P-matrix. However, A is not a $P^{\{\{1,2\}\}}$-matrix, since for the vector $v = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ we have $v^TAv = -2 < 0$.

Notation 2.10. Let A be an $n \times n$ matrix and let γ be a subset of $\{1, \dots, n\}$. We denote by $A[\gamma]$ the principal submatrix of A with rows and columns indexed by γ.

Notation 2.11. Let A be a real $n \times n$ matrix and let $\alpha = \{\alpha_1, \dots, \alpha_p\}$ be a partition of $\{1, \dots, n\}$. We denote by $\widetilde{A}^\alpha$ the matrix defined by

$$\widetilde{A}^\alpha_{ij} = \begin{cases} \frac{a_{ij}+a_{ji}}{2}, & i, j \in \alpha_k, \;\; k = 1, \dots, p \\ a_{ij}, & \textit{otherwise} \end{cases}.$$

Theorem 2.12. *Let α be a partition of $\{1, \dots, n\}$. A real $n \times n$ matrix A is a P^α-matrix if and only if the matrix $\widetilde{A}^\alpha$ is a P^α-matrix.*

Proof. Let S^α be the skew-symmetric $n \times n$ matrix defined by

$$S^\alpha_{ij} = \begin{cases} \frac{a_{ji}-a_{ij}}{2}, & i, j \in \alpha_k, \;\; k = 1, \dots, p \\ 0, & \textit{otherwise} \end{cases}.$$

Note that $\widetilde{A}^\alpha = A + S^\alpha$. Since S^α is a real skew-symmetric block diagonal matrix with diagonal blocks indexed by $\alpha_1, \ldots, \alpha_p$, it follows that for every vector v in $\mathbb{R}^n$ we have $v[\alpha_k](S^\alpha v)[\alpha_k] = 0$, $k = 1, \ldots, p$. Therefore, we have $v[\alpha_k](Av)[\alpha_k] = v[\alpha_k](\widetilde{A}^\alpha v)[\alpha_k]$, $k = 1, \ldots, p$, and in view of Definition 2.5 our claim follows.□

It follows from Theorem 2.12 and Remark 2.8 that if A is a P^α-matrix then $\widetilde{A}^\alpha$ is a P-matrix. The converse is not true even in the class of Z-matrices, that is, matrices with nonpositive off-diagonal elements, as is shown in the following example.

EXAMPLE 2.13. Let $\alpha = \{\{1,2\},\{3\}\}$. The Z-matrix

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1.1 \end{bmatrix}$$

is a P-matrix. It also satisfies $A = \widetilde{A}^\alpha$. Nevertheless, A is not a P^α-matrix, since for the vector $v = \begin{bmatrix} 1 \\ 1.3 \\ 1.1 \end{bmatrix}$ we have

$$v[\alpha_1]^T(Av)[\alpha_1] = -0.01 \ ; \qquad v[\alpha_2]^T(Av)[\alpha_2] = -0.099 \ .$$

3. Lyapunov scalar stability .

DEFINITION 3.1. A complex square matrix A is said to be *(positive) stable* if the spectrum of A lies in the open right half-plane.

Lyapunov, called by Gantmacher "the founder of the modern theory of stability", studied the asymptotic stability of solutions of differential systems. In 1892 he proved in his paper [13] a theorem which yields the following necessary and sufficient condition for stability of a complex matrix. The matrix formulation of Lyapunov's Theorem is apparently due to Gantmacher [8].

THEOREM 3.2. *A complex square matrix A is stable if and only if there exists a positive definite Hermitian matrix H such that the matrix $AH + HA^*$ is positive definite.*

We remark that Theorem 3.2 was proved in [8] for a real matrix A, however, as was also remarked in [8], the generalization to the complex case is immediate.

A special case of stable matrices which plays an important role in many disciplines, such as predator-prey systems in ecology, e.g. [9], economics, e.g. [10], and dynamical systems, e.g. [1], is the following.

DEFINITION 3.3. A real square matrix A is said to be *Lyapunov diagonally stable* if there exists a positive diagonal matrix D such that $AD + DA^T$ is positive definite.

In this section we generalize Lyapunov diagonal stability. For this purpose we define

DEFINITION 3.4. i) A scalar multiple of the identity matrix is said to be a *scalar matrix.*
ii) Let $\alpha = \{\alpha_1, \ldots, \alpha_p\}$ be a partition of $\{1, \ldots, n\}$. A diagonal $n \times n$ matrix D is said to be an *α-scalar* matrix if $D[\alpha_k]$ is a scalar matrix for every $k \in \{1, \ldots, p\}$.

NOTATION 3.5. For a positive definite Hermitian matrix H we say that $H > 0$.

DEFINITION 3.6. Let α be a partition of $\{1,\dots,n\}$. A complex $n\times n$ matrix A is said to be *Lyapunov α-scalar stable* if there exists a positive definite α-scalar matrix D such that $AD + DA^* > 0$.

REMARK 3.7. Note that the Lyapunov $\{\{1\},\dots,\{n\}\}$-scalar stable matrices are the (complex) Lyapunov diagonally stable matrices, while the Lyapunov $\{1,\dots,n\}$-scalar stable matrices are the (not necessarily Hermitian) positive definite matrices.

It is well known that real Lyapunov diagonally stable matrices are P-matrices, e.g. [6]. The generalization of this result to the α-scalar case is the following.

THEOREM 3.8. *Let α be a partition of $\{1,\dots,n\}$. Then every real Lyapunov α-scalar stable matrix is a P^α-matrix.*

Proof. Let $\alpha = \{\alpha_1,\dots,\alpha_p\}$ be a partition of $\{1,\dots,n\}$ and let A be a Lyapunov α-scalar stable matrix. By Definition 3.6 there exists a positive definite α-scalar matrix D such that $AD + DA^T > 0$. Let E be the positive definite α-scalar matrix defined by $e_{ii} = \sqrt{d_{ii}}$, $i = 1,\dots,n$. Since $AD + DA^T > 0$ it follows that $E^{-1}AE + EAE^{-1} = E^{-1}(AD+DA^T)E^{-1} > 0$, implying that $E^{-1}AE$ is positive definite. Now, let v be a nonzero vector in $\mathbb{R}^n$. We have $v^TE^{-1}AEv > 0$, implying that there exists $k \in \{1,\dots,p\}$ such that

$$v[\alpha_k]^T(E^{-1}AEv)[\alpha_k] > 0. \tag{1}$$

Since E is a diagonal matrix and since $E[\alpha_k]$ is a scalar matrix, the inequality (1) is $v[\alpha_k]^T(Av)[\alpha_k] > 0$, and hence A is a P^α-matrix. □

The converse of Theorem 3.8 does not hold when $\alpha = \{\{1\},\dots,\{n\}\}$, that is, P-matrices are not necessarily Lyapunov diagonally stable. They are not even necessarily stable, e.g. [11]. The converse of Theorem 3.8 does hold when $\alpha = \{1,\dots,n\}$ since in this case Lyapunov α-scalar stability and being a P^α-matrix are both equivalent to being a positive definite matrix.

The Lyapunov diagonally stable matrices were characterized in [3], where it is proven that

THEOREM 3.9. *A complex $n\times n$ matrix A is Lyapunov diagonally stable if and only if for every complex nonzero positive semidefinite Hermitian $n\times n$ matrix H the matrix HA has a diagonal element with positive real part.*

The following generalization of Theorem 3.9 provides a characterization for Lyapunov α-scalar stable matrices.

THEOREM 3.10. *Let $\alpha = \{\alpha_1,\dots,\alpha_p\}$ be a partition of $\{1,\dots,n\}$, and let A be a complex $n\times n$ matrix. The following are equivalent:*

(i) *The matrix A is Lyapunov α-scalar stable.*

(ii) *There exists a positive semidefinite α-scalar matrix D such that $AD + DA^* > 0$.*

(iii) *For every nonzero positive semidefinite Hermitian $n\times n$ matrix H there exists $k \in \{1,\dots,p\}$ such that the trace of the matrix $(HA)[\alpha_k]$ has a positive real part.*

Proof. Let V be the real vector space of all α-scalar real matrices, let C be the (convex) subset of V consisting of all nonnegative α-scalar matrices, let W be the real vector space of all complex Hermitian $n\times n$ matrices, let D be the (convex) subset of W consisting of all positive semidefinite complex Hermitian $n\times n$ matrices, and let $T : V \to W$ be the Lyapunov map L_A, defined by $L_A(H) = AH + HA^*$. We assume that V and W are inner product vector spaces with the usual inner product $\langle A, B\rangle = trace(AB)$ (note that A and B are both Hermitian). It is easy to verify that the dual set C^* of C, defined by

$$C^* = \{H \in V : \langle H, D\rangle \geq 0, \forall D \in C\},$$

is the set $\{H \in V : trace(H[\alpha_k]) \geq 0, k = 1, \ldots, p\}$. Also, it is easy to verify that $T^* = L_{A^*}$. Therefore, our statements (i),(ii) and (iii) are respectively statements (o), (i) and (iii) in Theorem (2.7) in [4] and the result follows.□

If in the proof of Theorem 3.10 we choose W to be the real vector space of all **real symmetric** $n \times n$ matrices and redefine D and T accordingly then we obtain the following characterization of **real** Lyapunov α-scalar stable matrices.

THEOREM 3.11. *Let $\alpha = \{\alpha_1, \ldots, \alpha_p\}$ be a partition of $\{1, \ldots, n\}$, and let A be a real $n \times n$ matrix. The following are equivalent:*
(i) *The matrix A is Lyapunov α-scalar stable.*
(ii) *There exists a positive semidefinite α-scalar matrix D such that $AD + DA^T > 0$.*
(iii) *For every nonzero positive semidefinite real symmetric $n \times n$ matrix H there exists $k \in \{1, \ldots, p\}$ such that the trace of the matrix $(HA)[\alpha_k]$ is positive.*

REMARK 3.12. Note that in the case that $\alpha = \{1, \ldots, n\}$, Theorem 3.10 asserts that a matrix A is positive definite if and only if for every nonzero positive semidefinite Hermitian matrix H the trace of the matrix HA has a positive real part. This is essentially the well known fact that the cone of positive semidefinite matrices is self dual in the real vector space of Hermitian matrices.

The following example illustrates the assertions of Theorems 3.8 and 3.10.

EXAMPLE 3.13. Let

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 2 & 0 \\ 3 & 3 & 1 \end{bmatrix}.$$

Since the matrix $A + A^T$ is not positive definite, the matrix A is not Lyapunov $\{\{1, 2, 3\}\}$-scalar stable. The matrix A is, however, Lyapunov $\{\{1, 2\}, \{3\}\}$-scalar stable since for the diagonal matrix $D = diag(1, 1, 2)$ the matrix $AD+DA^T$ is positive definite.

To see that A is a $P^{\{\{1,2\},\{3\}\}}$-matrix let $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ be a nonzero vector in $\mathbb{R}^n$.

We have

$$v[\{1, 2\}]^T(Av)[\{1, 2\}] = 2(v_1 + v_2)^2 - v_1v_2, \tag{2}$$

and

$$v[3]^T(Av)[3] = 3v_1v_3 + 3v_2v_3 + 3v_3^2. \tag{3}$$

Note that the right hand side of (2) is positive unless $v_1 = v_2 = 0$, in which case, since v is nonzero, we must have $v_3 \neq 0$ and the right hand side of (3) becomes $3v_3^2 > 0$. By Definition 2.5, A is a $P^{\{\{1,2\},\{3\}\}}$-matrix.

To demonstrate the implication (i)⇒(ii) in Theorem 3.10 let

$$H = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ \bar{h}_{12} & h_{22} & h_{23} \\ \bar{h}_{13} & \bar{h}_{23} & h_{33} \end{bmatrix}$$

be a nonzero positive semidefinite Hermitian matrix. The diagonal elements of HA are $(HA)_{11} = 2h_{11} + 2h_{12} + 3h_{13}$, $(HA)_{22} = \bar{h}_{12} + 2h_{22} + 3h_{23}$ and $(HA)_{33} = 3h_{33}$. If $h_{33} > 0$ then we are done. Otherwise, since H is positive semidefinite, it follows that $h_{33} = 0$ and hence also $h_{13} = h_{23} = 0$. We thus have

$$trace((HA)[\{1,2\}]) = 2h_{11} + 2h_{12} + \bar{h}_{12} + 2h_{22}. \tag{4}$$

Since H is positive semidefinite we have $h_{11}h_{22} \geq |h_{12}|^2$, implying that $Re(h_{12}) \leq |h_{12}| \leq \sqrt{h_{11}h_{22}} \leq \frac{h_{11}+h_{22}}{2}$, and it follows from (4) that $trace((HA)[\{1,2\}]) \geq 0$. Furthermore, it is easy to verify that $trace((HA)[\{1,2\}]) = 0$ only if $h_{11} = h_{12} = h_{22} = 0$, which is not the case since H is nonzero.

NOTATION 3.14. Let m and n be positive integers and let A and B be $m \times n$ matrices. We denote by $A \circ B$ the Hadamard product of A and B, that is, the $m \times n$ matrix whose elements are the products of the corresponding elements of A and B.

We now prove another characterization of Lyapunov α-scalar stability in terms of P^α-matrices, generalizing Theorem 1.2 in [12]. Our proof is similar to the proof given in [12].

THEOREM 3.15. *Let α be a partition of $\{1, \ldots, n\}$, and let A be a real $n \times n$ matrix. The following are equivalent:*
(i) *The matrix A is Lyapunov α-scalar stable.*
(ii) *The matrix $A \circ H$ is a P^α-matrix for every positive semidefinite real symmetric matrix H with nonzero diagonal elements.*
(iii) *The matrix $A \circ H$ is a P^α-matrix for every positive semidefinite real symmetric matrix H with diagonal elements all equal to* 1.

Proof. (i)$\Longrightarrow$(ii). The proof of this implication is essentially the same as the proof of the corresponding implication Theorem 1.2 of [12].

(ii)$\Longrightarrow$(iii) is clear.

(iii)$\Longrightarrow$(i). Let H be a nonzero positive semidefinite real symmetric $n \times n$ matrix and let D the diagonal matrix defined by $d_{ii} = \sqrt{h_{ii}}$, $i = 1, \ldots, n$. Since H is a positive semidefinite matrix, if for some i we have $h_{ii} = 0$ then also $h_{ij} = h_{ji} = 0$ for all j. Therefore, we can define a nonzero symmetric $n \times n$ matrix $\widetilde{H}$ by

$$\widetilde{h}_{ij} = \begin{cases} \frac{h_{ij}}{\sqrt{h_{ii}h_{jj}}}, & i \neq j \text{ and } h_{ij} \neq 0 \\ 0, & i \neq j \text{ and } h_{ij} = 0 \\ 1, & i = j \end{cases}.$$

The matrix $\widetilde{H}$ is positive semidefinite since $\widetilde{H} = D_1 H D_1 + D_2$, where D_1 and D_2 are the nonnegative diagonal matrices defined by

$$(D_1)_{ii} = \begin{cases} \frac{1}{\sqrt{h_{ii}}}\,, & h_{ii} \neq 0 \\ \\ 0\,, & h_{ii} = 0 \end{cases}, \quad (D_2)_{ii} = \begin{cases} 0\,, & h_{ii} \neq 0 \\ \\ 1\,, & h_{ii} = 0 \end{cases}.$$

Note that $H = D\widetilde{H}D$. Let e be the vector in $\mathbb{R}^n$ all of whose entries are 1 and let $v = De$. Also, let $\alpha = \{\alpha_1, \ldots, \alpha_p\}$. By (iii), The matrix $A \circ \widetilde{H}$ is a P^α-matrix. Therefore, in view of Definition 2.5 there exists $k \in \{1, \ldots, p\}$ such that $v[\alpha_k]^T((A \circ \widetilde{H})v)[\alpha_k] > 0$. Since

$$v[\alpha_k]^T((A \circ \widetilde{H})v)[\alpha_k] = (eD)[\alpha_k]^T((A \circ \widetilde{H})(De))[\alpha_k] = e[\alpha_k]^T((A \circ (D\widetilde{H}D))e)[\alpha_k] =$$

$$= e[\alpha_k]^T((A \circ H)e)[\alpha_k] = \sum_{i \in \alpha_k} \sum_{j=1}^{n} a_{ij} h_{ji} = trace((HA)[\alpha_k]),$$

(i) follows by Theorem 3.11.□

4. H-stability . In this section we study another specific type of matrix stability, which is defined in [14].

DEFINITION 4.1. A complex $n \times n$ matrix A is said to be *H-stable* if the product AH is stable for every positive definite Hermitian $n \times n$ matrix H.

DEFINITION 4.2. Let $\alpha = \{\alpha_1, \ldots, \alpha_p\}$ be a partition of $\{1, \ldots, n\}$. A block diagonal matrix with diagonal blocks indexed by $\alpha_1, \ldots, \alpha_p$ is said to be *α-diagonal.*

We now define a generalization of H-stability.

DEFINITION 4.3. Let $\alpha = \{\alpha_1, \ldots, \alpha_p\}$ be a partition of $\{1, \ldots, n\}$. A complex $n \times n$ matrix A is said to be *$H(\alpha)$-stable* if the product AH is stable for every α-diagonal positive definite Hermitian matrix H.

Note that the real $H(\{\{1\}, \ldots, \{n\}\})$-stable matrices are exactly the D-stable matrices, e.g. [10], while the real $H(\{\{1, \ldots, n\}\})$-stable matrices are the H-stable matrices.

It is well known that Lyapunov diagonal stability implies D-stability, e.g. [6]. In our terminology, Lyapunov $\{\{1\}, \ldots, \{n\}\}$-scalar stability implies $H(\{\{1\}, \ldots, \{n\}\})$-stability. A similar implication for the case of $\alpha = \{\{1, \ldots, n\}\}$ follows from Corollary 3 in [14]. We now show that the same implication holds for a general partition α.

THEOREM 4.4. *Let α be a partition of $\{1, \ldots, n\}$. Then every real Lyapunov α-scalar stable matrix is an $H(\alpha)$-stable matrix.*

Proof. Let A be real Lyapunov α-scalar stable matrix and let D be a positive definite α-scalar matrix such that $AD + DA^* > 0$. Let H be an α-diagonal positive definite Hermitian matrix. Note that H^{-1} is also an α-diagonal positive definite Hermitian matrix, and since D is a positive definite α-scalar matrix it follows that $H^{-1}D$ is an (α-diagonal) positive definite Hermitian matrix. Also, $H^{-1}D = DH^{-1}$. It now follows that

$$AH(H^{-1}D) + (H^{-1}D)(AH)^* = AD + DA^* > 0,$$

and by Theorem 3.2 the matrix AH is stable.□

Theorem 4.4 provides a sufficient condition for $H(\alpha)$-stability of real matrices. We do not know of any necessary and sufficient condition for $H(\alpha)$-stability. In particular, it would be interesting to generalize the following characterization of H-stability, proven in [5].

THEOREM 4.5. *A complex matrix A is H-stable if and only if the following conditions hold:*
(i) *The matrix $A + A^*$ is positive semidefinite.*
(ii) *For every vector x we have*

$$x^*(A + A^*)x = 0 \;\Rightarrow\; x^*(A - A^*)x = 0\,.$$

(iii) *A is nonsingular.*

An interesting result that follows from Theorem 4.5 is the following.

THEOREM 4.6. *Let A be a complex H-stable $n \times n$ matrix. Then $A + K$ is H-stable for every positive definite matrix K.*

Proof. Let A be a complex H-stable matrix and let K be a positive definite matrix. By Theorem 4.5.i we have

$$(A + K) + (A + K)^* = (A + A^*) + 2K > 0. \tag{5}$$

Now, let x be a vector satisfying $x^*((A+K)+(A+K)^*)x = 0$. By (5) it follows that $x = 0$ and so clearly $x^*((A + K) - (A + K)^*)x = 0$. Therefore, we have

$$x^*((A + K) + (A + K)^*)x = 0 \;\Rightarrow\; x^*((A + K) - (A + K)^*)x = 0. \tag{6}$$

Finally, we have

$$A + K = (AK^{-1} + I)K. \tag{7}$$

Since A is H-stable, the matrix AK^{-1} is stable. Therefore, the matrix $AK^{-1} + I$ is stable, and it follows from (7) that

$$A + K \text{ is nonsingular.} \tag{8}$$

By Theorem 4.5, it now follows from (5), (6) and (8) that $A + K$ is H-stable.□

We conclude by remarking that Theorem 4.6 does not hold in general when H-stability is replaced by $H(\alpha)$-stability and when K is α-diagonal. Moreover, it is shown in [2] that in the case $\alpha = \{\{1\}, \ldots, \{n\}\}$, if A is $H(\alpha)$-stable then $A + K$ is not even necessarily stable for every positive definite matrix α-diagonal matrix K.

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