

## DERIVATIVES OF THE DIAMETER AND THE AREA OF A CONNECTED COMPONENT OF THE PSEUDOSPECTRUM\*

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**Abstract.** The paper concerns the relation between the following two quantities.

- (i) The Hölder condition number of an eigenvalue  $\lambda$  of a square complex matrix.
- (ii) The rate of growth of the diameter and the area of the connected component of the  $\varepsilon$ -pseudospectrum containing  $\lambda$ .

**Key words.** Matrices, Pseudospectra, Condition number, Eigenvalues, Derivatives.

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**1. Introduction.** By  $\Lambda(A)$  we denote the spectrum of any  $A \in \mathbb{C}^{n \times n}$ . We denote by  $\|\cdot\|$  the 2-norm. Let  $\lambda$  be an eigenvalue of  $A$  of algebraic multiplicity  $m$ . For  $X \in \mathbb{C}^{n \times n}$ ,  $sv_{(A,\lambda)}(X)$  denotes the radius of the smallest circle centered at  $\lambda$  containing  $m$  of the eigenvalues of  $X$  counting multiplicities. The (Hölder) condition number of the eigenvalue  $\lambda$  of order  $\omega > 0$  is defined as

$$\text{cond}_\omega(A, \lambda) := \lim_{\varepsilon \rightarrow 0^+} \max_{0 < \|X - A\| \leq \varepsilon} \frac{sv_{(A,\lambda)}(X)}{\|X - A\|^\omega}.$$

The index of an eigenvalue  $\lambda$  of  $A$ ,  $\nu = \nu(\lambda)$ , is the size of the largest Jordan block associated with  $\lambda$ . The limit that defines  $\text{cond}_\omega(A, \lambda)$  is of interest just for  $\omega = 1/\nu$ .

On the other hand, for  $\varepsilon \geq 0$ , the  $\varepsilon$ -pseudospectrum of  $A$  consists of the eigenvalues of all matrices  $X$  within an  $\varepsilon$ -neighborhood of  $A$ , i.e.,

$$\Lambda_\varepsilon(A) := \bigcup_{\substack{X \in \mathbb{C}^{n \times n} \\ \|X - A\| \leq \varepsilon}} \Lambda(X).$$

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For any complex matrix  $M$  we denote by  $\sigma_1(M) \geq \sigma_2(M) \geq \dots$  its singular values arranged in decreasing order. It is well known that

$$\Lambda_\varepsilon(A) = \{z \in \mathbb{C} : \sigma_n(zI - A) \leq \varepsilon\}.$$

The subset  $\Lambda_\varepsilon(A)$  of the complex plane is a compact set consisting of at most  $r$  (disjoint) connected components, one around each eigenvalue, where  $\Lambda(A) = \{\lambda_1, \dots, \lambda_r\}$ . Denote the connected component of  $\Lambda_\varepsilon(A)$  around the eigenvalue  $\lambda$  by  $\mathcal{K}_\lambda(\varepsilon)$ , and consider the diameter  $\delta(\varepsilon)$  and the area  $a(\varepsilon)$  of this component as a function of  $\varepsilon$ .

We denote by  $\partial\mathcal{K}_\lambda(\varepsilon)$  the boundary of  $\mathcal{K}_\lambda(\varepsilon)$ . From [6, Proposition 2.6.5], we deduce that

$$c = \lim_{\varepsilon \rightarrow 0^+} \frac{\max_{z \in \partial\mathcal{K}_\lambda(\varepsilon)} |z - \lambda|}{\varepsilon^{1/\nu}},$$

where  $c$  denotes the Hölder condition number  $\text{cond}_{1/\nu}(A, \lambda)$ . Thus, calling  $\rho(\varepsilon) := \max_{z \in \partial\mathcal{K}_\lambda(\varepsilon)} |z - \lambda|$ , we see that  $\lim_{\varepsilon \rightarrow 0^+} \frac{\rho(\varepsilon)}{\varepsilon^{1/\nu}} = c$ . We extend this result to  $\delta(\varepsilon)$  and  $a(\varepsilon)$  instead of  $\rho(\varepsilon)$  in Theorems 4.1 and 6.1.

REMARK 1.1. When  $\lim_{\varepsilon \rightarrow 0^+} \frac{\delta(\varepsilon)}{\varepsilon} = \infty$  (resp.,  $\lim_{\varepsilon \rightarrow 0^+} \frac{a(\varepsilon)}{\varepsilon} = \infty$ ), strictly speaking the function  $\delta(\varepsilon)$  (resp.,  $a(\varepsilon)$ ) is not differentiable from the right-hand side at 0. However, in this case we put  $\delta'_+(0) = \infty$  (resp.,  $a'_+(0) = \infty$ ) in order to grasp the geometric meaning of the results.

*The main results of the paper are as follows.*

1.  $\delta'_+(0) = 2c$  if  $\nu = 1$ . Otherwise  $\delta'_+(0) = \infty$ . Here,  $\delta'_+$  denotes the right-derivative of  $\delta$ . See Theorem 5.2.

2.  $a'_+(0) = 0$  if  $\nu = 1$ , and  $a'_+(0) = \pi c^2$  if  $\nu = 2$ . Otherwise,  $a'_+(0) = \infty$ . See Theorem 7.2.

In Section 2, we work four examples of matrices  $A$  for which both the condition number and the geometry of the  $\varepsilon$ -pseudospectrum are known in detail; this let us corroborate our results. We demonstrate that an important result by Karow [6] allows us bound the  $\varepsilon$ -pseudospectrum by lower and upper closed disks; see Section 3. By the monotonicity of the diameter function, and its changes under a homothecy, we show how to transfer these inequalities with respect to the inclusion relation between sets to the diameters, in Section 4. In Section 5, we relate the first right-derivative of the diameter at  $\varepsilon = 0$  with the condition number. In Section 6, the monotonicity of the area function, and its changes under a homothecy, let us translate the bounds in Section 3 to numeric inequalities. In Section 7, we relate the first right-derivative of the area at  $\varepsilon = 0$  with the condition number  $c$ ; moreover, we establish a relation between  $c$  and the second right-derivative  $a''_+(0)$  whenever this derivative exists. Finally, in

Section 8, we formulate a conjecture about the semialgebraicity of the functions  $\delta(\varepsilon)$  and  $a(\varepsilon)$ ; if it were true, a de l'Hôpital inverse rule would let us prove the existence of  $a''_+(0)$ .

**2. Examples.** Next, we consider four examples where we compute the condition number of order  $1/\nu$  of an eigenvalue  $\lambda$  and the right-derivatives at  $\varepsilon = 0$  of the diameter  $\delta(\varepsilon)$  and the area  $a(\varepsilon)$  of the connected component  $\mathcal{K}_\lambda(\varepsilon)$ .

EXAMPLE 2.1. Let  $A \in \mathbb{C}^{n \times n}$  be a normal matrix. Then the  $\varepsilon$ -pseudospectrum of  $A$  is the union of the closed disks of radius  $\varepsilon$  centered at the eigenvalues of  $A$ . So, for sufficiently small  $\varepsilon \geq 0$ , we have

$$\mathcal{K}_\lambda(\varepsilon) = \mathcal{D}(\lambda, \varepsilon).$$

Therefore,  $\rho(\varepsilon) = \varepsilon$ , and since the eigenvalues of a normal matrix are semisimple (i.e., of index 1),

$$\text{cond}_1(A, \lambda) = \lim_{\varepsilon \rightarrow 0^+} \frac{\varepsilon}{\varepsilon} = 1.$$

The diameter of  $\mathcal{D}(\lambda, \varepsilon)$  is  $2\varepsilon$ . So,  $\delta'(\varepsilon) = 2$  and  $\delta'_+(0) = 2$ . If we denote by  $a(\varepsilon)$  the area of this circle,  $a(\varepsilon) = \pi\varepsilon^2$ ; hence,  $a'(\varepsilon) = 2\pi\varepsilon, a''(\varepsilon) = 2\pi$ . Therefore,  $a'_+(0) = 0, a''_+(0) = 2\pi$ .

EXAMPLE 2.2. Let

$$J_2(\lambda, d) = \begin{bmatrix} \lambda & d \\ 0 & \lambda \end{bmatrix}$$

be like a Jordan block, with complex numbers  $\lambda, d$  and  $d \neq 0$ . Karow proved in [6, Theorem 5.4.1, p. 74] that for each  $\varepsilon \geq 0$ ,

$$\Lambda_\varepsilon(J_2(\lambda, d))$$

is a closed disk centered at  $\lambda$  and with radius  $r_2(\varepsilon) = \max\{r > 0 \mid \sigma_2(J_2(r, |d|)) \leq \varepsilon\}$ . As

$$\sigma_2(J_2(r, |d|)) = \sqrt{r^2 + \frac{|d|^2}{2}} - \sqrt{r^2|d|^2 + \frac{|d|^4}{4}},$$

solving the equation in the unknown  $r$

$$r^2 + \frac{|d|^2}{2} - \sqrt{r^2|d|^2 + \frac{|d|^4}{4}} = \varepsilon^2,$$

we find that  $r_2(\varepsilon) = \sqrt{\varepsilon^2 + |d|\varepsilon}$ . This result has also been proved by Cui et al. [4, Proposition 2.1]. So,  $\delta(\varepsilon) = 2\sqrt{\varepsilon^2 + |d|\varepsilon}$ . Hence,

$$\delta'(\varepsilon) = \frac{2\varepsilon + |d|}{\sqrt{\varepsilon^2 + |d|\varepsilon}}, \text{ which implies } \delta'_+(0) = \infty.$$

It is obvious that  $\nu(\lambda) = 2$ . Let us remark that

$$(2.1) \quad \lim_{\varepsilon \rightarrow 0^+} \frac{\delta(\varepsilon)}{\varepsilon^{1/2}} = 2\sqrt{|d|}.$$

Now, let us see that  $c := \text{cond}_{1/2}(J_2(\lambda, d), \lambda) = \sqrt{|d|}$ . We need some previous considerations. For a general matrix  $A \in \mathbb{C}^{n \times n}$ , let  $\Lambda(A) = \{\lambda_1, \dots, \lambda_r\}$ . Let

$$A = \sum_{j=1}^r (\lambda_j P_j + N_j)$$

be the Jordan decomposition of  $A$ , where for each  $j \in \{1, \dots, r\}$ ,  $P_j$  is the Riesz projector onto the root subspace (or generalized eigenspace)  $\mathcal{R}_{\lambda_j}(A)$  of  $\lambda_j$  and along the sum of root subspaces associated with all eigenvalues of  $A$  different from  $\lambda_j$ ; and  $N_j := (A - \lambda_j I_n)P_j$  is the nilpotent matrix corresponding to  $\lambda_j$ . By [6, Theorem 5.4.4 (viii), p. 78], if  $\nu_j := \nu(\lambda_j) > 1$ , then

$$(2.2) \quad \text{cond}_{1/\nu_j}(A, \lambda_j) = \|N_j^{\nu_j-1}\|^{1/\nu_j}.$$

In our present example,  $A = J_2(\lambda, d) = \lambda I_2 + N$ , where

$$N = \begin{bmatrix} 0 & d \\ 0 & 0 \end{bmatrix},$$

and thus,  $c = \|N\|^{1/2} = \sqrt{|d|}$ . From (2.1), we have

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\delta(\varepsilon)}{\varepsilon^{1/2}} = 2c.$$

The area  $a(\varepsilon)$  is given by  $\pi(\varepsilon^2 + |d|\varepsilon)$ . So,  $a'(\varepsilon) = \pi(2\varepsilon + |d|)$ ; hence,  $a'_+(0) = \pi|d| = \pi c^2$ . This concludes Example 2.2.

Before discussing the third example, we need to introduce a result on the pseudospectra of nilpotent matrices of nilpotency index two. Here on, we denote by  $O_k$  the  $k \times k$  zero matrix.

**PROPOSITION 2.3** ([5], Theorem 3). *Let us assume that  $q, r$  are nonnegative integers such that  $n = 2q + r$ . Let  $N \in \mathbb{C}^{n \times n}$  be a matrix such that  $N^2 = O_n$ , whose nonzero singular values are  $\sigma_1(N) \geq \dots \geq \sigma_q(N)$ . Then there exists a unitary matrix  $U \in \mathbb{C}^{n \times n}$  such that*

$$U^* N U = \begin{bmatrix} 0 & \sigma_1(N) \\ 0 & 0 \end{bmatrix} \oplus \dots \oplus \begin{bmatrix} 0 & \sigma_q(N) \\ 0 & 0 \end{bmatrix} \oplus O_r.$$

By Example 2.2, or [4, Proposition 2.1] by Cui et al., the  $\varepsilon$ -pseudospectrum of

$$\begin{bmatrix} 0 & \sigma_i(N) \\ 0 & 0 \end{bmatrix}$$

is  $\mathcal{D}(0, \sqrt{\varepsilon^2 + \varepsilon\sigma_i(N)})$  for  $i = 1, \dots, q$ . So, by [6, Propositions 5.2.3 and 5.2.4], we have the following result.

PROPOSITION 2.4. *Under the hypotheses of Proposition 2.3 for  $\varepsilon \geq 0$ ,*

$$\begin{aligned} \Lambda_\varepsilon(N) &= \bigcup_{i=1}^q \mathcal{D}(0, \sqrt{\varepsilon^2 + \varepsilon\sigma_i(N)}) \cup \mathcal{D}(0, \varepsilon) \\ &= \mathcal{D}(0, \sqrt{\varepsilon^2 + \varepsilon\sigma_1(N)}). \end{aligned}$$

We will also need that for any  $\alpha \in \mathbb{C}$ ,  $A \in \mathbb{C}^{n \times n}$  and  $\varepsilon \geq 0$ ,

$$(2.3) \quad \Lambda_\varepsilon(\alpha I_n + A) = \alpha + \Lambda_\varepsilon(A).$$

EXAMPLE 2.5. This example is a small generalization of Example 2.2. Let  $A$  be any  $n$ -by- $n$  complex matrix with a unique eigenvalue  $\lambda$ . Moreover, let us assume that  $\nu(\lambda) = 2$ . Let  $N := A - \lambda I_n$ ; thus,  $A = \lambda I_n + N$  is the Jordan decomposition of  $A$ . Hence, by (2.2),  $c = \text{cond}_{1/2}(A, \lambda) = \|N\|^{1/2} = \sqrt{\sigma_1(A - \lambda I_n)}$ . By Proposition 2.4 and (2.3) we see that for  $\varepsilon \geq 0$ ,

$$\Lambda_\varepsilon(A) = \mathcal{D}(\lambda, \sqrt{\varepsilon^2 + \varepsilon\sigma_1(A - \lambda I_n)}).$$

So,  $\delta(\varepsilon) = 2\sqrt{\varepsilon^2 + \varepsilon\sigma_1(A - \lambda I_n)}$ ,  $a(\varepsilon) = \pi(\varepsilon^2 + \varepsilon\sigma_1(A - \lambda I_n))$ . Therefore,

$$\delta'(\varepsilon) = \frac{2\varepsilon + \sigma_1(A - \lambda I_n)}{\sqrt{\varepsilon^2 + \varepsilon\sigma_1(A - \lambda I_n)}}, \text{ which implies } \delta'_+(0) = \infty,$$

and

$$a'(\varepsilon) = \pi(2\varepsilon + \sigma_1(A - \lambda I_n)), \text{ which implies } a'_+(0) = \pi\sigma_1(A - \lambda I_n) = \pi c^2.$$

EXAMPLE 2.6. Let  $\lambda_1, \lambda_2$  be two different complex numbers. Let  $A \in \mathbb{C}^{2 \times 2}$  whose eigenvalues are  $\lambda_1$  and  $\lambda_2$ . Let us define

$$d(A) := \sqrt{\text{tr}(A^*A) - |\lambda_1|^2 - |\lambda_2|^2}.$$

The number  $d(A)$  is the departure from normality of  $A$ . In [6, Proposition 5.5.3, p. 80] was proved that for each  $\varepsilon > 0$ ,

$$(2.4) \quad \Lambda_\varepsilon(A) = \mathcal{D}(\lambda_1, \varepsilon) \cup \mathcal{D}(\lambda_2, \varepsilon) \cup \mathcal{M}_{\lambda_1, \lambda_2}(d(A), \varepsilon)$$

where

$$(2.5) \quad \mathcal{M}_{\lambda_1, \lambda_2}(d, \varepsilon) := \{z \in \mathbb{C} : (|z - \lambda_1|^2 - \varepsilon^2)(|z - \lambda_2|^2 - \varepsilon^2) \leq \varepsilon^2 d^2\}, \quad d \geq 0.$$

Let us consider Figure 2.1 that shows the  $\varepsilon$ -pseudospectrum for the values of  $\varepsilon = 1.00, 1.50, 1.75$  of the matrix

$$A_1 := \begin{bmatrix} 1 + 2i & 3 \\ 0 & -1 - 3i \end{bmatrix}.$$

So,  $\lambda_1 = 1 + 2i, \lambda_2 = -1 - 3i, \|A_1\|_F = \sqrt{|1 + 2i|^2 + 3^2 + |-1 - 3i|^2} = \sqrt{5 + 9 + 10} = \sqrt{24}$ . Thus,  $d(A_1) = \sqrt{24 - 5 - 10} = \sqrt{9} = 3$ .

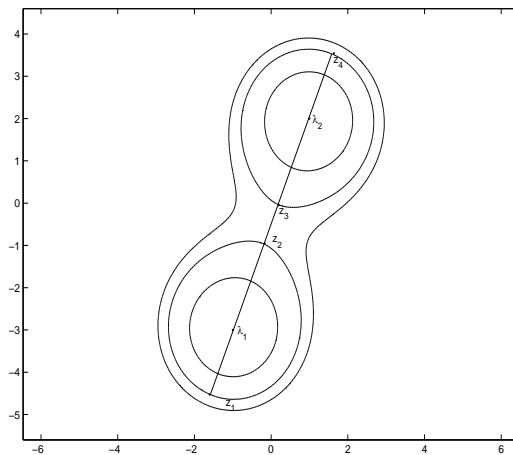


FIG. 2.1. Pseudospectra of  $A_1$  for  $\varepsilon = 1.00, 1.50, 1.75$ .

We see that for sufficiently small values of  $\varepsilon$  the  $\varepsilon$ -pseudospectrum of  $A_1$  has two connected components. Let us fix our attention on  $\varepsilon = 1.50$ . The figure contains a straight line that passes by  $\lambda_1$  and  $\lambda_2$ . This line intersects the boundary  $\partial\Lambda_\varepsilon(A_1)$  at the points  $z_1, z_2, z_3, z_4$ . If we consider the eigenvalue  $\lambda_2$ , the diameter  $\delta(\varepsilon)$  of the connected component  $\mathcal{K}_{\lambda_2}(\varepsilon)$  is equal to the distance between  $z_3$  and  $z_4$ . From (2.4) and (2.5), we can deduce that

$$\delta(\varepsilon) = \sqrt{1 + \varepsilon^2 + \sqrt{7}\varepsilon} - \sqrt{1 + \varepsilon^2 - \sqrt{7}\varepsilon}.$$

Instead of making a special reasoning for the matrix  $A_1$ , we are going to find a general expression for the diameter  $\delta(\varepsilon)$  of  $\mathcal{K}_{\lambda_2}(\varepsilon)$  for any matrix  $A \in \mathbb{C}^{2 \times 2}$  with eigenvalues  $\lambda_1$  and  $\lambda_2$ . By (2.5), the boundary  $\partial\Lambda_\varepsilon(A)$  is formed by quasi-Cassini ovals with foci the points  $\lambda_1$  and  $\lambda_2$ . In fact, the set  $\Lambda_\varepsilon(A)$  is symmetric about the straight line that joints  $\lambda_1$  and  $\lambda_2$ . This is a consequence of (2.4) and (2.5). Moreover,  $\delta(\varepsilon) = |z_4 - z_3| = |z_2 - z_1|$ . Later we will need the condition numbers of  $\lambda_1$  and  $\lambda_2$  of

order 1. These numbers are equal. In fact, by [6, Proposition 5.5.8, p. 83], we have

$$(2.6) \quad \text{cond}_1(A, \lambda_k) = \sqrt{1 + \left(\frac{d(A)}{|\lambda_2 - \lambda_1|}\right)^2}, \quad k = 1, 2.$$

Thus, let  $\varepsilon \geq 0$  be such that  $\Lambda_\varepsilon(A)$  has two connected components. Now, we compute the intersection points  $z_1, z_2, z_3, z_4$  of the line  $z(t) := (1-t)\lambda_1 + t\lambda_2, t \in \mathbb{R}$  with the curve  $\partial\Lambda_\varepsilon(A)$ . By (2.5), this curve is given by the equation

$$(2.7) \quad (|z - \lambda_1|^2 - \varepsilon^2)(|z - \lambda_2|^2 - \varepsilon^2) - \varepsilon^2 d(A)^2 = 0.$$

For determining the values of the parameter  $t$  that correspond to the points  $z_1, z_2, z_3$  and  $z_4$  we substitute  $z(t)$  into (2.7),

$$|\lambda_1 - \lambda_2|^4 t^4 - 2|\lambda_1 - \lambda_2|^4 t^3 + (|\lambda_1 - \lambda_2|^4 - 2\varepsilon^2|\lambda_1 - \lambda_2|^2) t^2 + 2\varepsilon^2|\lambda_1 - \lambda_2|^2 t - \varepsilon^2|\lambda_1 - \lambda_2|^2 + \varepsilon^4 - \varepsilon^2 d(A)^2 = 0.$$

For simplicity, we write  $\theta := |\lambda_1 - \lambda_2|$ ,

$$\theta^4 t^4 - 2\theta^4 t^3 + (\theta^4 - 2\varepsilon^2\theta^2) t^2 + 2\varepsilon^2\theta^2 t - \varepsilon^2\theta^2 + \varepsilon^4 - \varepsilon^2 d(A)^2 = 0.$$

With the command `solve` of MAPLE 13 we find the roots of this equation in  $t$  obtaining

$$\begin{aligned} t_1 &= \frac{1}{2} - \frac{1}{2\theta} \sqrt{\theta^2 + 4\varepsilon^2 + 4\varepsilon\sqrt{\theta^2 + d(A)^2}}, \\ t_2 &= \frac{1}{2} - \frac{1}{2\theta} \sqrt{\theta^2 + 4\varepsilon^2 - 4\varepsilon\sqrt{\theta^2 + d(A)^2}}, \\ t_3 &= \frac{1}{2} + \frac{1}{2\theta} \sqrt{\theta^2 + 4\varepsilon^2 - 4\varepsilon\sqrt{\theta^2 + d(A)^2}}, \\ t_4 &= \frac{1}{2} + \frac{1}{2\theta} \sqrt{\theta^2 + 4\varepsilon^2 + 4\varepsilon\sqrt{\theta^2 + d(A)^2}}, \end{aligned}$$

where  $t_1 < 0 < t_2 < \frac{1}{2} < t_3 < 1 < t_4$ . Hence,  $z_3 = z(t_3)$  and  $z_4 = z(t_4)$ . So,

$$\delta(\varepsilon) = |z_4 - z_3| = \frac{1}{2} \sqrt{\theta^2 + 4\varepsilon^2 + 4\varepsilon\sqrt{\theta^2 + d(A)^2}} - \frac{1}{2} \sqrt{\theta^2 + 4\varepsilon^2 - 4\varepsilon\sqrt{\theta^2 + d(A)^2}}.$$

Therefore,

$$\delta'(\varepsilon) = \frac{2\varepsilon + \sqrt{\theta^2 + d(A)^2}}{\sqrt{\theta^2 + 4\varepsilon^2 + 4\varepsilon\sqrt{\theta^2 + d(A)^2}}} - \frac{2\varepsilon - \sqrt{\theta^2 + d(A)^2}}{\sqrt{\theta^2 + 4\varepsilon^2 - 4\varepsilon\sqrt{\theta^2 + d(A)^2}}},$$

$$\delta'_+(0) = \frac{2\sqrt{\theta^2 + d(A)^2}}{\sqrt{\theta^2}} = 2\sqrt{1 + \left(\frac{d(A)}{\theta}\right)^2}.$$

By (2.6), we have  $\delta'_+(0) = 2 \operatorname{cond}_1(A, \lambda_2)$ .

REMARK 2.7. As discussed in the introduction, we will see that the results in these examples are not casual. In fact, we will prove that the condition number of  $\lambda$  of order  $1/\nu$  is related with the functions  $\delta$  and  $a$ .

**3. Bounds by closed disks.** From Theorems 2.6.6 and 5.4.4 of Karow [6], we infer the following theorem.

THEOREM 3.1. *Let  $\lambda$  be an eigenvalue of  $A \in \mathbb{C}^{n \times n}$  of index  $\nu$ . For each  $\eta \in (0, 1]$  there exists an  $\varepsilon_\eta > 0$  such that for every  $\varepsilon \in (0, \varepsilon_\eta]$ ,*

$$(3.1) \quad \mathcal{D}(\lambda, ((1 - \eta)\varepsilon)^{1/\nu} c) \subset \mathcal{K}_\lambda(\varepsilon) \subset \mathcal{D}(\lambda, ((1 + \eta)\varepsilon)^{1/\nu} c)$$

$c$  being the condition number of  $\lambda$  of order  $1/\nu$ .

From (3.1), we deduce that

$$\mathcal{D}(0, ((1 - \eta)\varepsilon)^{1/\nu} c) \subset \mathcal{K}_\lambda(\varepsilon) - \lambda \subset \mathcal{D}(0, ((1 + \eta)\varepsilon)^{1/\nu} c).$$

Considering the homothety

$$z \mapsto \frac{z}{\varepsilon^{1/\nu}}, \quad z \in \mathbb{C},$$

we see

$$(3.2) \quad \mathcal{D}(0, (1 - \eta)^{1/\nu} c) \subset \frac{\mathcal{K}_\lambda(\varepsilon) - \lambda}{\varepsilon^{1/\nu}} \subset \mathcal{D}(0, (1 + \eta)^{1/\nu} c).$$

**4. Bounds by diameters.** Since the diameter function is monotone increasing with respect to  $\subset$ , by (3.2),

$$(4.1) \quad 2c(1 - \eta)^{1/\nu} \leq \frac{\delta(\varepsilon)}{\varepsilon^{1/\nu}} \leq 2c(1 + \eta)^{1/\nu}.$$

Thus, we arrive at the following theorem.

THEOREM 4.1. *Let  $\nu$  be the index of  $\lambda$ , and let  $c$  be the condition number of  $\lambda$  of order  $1/\nu$ . Then*

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\delta(\varepsilon)}{\varepsilon^{1/\nu}} = 2c.$$

*Proof.* Let us define the functions

$$\begin{aligned} f_1(\eta) &:= 1 - (1 - \eta)^{1/\nu}, \\ f_2(\eta) &:= (1 + \eta)^{1/\nu} - 1, \end{aligned}$$



where  $\eta \in [0, 1]$ . If  $\nu = 1$ , then

$$\begin{aligned} f_1(\eta) &:= 1 - 1 + \eta = \eta, \\ f_2(\eta) &:= 1 + \eta - 1 = \eta; \end{aligned}$$

so,  $f_1(\eta) = f_2(\eta)$  on  $[0, 1]$ .

If  $\nu \geq 2$ , we will deduce that  $f_2(\eta) \leq f_1(\eta)$ . This last inequality is equivalent to

$$\begin{aligned} (1 + \eta)^{1/\nu} - 1 &\leq 1 - (1 - \eta)^{1/\nu}, \quad \forall \eta \in [0, 1] \\ \iff (1 + \eta)^{1/\nu} + (1 - \eta)^{1/\nu} &\leq 2, \quad \forall \eta \in [0, 1]. \end{aligned}$$

Let us define

$$g(\eta) := (1 + \eta)^{1/\nu} + (1 - \eta)^{1/\nu}, \quad \eta \in [0, 1].$$

Then

$$g'(\eta) = \frac{1}{\nu} [(1 + \eta)^{1/\nu-1} - (1 - \eta)^{1/\nu-1}].$$

Since  $x^\alpha := e^{\alpha \ln x}$ , when  $\alpha < 0$  the function  $x \mapsto x^\alpha$  is decreasing in  $(0, \infty)$ ; therefore,  $x \mapsto x^{1/\nu-1}$  is decreasing in  $(0, \infty)$ . Hence, if  $0 < \eta < 1$ , we see that  $1 - \eta < 1 + \eta$ ; what implies

$$(1 - \eta)^{1/\nu-1} > (1 + \eta)^{1/\nu-1}.$$

Thus,  $g'(\eta) < 0$ . So,  $g$  is decreasing on  $[0, 1]$ . Therefore,  $g(\eta) < g(0) = 2$ . That is,  $f_2(\eta) \leq f_1(\eta)$ .

Given that  $2c - 2cf_1(\eta) = 2c(1 - \eta)^{1/\nu}$  and  $2c + 2cf_2(\eta) = 2c(1 + \eta)^{1/\nu}$ , by (4.1), we have

$$2c - 2cf_1(\eta) \leq \frac{\delta(\varepsilon)}{\varepsilon^{1/\nu}} \leq 2c + 2cf_2(\eta).$$

For every  $\nu \geq 1$ ,  $f_2(\eta) \leq f_1(\eta)$  for  $\eta \in [0, 1]$ . Then,

$$2c - 2cf_1(\eta) \leq \frac{\delta(\varepsilon)}{\varepsilon^{1/\nu}} \leq 2c + 2cf_1(\eta)$$

$$\iff -2cf_1(\eta) \leq \frac{\delta(\varepsilon)}{\varepsilon^{1/\nu}} - 2c \leq 2cf_1(\eta),$$

or

$$\left| \frac{\delta(\varepsilon)}{\varepsilon^{1/\nu}} - 2c \right| \leq 2cf_1(\eta).$$

But  $\lim_{\eta \rightarrow 0^+} f_1(\eta) = 0$  and  $f_1(\eta) > 0$  for  $\eta > 0$ . Thus, for a fixed  $\eta_0 > 0$ , there exists an  $\eta_1 > 0$  such that  $2cf_1(\eta_1) < \eta_0$ . For this  $\eta_1$ , there is an  $\varepsilon_{\eta_1} > 0$  such that for all  $\varepsilon \in (0, \varepsilon_{\eta_1})$ ,

$$\left| \frac{\delta(\varepsilon)}{\varepsilon^{1/\nu}} - 2c \right| \leq 2cf_1(\eta) < \eta_0.$$

So, there exists the limit

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\delta(\varepsilon)}{\varepsilon^{1/\nu}}$$

and it is equal to  $2c$ .  $\square$

**5. Derivatives of the diameter.** In this section, we relate the right-derivative of the diameter  $\delta$  at 0 with the condition number of the eigenvalue  $\lambda$ , when  $\nu = 1$ . First, we have the following lemma.

LEMMA 5.1. *Let  $A \in \mathbb{C}^{n \times n}$  and  $\lambda$  be an eigenvalue of  $A$  of index  $\nu$ . Let  $\delta(\varepsilon)$  be the diameter of the connected component of  $\Lambda_\varepsilon(A)$  that contains  $\lambda$ . Let  $c$  be the condition number of  $\lambda$  of order  $1/\nu$ . Then, there exists the limit*

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\delta(\varepsilon)}{\varepsilon}$$

and it is equal to

$$\begin{cases} 2c & \text{if } \nu = 1, \\ \infty & \text{if } \nu \geq 2. \end{cases}$$

*Proof.* If  $\nu = 1$ , Theorem 4.1 implies

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\delta(\varepsilon)}{\varepsilon} = 2c.$$

If  $\nu \geq 2$ , from the same Theorem,

$$\frac{\delta(\varepsilon)}{\varepsilon} = \frac{\delta(\varepsilon)}{\varepsilon^{1/\nu}} \frac{1}{\varepsilon^{1-1/\nu}} \rightarrow 2c \cdot \infty,$$

when  $\varepsilon \rightarrow 0^+$ .  $\square$

From this lemma, the next theorem follows immediately.

THEOREM 5.2. *Let  $A \in \mathbb{C}^{n \times n}$  and  $\lambda$  be an eigenvalue of  $A$  of index  $\nu$ . Let  $\delta(\varepsilon)$  be the diameter of the connected component of  $\Lambda_\varepsilon(A)$  that contains  $\lambda$ . Let  $c$  be the condition number of  $\lambda$  of order  $1/\nu$ . Then,*

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\delta(\varepsilon)}{\varepsilon} = \begin{cases} 2c & \text{if } \nu = 1, \\ \infty & \text{if } \nu \geq 2. \end{cases}$$

Therefore,  $\delta'_+(0) = 2c$  if  $\nu = 1$ .

**6. Bounds by areas.** Since the area function is monotone increasing with respect to  $\subset$ , by (3.2),

$$(6.1) \quad \pi c^2(1 - \eta)^{2/\nu} \leq \frac{a(\varepsilon)}{\varepsilon^{2/\nu}} \leq \pi c^2(1 + \eta)^{2/\nu},$$

where  $a(\varepsilon) := \text{area or Lebesgue measure of } \mathcal{K}_\lambda(\varepsilon)$ . Thus, we arrive at the following theorem.

**THEOREM 6.1.** *Let  $\nu$  be the index of  $\lambda$ , and let  $c$  be the condition number of  $\lambda$  of order  $1/\nu$ . Then*

$$\lim_{\varepsilon \rightarrow 0^+} \frac{a(\varepsilon)}{\varepsilon^{2/\nu}} = \pi c^2.$$

*Proof.* Let us define the functions

$$(6.2) \quad \varphi_1(\eta) := 1 - (1 - \eta)^{2/\nu},$$

$$(6.3) \quad \varphi_2(\eta) := (1 + \eta)^{2/\nu} - 1.$$

If  $\nu = 1$ , then

$$\begin{aligned} \varphi_1(\eta) &= 1 - (1 - \eta)^2 = 2\eta - \eta^2, \\ \varphi_2(\eta) &= (1 + \eta)^2 - 1 = 2\eta + \eta^2; \end{aligned}$$

it is obvious that  $\varphi_1(\eta) \leq \varphi_2(\eta)$  when  $0 \leq \eta$ .

If  $\nu = 2$ , then

$$\varphi_1(\eta) = 1 - (1 - \eta) = \eta = (1 + \eta) - 1 = \varphi_2(\eta).$$

If  $\nu \geq 3$ , we will see that

$$\varphi_2(\eta) \leq \varphi_1(\eta), \quad \eta \in [0, 1],$$

or, equivalently,

$$(1 + \eta)^{2/\nu} - 1 \leq 1 - (1 - \eta)^{2/\nu}, \quad \eta \in [0, 1]$$

$$\iff (1 + \eta)^{2/\nu} + (1 - \eta)^{2/\nu} \leq 2, \quad \eta \in [0, 1].$$

Let us define

$$\psi(\eta) := (1 + \eta)^{2/\nu} + (1 - \eta)^{2/\nu}, \quad \eta \in [0, 1].$$

Then

$$\psi'(\eta) = \frac{2}{\nu} \left[ (1 + \eta)^{2/\nu-1} - (1 - \eta)^{2/\nu-1} \right].$$

As the function  $x \mapsto x^{2/\nu-1}$  is decreasing in  $(0, \infty)$ , if  $0 < \eta < 1$ , then  $1 - \eta < 1 + \eta$  implies  $(1 - \eta)^{2/\nu-1} > (1 + \eta)^{2/\nu-1}$  and  $\psi'(\eta) < 0$ . In consequence,  $\psi$  is decreasing in  $[0, 1]$ . Thus, for  $\eta \in (0, 1]$ ,  $\psi(\eta) < \psi(0) = 2$ . Accordingly,

$$\varphi_2(\eta) \leq \varphi_1(\eta), \quad \eta \in [0, 1].$$

By (6.2) and (6.3),

$$\begin{aligned} (1 - \eta)^{2/\nu} &= 1 - \varphi_1(\eta), \\ (1 + \eta)^{2/\nu} &= 1 + \varphi_2(\eta). \end{aligned}$$

Inequalities (6.1) imply

$$(6.4) \quad \pi c^2 - \pi c^2 \varphi_1(\eta) \leq \frac{a(\varepsilon)}{\varepsilon^{2/\nu}} \leq \pi c^2 + \pi c^2 \varphi_2(\eta).$$

**The  $\nu \leq 2$  case.** Since  $\varphi_1(\eta) \leq \varphi_2(\eta)$  in  $[0, 1]$ ,

$$-\pi c^2 \varphi_2(\eta) \leq -\pi c^2 \varphi_1(\eta).$$

Hence, by (6.4),

$$\pi c^2 - \pi c^2 \varphi_2(\eta) \leq \frac{a(\varepsilon)}{\varepsilon^{2/\nu}} \leq \pi c^2 + \pi c^2 \varphi_2(\eta);$$

which is equivalent to

$$\left| \frac{a(\varepsilon)}{\varepsilon^{2/\nu}} - \pi c^2 \right| \leq \pi c^2 \varphi_2(\eta).$$

Bearing

$$\lim_{\eta \rightarrow 0^+} \pi c^2 \varphi_2(\eta) = 0$$

in mind, we deduce that for a fixed  $\eta_0 > 0$  there exists an  $\eta_1 > 0$  such that  $\pi c^2 \varphi_2(\eta_1) < \eta_0$ . For this  $\eta_1$  there is an  $\varepsilon_{\eta_1} > 0$  such that for  $\varepsilon \in (0, \varepsilon_{\eta_1}]$ ,

$$\left| \frac{a(\varepsilon)}{\varepsilon^{2/\nu}} - \pi c^2 \right| \leq \pi c^2 \varphi_2(\eta_1) < \eta_0.$$

This proves that there exists the limit

$$\lim_{\varepsilon \rightarrow 0^+} \frac{a(\varepsilon)}{\varepsilon^{2/\nu}}$$

and it is equal to  $\pi c^2$ .

**The  $\nu \geq 3$  case.** As  $\varphi_2(\eta) \leq \varphi_1(\eta)$  in  $[0, 1]$ , from (6.4) we deduce that

$$\pi c^2 - \pi c^2 \varphi_1(\eta) \leq \frac{a(\varepsilon)}{\varepsilon^{2/\nu}} \leq \pi c^2 + \pi c^2 \varphi_1(\eta)$$

and, as  $\varphi_1(\eta) > 0$  if  $\eta > 0$  and

$$\lim_{\eta \rightarrow 0^+} \varphi_1(\eta) = 0,$$

by a reasoning analogous to the former one we infer that there exists the limit

$$\lim_{\varepsilon \rightarrow 0^+} \frac{a(\varepsilon)}{\varepsilon^{2/\nu}} = \pi c^2. \quad \square$$

**7. Derivatives of the area.** In this section, we establish the relation between the first and second right-derivatives of  $a$  at 0 and the condition number of the eigenvalue  $\lambda$  of order  $1/\nu$ , when  $\nu = 1$  or 2. First, we prove the following lemma.

LEMMA 7.1. *Let  $A \in \mathbb{C}^{n \times n}$  and  $\lambda$  be an eigenvalue of  $A$  of index  $\nu$ . Let  $a(\varepsilon)$  be the area of the connected component of  $\Lambda_\varepsilon(A)$  that contains  $\lambda$ . Let  $c$  be the condition number of  $\lambda$  of order  $1/\nu$ . Then, there exists the limit*

$$\lim_{\varepsilon \rightarrow 0^+} \frac{a(\varepsilon)}{\varepsilon}$$

and it is equal to

$$\begin{cases} 0 & \text{if } \nu = 1, \\ \pi c^2 & \text{if } \nu = 2, \\ \infty & \text{if } \nu \geq 3. \end{cases}$$

*Proof.* First, let us suppose that  $\nu = 1$ . Then Theorem 6.1 implies

$$\lim_{\varepsilon \rightarrow 0^+} \frac{a(\varepsilon)}{\varepsilon^2} = \pi c^2.$$

Hence,

$$\lim_{\varepsilon \rightarrow 0^+} \frac{a(\varepsilon)}{\varepsilon} = \lim_{\varepsilon \rightarrow 0^+} \varepsilon \frac{a(\varepsilon)}{\varepsilon^2} = \left( \lim_{\varepsilon \rightarrow 0^+} \varepsilon \right) \left( \lim_{\varepsilon \rightarrow 0^+} \frac{a(\varepsilon)}{\varepsilon^2} \right) = 0 \cdot \pi c^2 = 0.$$

Second, let us assume now that  $\nu = 2$ . From Theorem 6.1,

$$\lim_{\varepsilon \rightarrow 0^+} \frac{a(\varepsilon)}{\varepsilon} = \pi c^2.$$

Finally, when  $\nu \geq 3$ , it is obvious, by Theorem 6.1 and

$$\lim_{\varepsilon \rightarrow 0^+} \frac{a(\varepsilon)}{\varepsilon} = \lim_{\varepsilon \rightarrow 0^+} \frac{a(\varepsilon)}{\varepsilon^{2/\nu}} \cdot \frac{1}{\varepsilon^{(\nu-2)/\nu}},$$

that

$$\lim_{\varepsilon \rightarrow 0^+} \frac{a(\varepsilon)}{\varepsilon} = \infty. \quad \square$$

**THEOREM 7.2.** *Let  $A \in \mathbb{C}^{n \times n}$  and  $\lambda$  be an eigenvalue of  $A$  of index  $\nu$ . Let  $a(\varepsilon)$  be the area of the connected component  $\Lambda_\varepsilon(A)$  that contains  $\lambda$ . Then, there exists the right-derivative of  $a$  at 0,  $a'_+(0)$ , and*

$$a'_+(0) = \begin{cases} 0 & \text{if } \nu = 1, \\ \pi c^2 & \text{if } \nu = 2, \\ \infty & \text{if } \nu \geq 3. \end{cases}$$

*Proof.* By Lemma 7.1 we deduce that

if  $\nu = 1$ ,

$$a'_+(0) = \lim_{\varepsilon \rightarrow 0^+} \frac{a(\varepsilon) - a(0)}{\varepsilon} = \lim_{\varepsilon \rightarrow 0^+} \frac{a(\varepsilon)}{\varepsilon} = 0;$$

if  $\nu = 2$ ,

$$a'_+(0) = \lim_{\varepsilon \rightarrow 0^+} \frac{a(\varepsilon) - a(0)}{\varepsilon} = \pi c^2;$$

if  $\nu \geq 3$ ,

$$a'_+(0) = \lim_{\varepsilon \rightarrow 0^+} \frac{a(\varepsilon) - a(0)}{\varepsilon} = \infty. \quad \square$$

**THEOREM 7.3.** *Let  $A \in \mathbb{C}^{n \times n}$  and  $\lambda$  be a semisimple eigenvalue. Let  $a(\varepsilon)$  be the area of the connected component  $\Lambda_\varepsilon(A)$  that contains  $\lambda$ . Let us assume that there exists  $a''_+(0)$ . Then*

$$a''_+(0) = 2\pi c^2.$$

*Proof.* Let us define the function

$$A(\varepsilon) := \begin{cases} a(\varepsilon) & \text{if } \varepsilon \geq 0, \\ a(-\varepsilon) & \text{if } \varepsilon < 0. \end{cases}$$

By Theorem 7.2, as  $\nu = 1$ , there exists  $A'(0)$  and  $A'(0) = 0$ . If we suppose that there exists the derivative  $A''(0)$ , then

$$A''(0) = \lim_{\varepsilon \rightarrow 0} \frac{A(\varepsilon) - 2A(0) + A(-\varepsilon)}{\varepsilon^2}.$$

But, the existence of this limit does not imply the existence of  $A''(0)$ . See [1, Exercise 5–20.].

When  $\varepsilon > 0$ ,  $A(\varepsilon) = a(\varepsilon)$ ,  $A(-\varepsilon) = a(-(-\varepsilon)) = a(\varepsilon)$ ; therefore,  $A(\varepsilon) + A(-\varepsilon) = 2a(\varepsilon)$ .

When  $\varepsilon < 0$ ,  $A(\varepsilon) = a(-\varepsilon)$ ,  $A(-\varepsilon) = a(-\varepsilon)$ ; hence,  $A(\varepsilon) + A(-\varepsilon) = 2a(-\varepsilon)$ .

Then, by Theorem 6.1,

$$\lim_{\varepsilon \rightarrow 0^+} \frac{A(\varepsilon) - 2A(0) + A(-\varepsilon)}{\varepsilon^2} = \lim_{\varepsilon \rightarrow 0^+} \frac{2a(\varepsilon)}{\varepsilon^2} = 2\pi c^2,$$

$$\lim_{\varepsilon \rightarrow 0^-} \frac{A(\varepsilon) - 2A(0) + A(-\varepsilon)}{\varepsilon^2} = \lim_{\varepsilon \rightarrow 0^-} \frac{2a(-\varepsilon)}{(-\varepsilon)^2} = \lim_{\beta \rightarrow 0^+} \frac{2a(\beta)}{\beta^2} = 2\pi c^2.$$

Consequently, there exists the limit

$$\lim_{\varepsilon \rightarrow 0} \frac{A(\varepsilon) - 2A(0) + A(-\varepsilon)}{\varepsilon^2}$$

and is equal to  $2\pi c^2$ . So,  $A''(0) = 2\pi c^2$ , and  $a''_+(0) = 2\pi c^2$ .  $\square$

**8. A conjecture.** Let  $\lambda$  be a semisimple eigenvalue of a matrix  $A \in \mathbb{C}^{n \times n}$ . We need the definition of a semialgebraic set  $S \subset \mathbb{R}^n$  and of a semialgebraic function  $f : S \rightarrow \mathbb{R}$ . These concepts can be seen in [6, Chapter 3, p. 39]. A classical reference is [2, Chapter 2, p. 23].

CONJECTURE 8.1. The functions  $\delta, a : [0, \infty) \rightarrow \mathbb{R}$  are semialgebraic.

The following theorem is proved in [3, Lemma 3.1 (ii)].

THEOREM 8.2 (de l'Hôpital inverse rule). *If  $f, g : [0, \infty) \rightarrow \mathbb{R}$  are semialgebraic functions,  $f(0) = g(0) = 0$ , and there is an  $\varepsilon_0 > 0$  such that  $g'(\varepsilon) > 0$  for  $\varepsilon \in (0, \varepsilon_0)$ , then*

$$\lim_{\varepsilon \rightarrow 0^+} \frac{f(\varepsilon)}{g(\varepsilon)} = \ell \in \mathbb{R} \implies \lim_{\varepsilon \rightarrow 0^+} \frac{f'(\varepsilon)}{g'(\varepsilon)} = \ell.$$

If Conjecture 8.1 were true, then

$$\pi c^2 = \lim_{\varepsilon \rightarrow 0^+} \frac{a(\varepsilon)}{\varepsilon^2} = \lim_{\varepsilon \rightarrow 0^+} \frac{a'(\varepsilon)}{2\varepsilon} = \lim_{\varepsilon \rightarrow 0^+} \frac{a''(\varepsilon)}{2}.$$

Thus, the derivative  $a''_+(0)$  would exist and it would be equal to  $2\pi c^2$  because  $a''_+(0) = \lim_{\varepsilon \rightarrow 0^+} a''(\varepsilon)$ .

As the area of a region can be expressed by means of a line integral, and taking into account that a parametric integral is differentiable with respect to the parameter when the integrand is, the derivatives  $a'(\varepsilon)$  and  $a''(\varepsilon)$  exist for sufficiently small  $\varepsilon > 0$ .

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