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A DIAZ–METCALF TYPE INEQUALITY FOR POSITIVE LINEAR MAPS AND ITS APPLICATIONS*

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Abstract. We present a Diaz–Metcalf type operator inequality as a reverse Cauchy–Schwarz inequality and then apply it to get some operator versions of Pólya–Szegö's, Greub–Rheinboldt's, Kantorovich's, Shisha–Mond's, Schweitzer's, Cassels' and Klamkin–McLenaghan's inequalities via a unified approach. We also give some operator Grüss type inequalities and an operator Ozeki–Izumino–Mori–Seo type inequality. Several applications are included as well.

Key words. Diaz–Metcalf type inequality, Reverse Cauchy–Schwarz inequality, Positive map, Ozeki–Izumino–Mori–Seo inequality, Operator inequality.

AMS subject classifications. 46L08, 26D15, 46L05, 47A30, 47A63.

1. Introduction. The Cauchy–Schwarz inequality plays an essential role in mathematical analysis and its applications. In a semi-inner product space $(\mathscr{H}, \langle \cdot, \cdot \rangle)$ the Cauchy–Schwarz inequality reads as follows

$$|\langle x, y \rangle| \leq \langle x, x \rangle^{1/2} \langle y, y \rangle^{1/2} \qquad (x, y \in \mathscr{H}).$$

There are interesting generalizations of the Cauchy–Schwarz inequality in various frameworks, e.g., finite sums, integrals, isotone functionals, inner product spaces, C^*-algebras and Hilbert C^*-modules; see [5, 6, 7, 9, 11, 13, 17, 20] and references therein. There are several reverses of the Cauchy–Schwarz inequality in the literature: Diaz–Metcalf's, Pólya–Szegö's, Greub–Rheinboldt's, Kantorovich's, Shisha–Mond's, Ozeki–Izumino–Mori–Seo's, Schweitzer's, Cassels' and Klamkin–McLenaghan's inequalities.

Inspired by the work of J.B. Diaz and F.T. Metcalf [4], we present several reverse Cauchy–Schwarz type inequalities for positive linear maps. We give a unified treatment of some reverse inequalities of the classical Cauchy–Schwarz type for positive

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linear maps.

Throughout the paper $\mathbb{B}(\mathscr{H})$ stands for the algebra of all bounded linear operators acting on a Hilbert space $\mathscr{H}$. We simply denote by α the scalar multiple αI of the identity operator $I \in \mathbb{B}(\mathscr{H})$. For self-adjoint operators A, B the partially ordered relation $B \le A$ means that $\langle B\xi, \xi\rangle \le \langle A\xi, \xi\rangle$ for all $\xi \in \mathscr{H}$. In particular, if $0 \le A$, then A is called positive. If A is a positive invertible operator, then we write $0 < A$. A linear map $\Phi : \mathscr{A} \to \mathscr{B}$ between C^*-algebras is said to be positive if $\Phi(A)$ is positive whenever A is. We say that Φ is unital if Φ preserves the identity. The reader is referred to [9, 19] for undefined notations and terminologies.

2. Operator Diaz–Metcalf type inequality. We start this section with our main result. Recall that the geometric operator mean $A \sharp B$ for positive operators $A, B \in \mathbb{B}(\mathscr{H})$ is defined by

$$A \sharp B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{\frac{1}{2}} A^{\frac{1}{2}}$$

if $0 < A$.

THEOREM 2.1. *Let $A, B \in \mathbb{B}(\mathscr{H})$ be positive invertible operators and $\Phi : \mathbb{B}(\mathscr{H}) \to \mathbb{B}(\mathscr{K})$ be a positive linear map.*

(i) *If $m^2A \le B \le M^2A$ for some positive real numbers $m < M$, then the following inequalities hold:*

- *Operator Diaz–Metcalf inequality of first type*

$$Mm\Phi(A) + \Phi(B) \le (M + m)\Phi(A\sharp B)\,;$$

- *Operator Cassels inequality*

$$\Phi(A)\sharp\Phi(B) \le \frac{M + m}{2\sqrt{Mm}}\Phi(A\sharp B)\,;$$

- *Operator Klamkin–McLenaghan inequality*

$$\Phi(A\sharp B)^{\frac{-1}{2}}\Phi(B)\Phi(A\sharp B)^{\frac{-1}{2}} - \Phi(A\sharp B)^{\frac{1}{2}}\Phi(A)^{-1}\Phi(A\sharp B)^{\frac{1}{2}} \le (\sqrt{M} - \sqrt{m})^2\,;$$

- *Operator Kantorovich inequality*

$$\Phi(A)\sharp\Phi(A^{-1}) \le \frac{M^2 + m^2}{2Mm}\,.$$

(ii) *If $m_1^2 \le A \le M_1^2$ and $m_2^2 \le B \le M_2^2$ for some positive real numbers $m_1 < M_1$ and $m_2 < M_2$, then the following inequalities hold:*

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- *Operator Diaz–Metcalf inequality of second type*

$$\frac{M_2 m_2}{M_1 m_1}\Phi(A) + \Phi(B) \le \left(\frac{M_2}{m_1} + \frac{m_2}{M_1}\right)\Phi(A\sharp B)\,;$$

- *Operator Pólya–Szegö inequality*

$$\Phi(A)\sharp\Phi(B) \le \frac{1}{2}\left(\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}}\right)\Phi(A\sharp B)\,;$$

- *Operator Shisha–Mond inequality*

$$\Phi(A\sharp B)^{\frac{-1}{2}}\Phi(B)\Phi(A\sharp B)^{\frac{-1}{2}} - \Phi(A\sharp B)^{\frac{1}{2}}\Phi(A)^{-1}\Phi(A\sharp B)^{\frac{1}{2}} \le \left(\sqrt{\frac{M_2}{m_1}} - \sqrt{\frac{m_2}{M_1}}\right)^2\,;$$

- *Operator Grüss type inequality*

$$\Phi(A)\sharp\Phi(B) - \Phi(A\sharp B) \le \frac{\sqrt{M_1 M_2}\left(\sqrt{M_1 M_2} - \sqrt{m_1 m_2}\right)^2}{2\sqrt{m_1 m_2}} \min\left\{\frac{M_1}{m_1}, \frac{M_2}{m_2}\right\}\,.$$

Proof. (i) If $m^2 A \le B \le M^2 A$ for some positive real numbers $m < M$, then $m^2 \le A^{\frac{-1}{2}} B A^{\frac{-1}{2}} \le M^2$.

(ii) If $m_1^2 \le A \le M_1^2$ and $m_2^2 \le B \le M_2^2$ for some positive real numbers $m_1 < M_1$ and $m_2 < M_2$, then

$$m^2 = \frac{m_2^2}{M_1^2} \le A^{\frac{-1}{2}} B A^{\frac{-1}{2}} \le \frac{M_2^2}{m_1^2} = M^2\,. \tag{2.1}$$

In any case we then have

$$\left(M - \left(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}\right)^{1/2}\right)\left(\left(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}\right)^{1/2} - m\right) \ge 0\,,$$

whence

$$Mm + A^{\frac{-1}{2}} B A^{\frac{-1}{2}} \le (M+m)\left(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}\right)^{\frac{1}{2}}\,.$$

Hence

$$MmA + B \le (M+m)A^{1/2}\left(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}\right)^{\frac{1}{2}} A^{1/2} = (M+m)A\sharp B\,. \tag{2.2}$$

Since Φ is a positive linear map, (2.2) yields the *operator Diaz–Metcalf inequality of first type* as follows:

$$Mm\Phi(A) + \Phi(B) \le (M+m)\Phi(A\sharp B)\,. \tag{2.3}$$

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In the case when (ii) holds we get the following, which is called the *operator Diaz–Metcalf inequality of second type*:

$$\frac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B)\le\left(\frac{M_2}{m_1}+\frac{m_2}{M_1}\right)\Phi(A\sharp B)\,.$$

Following the strategy of [21], we apply the operator geometric–arithmetic inequality to $Mm\Phi(A)$ and $\Phi(B)$ to get:

$$\sqrt{Mm}(\Phi(A)\sharp\Phi(B))=(Mm\Phi(A))\sharp\Phi(B)\le\frac{1}{2}\left(Mm\Phi(A)+\Phi(B)\right). \tag{2.4}$$

It follows from (2.3) and (2.4) that

$$\Phi(A)\sharp\Phi(B)\le\frac{M+m}{2\sqrt{Mm}}\Phi(A\sharp B)\,,$$

which is said to be the *operator Cassels inequality* under the assumption (i); see also [16]. Under the case (ii) we can represent it as the following inequality being called the *operator Pólya–Szegö inequality* or the *operator Greub–Rheinboldt inequality*:

$$\Phi(A)\sharp\Phi(B)\le\frac{1}{2}\left(\sqrt{\frac{M_1M_2}{m_1m_2}}+\sqrt{\frac{m_1m_2}{M_1M_2}}\right)\Phi(A\sharp B)\,. \tag{2.5}$$

It follows from (2.5) that

$$\begin{aligned}\Phi(A)\sharp\Phi(B)-\Phi(A\sharp B)&\le\left(\frac{1}{2}\left(\sqrt{\frac{M_1M_2}{m_1m_2}}+\sqrt{\frac{m_1m_2}{M_1M_2}}\right)-1\right)\Phi(A\sharp B)\\&=\frac{\left(\sqrt{M_1M_2}-\sqrt{m_1m_2}\right)^2}{2\sqrt{m_1m_2}\sqrt{M_1M_2}}\Phi(A\sharp B)\,.\end{aligned} \tag{2.6}$$

It follows from (2.1) that

$$\frac{m_2}{M_1}A\le A^{1/2}\left(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\right)^{1/2}A^{1/2}\le\frac{M_2}{m_1}A\,,$$

so

$$\frac{m_1^2m_2}{M_1}\le A\sharp B\le\frac{M_1^2M_2}{m_1}\,. \tag{2.7}$$

Now, (2.6) and (2.7) yield that

$$\Phi(A)\sharp\Phi(B)-\Phi(A\sharp B)\le\frac{\left(\sqrt{M_1M_2}-\sqrt{m_1m_2}\right)^2}{2\sqrt{m_1m_2}\sqrt{M_1M_2}}\frac{M_1^2M_2}{m_1}\,.$$

An easy symmetric argument then follows that

$$\Phi(A)\sharp\Phi(B)-\Phi(A\sharp B)\le\frac{\sqrt{M_1M_2}\left(\sqrt{M_1M_2}-\sqrt{m_1m_2}\right)^2}{2\sqrt{m_1m_2}}\min\left\{\frac{M_1}{m_1},\frac{M_2}{m_2}\right\}\,,$$

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presenting a Grüss type inequality.

If A is invertible and Φ is unital and $m_1^2 = m^2 \leq A \leq M^2 = M_1^2$, then by putting $m_2^2 = 1/M^2 \leq B = A^{-1} \leq 1/m^2 = M_2^2$ in (2.5) we get the following *operator Kantorovich inequality*:

$$\Phi(A)\sharp\Phi(A^{-1}) \leq \frac{M^2+m^2}{2Mm}\,.$$

It follows from (2.3) that

$$\begin{aligned}
&\Phi(A\sharp B)^{\frac{-1}{2}}\Phi(B)\Phi(A\sharp B)^{\frac{-1}{2}} - \Phi(A\sharp B)^{\frac{1}{2}}\Phi(A)^{-1}\Phi(A\sharp B)^{\frac{1}{2}}\\
&\leq M+m-Mm\Phi(A\sharp B)^{\frac{-1}{2}}\Phi(A)\Phi(A\sharp B)^{\frac{-1}{2}} - \Phi(A\sharp B)^{\frac{1}{2}}\Phi(A)^{-1}\Phi(A\sharp B)^{\frac{1}{2}}\\
&\leq M+m-2\sqrt{Mm} - \left(\sqrt{Mm}\left(\Phi(A\sharp B)^{\frac{-1}{2}}\Phi(A)\Phi(A\sharp B)^{\frac{-1}{2}}\right)^{1/2}\right.\\
&\quad \left. - \left(\Phi(A\sharp B)^{\frac{1}{2}}\Phi(A)^{-1}\Phi(A\sharp B)^{\frac{1}{2}}\right)^{1/2}\right)^2\\
&\leq (\sqrt{M}-\sqrt{m})^2\,, \qquad (2.8)
\end{aligned}$$

that is, an *operator Klakmin–Mclenaghan inequality* when (i) holds. Under (ii), we get the following *operator Shisha–Szegö inequality* from (2.8):

$$\Phi(A\sharp B)^{\frac{-1}{2}}\Phi(B)\Phi(A\sharp B)^{\frac{-1}{2}} - \Phi(A\sharp B)^{\frac{1}{2}}\Phi(A)^{-1}\Phi(A\sharp B)^{\frac{1}{2}} \leq \left(\sqrt{\frac{M_2}{m_1}} - \sqrt{\frac{m_2}{M_1}}\right)^2\,. \quad \square$$

3. Applications. If $(a_1,\ldots,a_n)$ and $(b_1,\ldots,b_n)$ are n-tuples of real numbers with $0 < m_1 \leq a_i \leq M_1$ $(1 \leq i \leq n), 0 < m_2 \leq b_i \leq M_2$ $(1 \leq i \leq n)$, we can consider the positive linear map $\Phi(T) = \langle Tx, x\rangle$ on $\mathbb{B}(\mathbb{C}^n) = M_n(\mathbb{C})$ and let $A = \mathrm{diag}(a_1^2,\ldots,a_n^2)$, $B = \mathrm{diag}(b_1^2,\ldots,b_n^2)$ and $x = (1,\ldots,1)^t$ in the operator inequalities above to get the following classical inequalities:

- Diaz–Metcalf inequality [4]

$$\sum_{k=1}^n b_k^2 + \frac{m_2M_2}{m_1M_1}\sum_{k=1}^n a_k^2 \leq \left(\frac{M_2}{m_1}+\frac{m_2}{M_1}\right)\sum_{k=1}^n a_kb_k\,.$$

- Pólya–Szegö inequality [23]

$$\frac{\sum_{k=1}^n a_k^2 \sum_{k=1}^n b_k^2}{\left(\sum_{k=1}^n a_kb_k\right)^2} \leq \frac{1}{4}\left(\sqrt{\frac{M_1M_2}{m_1m_2}} + \sqrt{\frac{m_1m_2}{M_1M_2}}\right)^2;$$

- Shisha–Mond inequality [24]

$$\frac{\sum_{k=1}^n a_k^2}{\sum_{k=1}^n a_kb_k} - \frac{\sum_{k=1}^n a_kb_k}{\sum_{k=1}^n b_k^2} \leq \left(\sqrt{\frac{M_1}{m_2}} - \sqrt{\frac{m_1}{M_2}}\right)^2;$$

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- A Grüss type inequality

$$\left(\sum_{k=1}^{n} a_k^2\right)^{1/2}\left(\sum_{k=1}^{n} b_k^2\right)^{1/2} - \sum_{k=1}^{n} a_k b_k \le \frac{\sqrt{M_1M_2}\left(\sqrt{M_1M_2}-\sqrt{m_1m_2}\right)^2}{2\sqrt{m_1m_2}} \min\left\{\frac{M_1}{m_1}, \frac{M_2}{m_2}\right\}.$$

Using the same argument with a positive n-tuple $(a_1, \ldots, a_n)$ of real numbers with $0 < m \le a_i \le M$ $(1 \le i \le n)$, $x = \frac{1}{\sqrt{n}}(1, \ldots, 1)^t$, we get from Kantorovich inequality that

- Schweitzer inequality [2]

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i^2\right)\left(\frac{1}{n}\sum_{i=1}^{n} a_i^{-2}\right) \le \frac{(M^2+m^2)^2}{4M^2m^2}.$$

If $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are n-tuples of real numbers with $0 < m \le a_i/b_i \le M$ $(1 \le i \le n)$, we can consider the positive linear map $\Phi(T) = \langle Tx, x\rangle$ on $\mathbb{B}(\mathbb{C}^n) = M_n(\mathbb{C})$ and let $A = \mathrm{diag}(a_1^2, \ldots, a_n^2)$, $B = \mathrm{diag}(b_1^2, \ldots, b_n^2)$ and $x = (\sqrt{w_1}, \ldots, \sqrt{w_n})^t$ based on the weight $\bar{\mathbf{w}} = (w_1, \ldots, w_n)$, in the operator inequalities above to get the following classical inequalities:

- Cassels inequality [25]

$$\frac{\sum_{k=1}^n w_k a_k^2 \sum_{k=1}^n w_k b_k^2}{\left(\sum_{k=1}^n w_k a_k b_k\right)^2} \le \frac{(M+m)^2}{4mM};$$

- Klamkin–McLenaghan inequality [14]

$$\sum_{k=1}^{n} w_k a_k^2 \sum_{k=1}^{n} w_k b_k^2 - \left(\sum_{k=1}^{n} w_k a_k b_k\right)^2 \le \left(\sqrt{M}-\sqrt{m}\right)^2 \sum_{k=1}^{n} w_k a_k b_k \sum_{k=1}^{n} w_k a_k^2.$$

Using the same argument, we obtain a weighted form of the Pólya–Szegö inequality as follows:

- Grueb–Rheinboldt inequality [10]

$$\frac{\sum_{k=1}^n w_k a_k^2 \sum_{k=1}^n w_k b_k^2}{\left(\sum_{k=1}^n w_k a_k b_k\right)^2} \le \frac{(M_1M_2+m_1m_2)^2}{4m_1m_2M_1M_2}.$$

One can assert the integral versions of discrete results above by considering $L^2(X,\mu)$, where (X,μ) is a probability space, as a Hilbert space via $\langle h_1, h_2\rangle = \int_X h_1\overline{h_2}d\mu$, multiplication operators $A, B \in \mathbb{B}(L^2(X,\mu))$ defined by $A(h) = f^2h$

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and $B(h) = g^2h$ for bounded $f, g \in L^2(X, \mu)$ and a positive linear map Φ by $\Phi(T) = \int_X T(1)d\mu$ on $\mathbb{B}(L^2(X, \mu)))$. For instance, let us state integral versions of the Cassels and Klamkin–McLenaghan inequalities. These two inequalities are obtained, first for bounded positive functions $f, g \in L^2(X, \mu)$ and next for general positive functions $f, g \in L^2(X, \mu)$ as the limits of sequences of bounded positive functions.

Corollary 3.1. *Let (X, μ) be a probability space and $f, g \in L^2(X, \mu)$ with $0 \le mg \le f \le Mg$ for some scalars $0 < m < M$. Then*

$$\int_X f^2 d\mu \int_X g^2 d\mu \le \frac{(M+m)^2}{4Mm} \left(\int_X fg d\mu \right)^2$$

and

$$\int_X f^2 d\mu \int_X g^2 d\mu - \left(\int_X fg d\mu \right)^2 \le \left(\sqrt{M} - \sqrt{m} \right)^2 \int_X fg d\mu \int_X f^2 d\mu \,.$$

Considering the positive linear functional $\Phi(R) = \sum_{i=1}^n \langle R\xi_i, \xi_i \rangle$ on $\mathbb{B}(\mathscr{H})$, where $\xi_1, \ldots, \xi_n \in \mathscr{H}$, we get the following versions of the Diaz–Metcalf and Pólya–Szegö inequalities in a Hilbert space.

Corollary 3.2. *Let $\mathscr{H}$ be a Hilbert space, let $\xi_1, \ldots, \xi_n \in \mathscr{H}$ and let $T, S \in \mathbb{B}(\mathscr{H})$ be positive operators satisfying $0 < m_1 \le T \le M_1$ and $0 < m_2 \le S \le M_2$. Then*

$$\frac{M_2 m_2}{M_1 m_1} \sum_{i=1}^n \|T\xi_i\|^2 + \sum_{i=1}^n \|S\xi_i\|^2 \le \left(\frac{M_2}{m_1} + \frac{m_2}{M_1} \right) \sum_{i=1}^n \|(T^2 \sharp S^2)^{1/2} \xi_i\|^2$$

and

$$\left(\sum_{i=1}^n \|T\xi_i\|^2 \right)^{1/2} \left(\sum_{i=1}^n \|S\xi_i\|^2 \right)^{1/2} \le \frac{1}{2} \left(\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}} \right) \sum_{i=1}^n \|(T^2 \sharp S^2)^{1/2} \xi_i\|^2 \,.$$

4. A Grüss type inequality. In this section we obtain another Grüss type inequality, see also [18]. Let $\mathscr{A}$ be a C^*-algebra and let $\mathscr{B}$ be a C^*-subalgebra of $\mathscr{A}$. Following [1], a positive linear map $\Phi : \mathscr{A} \to \mathscr{B}$ is called a left multiplier if $\Phi(XY) = \Phi(X)Y$ for every $X \in \mathscr{A}$, $Y \in \mathscr{B}$.

The following lemma is interesting on its own right.

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LEMMA 4.1. *Let Φ be a unital positive linear map on $\mathscr{A}$, $A \in \mathscr{A}$ and M, m be complex numbers such that*

$$\mathrm{Re}\,((M-A)^*(A-m)) \geq 0\,. \tag{4.1}$$

Then

$$\Phi(|A|^2) - |\Phi(A)|^2 \leq \frac{1}{4}|M-m|^2\,.$$

Proof. For any complex number $c \in \mathbb{C}$, we have

$$\Phi(|A|^2) - |\Phi(A)|^2 = \Phi(|A-c|^2) - |\Phi(A-c)|^2. \tag{4.2}$$

Since for any $T \in \mathscr{A}$ the operator equality

$$\frac{1}{4}|M-m|^2 - \left|T - \frac{M+m}{2}\right|^2 = \mathrm{Re}\,((M-T)(T-m)^*)$$

holds, the condition (4.1) implies that

$$\Phi\left(\left|A - \frac{M+m}{2}\right|^2\right) \leq \frac{1}{4}|M-m|^2\,. \tag{4.3}$$

Therefore, it follows from (4.2) and (4.3) that

$$\begin{aligned}\Phi(|A|^2) - |\Phi(A)|^2 &\leq \Phi(|A - \frac{M+m}{2}|^2)\\ &\leq \frac{1}{4}|M-m|^2. \quad \square\end{aligned}$$

REMARK 4.2. If (i) Φ is a unital positive linear map and A is a normal operator or (ii) Φ is a 2-positive linear map and A is an arbitrary operator, then it follows from [3] that

$$0 \leq \Phi(|A|^2) - |\Phi(A)|^2\,. \tag{4.4}$$

Condition (4.4) is stronger than positivity and weaker than 2-positivity; see [8]. Another class of positive linear maps satisfying (4.4) are left multipliers, cf. [1, Corollary 2.4].

LEMMA 4.3. *Let a positive linear map $\Phi : \mathscr{A} \to \mathscr{B}$ be a unital left multiplier. Then*

$$|\Phi(A^*B) - \Phi(A)^*\Phi(B)|^2 \leq \left\|\Phi(|A|^2) - |\Phi(A)|^2\right\| \left(\Phi(|B|^2) - |\Phi(B)|^2\right) \tag{4.5}$$

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Proof. If we put $[X,Y] := \Phi(X^*Y) - \Phi(X)^*\Phi(Y)$, then $\mathscr{A}$ is a right pre-inner product C^*-module over $\mathscr{B}$, since $\Phi(X^*Y)$ is a right pre-inner product $\mathscr{B}$-module, see [1, Corollary 2.4]. It follows from the Cauchy–Schwarz inequality in pre-inner product C^*-modules (see [15, Proposition 1.1]) that

$$\begin{aligned}|\Phi(A^*B) - \Phi(A)^*\Phi(B)|^2 &= [B,A][A,B]\\ &\le \|[A,A]\|[B,B]\\ &= \|\Phi(A^*A) - \Phi(A)^*\Phi(A)\|\left(\Phi(B^*B) - \Phi(B)^*\Phi(B)\right)\end{aligned}$$

and hence (4.5) holds. □

THEOREM 4.4. *Let a positive linear map $\Phi : \mathscr{A} \to \mathscr{B}$ be a unital left multiplier. If $M_1, m_1, M_2, m_2 \in \mathbb{C}$ and $A, B \in \mathscr{A}$ satisfy the following conditions:*

$$\mathrm{Re}(M_1 - A)^*(A - m_1) \ge 0 \quad \textit{and} \quad \mathrm{Re}(M_2 - B)^*(B - m_2) \ge 0,$$

then

$$|\Phi(A^*B) - \Phi(A)^*\Phi(B)| \le \frac{1}{4}|M_1 - m_1|\,|M_2 - m_2|.$$

Proof. By Löwner–Heinz theorem, we have

$$\begin{aligned}&|\Phi(A^*B) - \Phi(A)^*\Phi(B)|\\ &\le \left\|\Phi(|A|^2) - |\Phi(A)|^2\right\|^{\frac12}\left(\Phi(|B|^2) - |\Phi(B)|^2\right)^{\frac12} \qquad \text{(by Lemma 4.3)}\\ &\le \frac{1}{4}|M_1 - m_1|\,|M_2 - m_2| \qquad \text{(by Lemma 4.1)}.\end{aligned}$$ □

5. Ozeki–Izumino–Mori–Seo type inequality. Let $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ be n-tuples of real numbers satisfying

$$0 \le m_1 \le a_i \le M_1 \quad \text{and} \quad 0 \le m_2 \le b_i \le M_2 \quad (i = 1, \dots, n).$$

Then Ozeki–Izumino–Mori–Seo inequality [12, 22] asserts that

$$\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - \left(\sum_{i=1}^n a_i b_i\right)^2 \le \frac{n^2}{3}\left(M_1M_2 - m_1m_2\right)^2. \tag{5.1}$$

In [12] they also showed the following operator version of (5.1): If A and B are positive operators in $\mathbb{B}(\mathscr{H})$ such that $0 < m_1 \le A \le M_1$ and $0 < m_2 \le B \le M_2$ for some scalars $m_1 \le M_1$ and $m_2 \le M_2$, then

$$(A^2x,x)(B^2x,x) - (A^2 \sharp B^2x,x)^2 \le \frac{1}{4\gamma^2}\left(M_1M_2 - m_1m_2\right)^2 \tag{5.2}$$

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for every unit vector $x \in H$, where $\gamma = \max\{\frac{m_1}{M_1}, \frac{m_2}{M_2}\}$.

Based on the Kantorovich inequality for the difference, we present an extension of Ozeki–Izumino–Mori–Seo inequality (5.2) as follows.

Theorem 5.1. *Suppose that $\Phi : \mathbb{B}(\mathscr{H}) \to \mathbb{B}(\mathscr{K})$ is a positive linear map such that $\Phi(I)$ is invertible and $\Phi(I) \le I$. Assume that $A, B \in \mathbb{B}(\mathscr{H})$ are positive invertible operators such that $0 < m_1 \le A \le M_1$ and $0 < m_2 \le B \le M_2$ for some scalars $m_1 \le M_1$ and $m_2 \le M_2$. Then*

$$\Phi(B^2)^{\frac{1}{2}}\Phi(A^2)\Phi(B^2)^{\frac{1}{2}} - |\Phi(B^2)^{-\frac{1}{2}}\Phi(A^2\sharp B^2)\Phi(B^2)^{\frac{1}{2}}|^2 \le \frac{(M_1M_2 - m_1m_2)^2}{4} \times \frac{M_2^2}{m_2^2} \tag{5.3}$$

and

$$\Phi(A^2)^{\frac{1}{2}}\Phi(B^2)\Phi(A^2)^{\frac{1}{2}} - |\Phi(A^2)^{-\frac{1}{2}}\Phi(A^2\sharp B^2)\Phi(A^2)^{\frac{1}{2}}|^2 \le \frac{(M_1M_2 - m_1m_2)^2}{4} \times \frac{M_1^2}{m_1^2}. \tag{5.4}$$

Proof. Define a normalized positive linear map Ψ by

$$\Psi(X) := \Phi(A)^{-\frac{1}{2}}\Phi(A^{\frac{1}{2}}XA^{\frac{1}{2}})\Phi(A)^{-\frac{1}{2}}.$$

By using the Kantorovich inequality for the difference, it follows that

$$\Psi(X^2) - \Psi(X)^2 \le \frac{(M-m)^2}{4} \tag{5.5}$$

for all $0 < m \le X \le M$ with some scalars $m \le M$. As a matter of fact, we have

$$\begin{aligned}\Psi(X^2) - \Psi(X)^2 &\le \Psi((M+m)X - Mm) - \Psi(X)^2 \\ &= -\left(\Psi(X) - \frac{M+m}{2}\right)^2 + \frac{(M-m)^2}{4} \\ &\le \frac{(M-m)^2}{4}.\end{aligned}$$

If we put $X = (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}$, then due to

$$0 < (m =)\sqrt{\frac{m_2}{M_1}} \le X \le \sqrt{\frac{M_2}{m_1}}(= M)$$

we deduce from (5.5) that

$$\Phi(A)^{-\frac{1}{2}}\Phi(B)\Phi(A)^{-\frac{1}{2}} - \left(\Phi(A)^{-\frac{1}{2}}\Phi(A\sharp B)\Phi(A)^{-\frac{1}{2}}\right)^2 \le \frac{(\sqrt{M_1M_2} - \sqrt{m_1m_2})^2}{4M_1m_1}.$$

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Pre- and post-multiplying both sides by $\Phi(A)$, we obtain

$$\begin{aligned}\Phi(A)^{\frac{1}{2}}\Phi(B)\Phi(A)^{\frac{1}{2}} - |\Phi(A)^{-\frac{1}{2}}\Phi(A\sharp B)\Phi(A)^{\frac{1}{2}}|^2 &\le \frac{(\sqrt{M_1M_2}-\sqrt{m_1m_2})^2}{4M_1m_1}\Phi(A)^2\\ &\le \frac{(\sqrt{M_1M_2}-\sqrt{m_1m_2})^2}{4}\times\frac{M_1}{m_1},\end{aligned}$$

since $0 \le \Phi(A)^2 \le M_1^2$. Replacing A and B by A^2 and B^2 respectively, we have the desired inequality (5.4). Similarly, one can obtain (5.3). □

Remark 5.2. If Φ is a vector state in (5.3) and (5.4), then we get Ozeki–Izumino–Mori–Seo inequality (5.2).

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REFERENCES

[1] Lj. Arambašić, D. Bakić and M.S. Moslehian. A treatment of the Cauchy–Schwarz inequality in C^*-modules. *J. Math. Anal. Appl.*, to appear, doi:10.1016/j.jmaa.2011.02.062.
[2] P.S. Bullen. *A Dictionary of Inequalities.* Pitman Monographs and Surveys in Pure and Applied Mathematics, 97, Longman, Harlow, 1998.
[3] M.-D. Choi. Some assorted inequalities for positive linear maps on C^*-algebras. *J. Operator Theory*, 4:271–285, 1980.
[4] J.B. Diaz and F.T. Metcalf. Stronger forms of a class of inequalities of G. Pólya-G. Szegö and L.V. Kantorovich. *Bull. Amer. Math. Soc.*, 69:415–418, 1963.
[5] S.S. Dragomir. A survey on Cauchy–Bunyakovsky–Schwarz type discrete inequalities. *JIPAM J. Inequal. Pure Appl. Math.*, 4, 2003, Article 63.
[6] S.S. Dragomir. A counterpart of Schwarz inequality in inner product spaces. *RGMIA Res. Rep. Coll.*, 6, 2003, Article 18.
[7] S.S. Dragomir. *Advances in Inequalities of the Schwarz, Triangle and Heisenberg Type in Inner Product Spaces.* Nova Science Publishers, Inc., New York, 2007.
[8] D.E. Evans and J.T. Lewis, *Dilations of Irreversible Evolutions in Algebraic Quantum Theory.* Comm. Dublin Inst. Adv. Studies. Ser. A, no. 24, 1977.
[9] T. Furuta, J.M. Hot, J.E. Pečarić and Y. Seo. *Mond–Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space.* Monographs in Inequalities 1, Element, Zagreb, 2005.
[10] W. Greub and W. Rheinboldt. On a generalization of an inequality of L.V. Kantorovich. *Proc. Amer. Math. Soc.*, 10:407–415, 1959.
[11] D. Ilišević and S. Varošanec. On the Cauchy-Schwarz inequality and its reverse in semi-inner product C^*-modules. *Banach J. Math. Anal.*, 1:78–84, 2007.
[12] S. Izumino, H. Mori and Y. Seo. On Ozeki's inequality. *J. Inequal. Appl.*, 2:235–253, 1998.
[13] M. Joiţa. On the Cauchy–Schwarz inequality in C^*-algebras. *Math. Rep. (Bucur.)*, 3(53):243–246, 2001.
[14] M.S. Klamkin and R.G. Mclenaghan. An ellipse inequality. *Math. Mag.*, 50:261–263, 1977.
[15] E.C. Lance. *Hilbert C^*-Modules.* London Math. Soc. Lecture Note Series 210, Cambridge Univ. Press, 1995.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 179-190, March 2011

[16] E.-Y. Lee. A matrix reverse Cauchy-Schwarz inequality. *Linear Algebra Appl.*, 430:805–810, 2009.

[17] M.S. Moslehian and L.-E. Persson. Reverse Cauchy-Schwarz inequalities for positive C^*-valued sesquilinear forms. *Math. Inequal. Appl.*, 12:701–709, 2009.

[18] M.S. Moslehian and R. Rajić. Grüss inequality for n-positive linear maps. *Linear Alg. Appl.*, 433:1555–1560, 2010.

[19] G.J. Murphy, *C^*-algebras and Operator Theory.* Academic Press, Boston, 1990.

[20] C.P. Niculescu. Converses of the Cauchy–Schwarz inequality in the C^*-framework. *An. Univ. Craiova Ser. Mat. Inform.*, 26:22–28, 1999.

[21] M. Niezgoda. Accretive operators and Cassels inequality. *Linear Algebra Appl.*, 433:136–142, 2010.

[22] N. Ozeki. On the estimation of the inequalities by the maximum, or minimum values (in Japanese). *J. College Arts Sci. Chiba Univ.*, 5:199–203, 1968.

[23] G. Pólya and G. Szegö. *Aufgaben und Lehrsätze aus der Analysis*, Vol. 1, Berlin, 213–214, 1925.

[24] O. Shisha and B. Mond. Bounds on differences of means. *Inequalities I*, 293–308, 1967.

[25] G.S. Watson. Serial correlation in regression analysis. I. *Biometrika*, 42:327–342, 1955.