Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 20, pp. 586-594, September 2010

ELA
http://math.technion.ac.il/iic/ela

THE SPECTRUM OF THE EDGE CORONA OF TWO GRAPHS*

YAOPING HOU† AND WAI-CHEE SHIU‡

Abstract. Given two graphs G_1, with vertices $1, 2, ..., n$ and edges $e_1, e_2, ..., e_m$, and G_2, the edge corona $G_1 \diamond G_2$ of G_1 and G_2 is defined as the graph obtained by taking m copies of G_2 and for each edge $e_k = ij$ of G, joining edges between the two end-vertices i, j of e_k and each vertex of the k-copy of G_2. In this paper, the adjacency spectrum and Laplacian spectrum of $G_1 \diamond G_2$ are given in terms of the spectrum and Laplacian spectrum of G_1 and G_2, respectively. As an application of these results, the number of spanning trees of the edge corona is also considered.

Key words. Spectrum, Adjacency matrix, Laplacian matrix, Corona of graphs.

AMS subject classifications. 05C05, 05C50.

1. Introduction. Throughout this paper, we consider only simple graphs. Let $G = (V, E)$ be a graph with vertex set $V = \{1, 2, ..., n\}$. The adjacency matrix of G denoted by $A(G)$ is defined as $A(G) = (a_{ij})$, where $a_{ij} = 1$ if i and j are adjacent in G, 0 otherwise. The spectrum of G is defined as

$$\sigma(G) = (\lambda_1(G), \lambda_2(G), ..., \lambda_n(G)),$$

where $\lambda_1(G) \leq \lambda_2(G) \leq ... \leq \lambda_n(G)$ are the eigenvalues of $A(G)$. The Laplacian matrix of G, denoted by $L(G)$ is defined as $D(G) - A(G)$, where $D(G)$ is the diagonal degree matrix of G. The Laplacian spectrum of G is defined as

$$S(G) = (\theta_1(G), \theta_2(G), ..., \theta_n(G)),$$

where $0 = \theta_1(G) \leq \theta_2(G) \leq ... \leq \theta_n(G)$ are the eigenvalues of $L(G)$. We call $\lambda_n(G)$ and $\theta_n(G)$ the spectral radius and Laplacian spectral radius, respectively. There is extensive literature available on works related to spectrum and Laplacian spectrum of a graph. See [2, 5, 6] and the references therein to know more.

The corona of two graphs is defined in [4] and there have been some results on the corona of two graphs [3]. The complete information about the spectrum of the corona of two graphs G, H in terms of the spectrum of G, H are given in [1]. In this

* Received by the editors September 11, 2009. Accepted for publication August 22, 2010. Handling Editor: Richard A. Brualdi.

†Department of Mathematics, Hunan Normal University, Changsha, Hunan 410081, China (yphou@hunnu.edu.cn). Research supported by NSFC(10771061).

‡Department of Mathematics, Hong Kong Baptist University, Hong Kong, China. (wcshiu@hkbu.edu.hk).

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 20, pp. 586-594, September 2010

ELA
http://math.technion.ac.il/iic/ela

paper, we consider a variation of the corona of two graphs and discuss its spectrum and the number of spanning trees.

DEFINITION 1.1. Let G_1 and G_2 be two graphs on disjoint sets of n_1 and n_2 vertices, m_1 and m_2 edges, respectively. The *edge corona* $G_1 \diamond G_2$ of G_1 and G_2 is defined as the graph obtained by taking one copy of G_1 and m_1 copies of G_2, and then joining two end-vertices of the i-th edge of G_1 to every vertex in the i-th copy of G_2.

Note that the edge corona $G_1 \diamond G_2$ of G_1 and G_2 has $n_1 + m_1n_2$ vertices and $m_1 + 2m_1n_2 + m_1m_2$ edges.

EXAMPLE 1.2. Let G_1 be the cycle of order 4 and G_2 be the complete graph K_2 of order 2. The two edge coronas $G_1 \diamond G_2$ and $G_2 \diamond G_1$ are depicted in Figure 1.

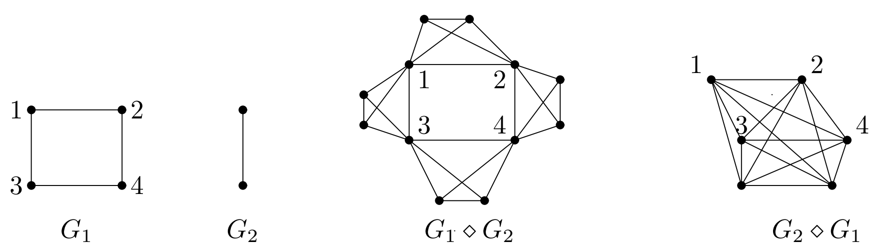

Figure 1: An example of edge corona graphs

Throughout this paper, G_1 is assumed to be a connected graph with at least one edge. In this paper, we give a complete description of the eigenvalues and the corresponding eigenvectors of the adjacency matrix of $G_1 \diamond G_2$ when G_1 and G_2 are both regular graphs and give a complete description of the eigenvalues and the corresponding eigenvectors of the Laplacian matrix of $G_1 \diamond G_2$ for a regular graph G_1 and arbitrary graph G_2. As an application of these results, we also consider the number of spanning trees of the edge corona.

2. The spectrum of the graph $G_1 \diamond G_2$.

Let the vertex set and edge set of a graph G be $V = \{1, 2, ..., n\}$ and $E = \{e_1, e_2, ..., e_m\}$, respectively. The vertex-edge incidence matrix $R(G) = (r_{ij})$ is an $n \times m$ matrix with entry $r_{ij} = 1$ if the vertex i is incident the edge e_j and 0 otherwise.

LEMMA 2.1. [2, P. 114] *Let G be a connected graph with n vertices and R be the vertex-edge incident matrix. Then $rank(R) = n-1$ if G is bipartite and n otherwise.*

LEMMA 2.2. [2] *Let G be a connected graph with spectral radius ρ. Then $-\rho$ is also an eigenvalue of $A(G)$ if and only if G is bipartite. Moreover, if G is a*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 20, pp. 586-594, September 2010

connected bipartite graph with vertex partition $V = V_1 \cup V_2$ *and* $X = (X_1, X_2)^T$ *is an eigenvector corresponding eigenvalue* λ *of* $A(G)$ *then* $X = (X_1, -X_2)^T$ *is an eigenvector corresponding eigenvalue* $-\lambda$ *of* $A(G)$.

Let $A = (a_{ij}), B$ be matrices. Then the Kronecker product of A and B is defined the partition matrix $(a_{ij}B)$ and is denoted by $A \otimes B$. The row vector of size n with all entries equal to one is denoted by $\mathbf{j}_n$ and the identity matrix of order n is denoted by $\mathbf{I}_n$.

Let G_1 and G_2 be graphs with n_1, n_2 vertices and m_1, m_2 edges,respectively. Then the adjacency matrix of $G = G_1 \diamond G_2$ can be written as

$$A(G) = \left(\begin{array}{c|c} A(G_1) & R(G_1) \otimes \mathbf{j}_{n_2} \\ \hline (R(G_1) \otimes \mathbf{j}_{n_2})^T & \mathbf{I}_{m_1} \otimes A(G_2) \end{array}\right),$$

where $A(G_1)$ and $A(G_2)$ are the adjacency matrices of the graphs G_1 and G_2, respectively, and $R(G_1)$ is the vertex-edge incidence matrix of G_1. A complete characterization of the eigenvalues and eigenvectors of $G_1 \diamond G_2$ will be given when both G_1 and G_2 are regular.

Let G_1 be an r_1-regular graph and G_2 be an r_2-regular graph and

$$\sigma(G_1) = (\mu_1, \mu_2, ..., \mu_{n_1}), \quad \sigma(G_2) = (\eta_1, \eta_2, ..., \eta_{n_2}) \tag{2.1}$$

be their adjacency spectrum, respectively. If G_1 is 1-regular then $G_1 = K_2$ as G_1 is connected. In this case, $G_1 \diamond G_2$ is the complete product of K_2 and G_2. By Theorem 2.8 of [2], or by some direct computations, we can obtain the spectrum of $G = G_1 \diamond G_2$ as $(\eta_1, ..., \eta_{n_2-1}, \mu_1 = \frac{r_2+\mu_1-\sqrt{(r_2-\mu_1)^2+4(r_1+\mu_1)n_2}}{2}, \frac{r_2+\mu_2\pm\sqrt{(r_2-\mu_2)^2+4(r_1+\mu_2)n_2}}{2} \quad)$, where $\mu_1 = -1, \mu_2 = 1$ are the spectrum of K_2.

THEOREM 2.3. *Let* G_1 *be an* r_1*-regular* $(r_1 \geq 2)$ *graph and* G_2 *be an* r_2*-regular graph and their spectra are as in* (2.1)*. Then the spectrum* $\sigma(G)$ *of* G *is*

$$\begin{pmatrix} \eta_1 & \eta_2 & \cdots & \eta_{n_2} = r_2 & \frac{r_2+\mu_1\pm\sqrt{(r_2-\mu_1)^2+4(r_1+\mu_1)n_2}}{2} & \cdots & \frac{r_2+\mu_{n_1}\pm\sqrt{(r_2-\mu_{n_1})^2+4(r_1+\mu_{n_1})n_2}}{2} \\ m_1 & m_1 & \cdots & m_1 - n_1 & 1 & \cdots & 1 \end{pmatrix}$$

where entries in the first row are the eigenvalues with the number of repetitions written below, respectively.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 20, pp. 586-594, September 2010

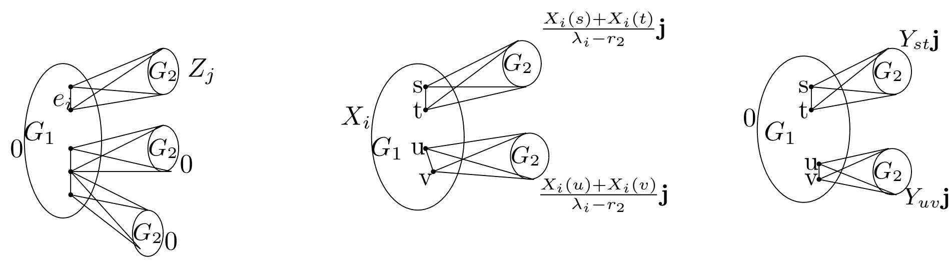

Figure 2: Description of adjacency eigenvectors

Proof. Let $Z_1, Z_2, ..., Z_{n_2}$ be the orthogonal eigenvectors of $A(G_2)$ corresponding to the eigenvalue $\eta_1, \eta_2, ..., \eta_{n_2} = r_2$, respectively. Note that G_2 is r_2-regular and $Z_j \perp \mathbf{j}$ for $j = 1, 2, ..., n_2 - 1$. Then for $i = 1, 2, ..., m_1$ and for $j = 1, 2, ..., n_2 - 1$, we have (see Figure 2, picture on the left) that $(n_1 + m_1 n_2)$-dimension vectors $(0, 0, ..., 0, Z_j, 0, ..., 0)^T$, where $(i+1)$-th block is Z_j are eigenvectors of G corresponding to eigenvalue η_j. Thus we obtain $m_1(n_2 - 1)$ eigenvalues and corresponding eigenvectors of G.

Let $X_1, X_2, ..., X_{n_1}$ be the orthogonal eigenvectors of $A(G_1)$ corresponding to the eigenvalues $\mu_1, \mu_2, ...\mu_{n_1}$, respectively. For $i = 1, 2, ..., n_1$, let

$$\lambda_i = \frac{r_2 + \mu_{n_i} + \sqrt{(r_2 - \mu_{n_i})^2 + 4(r_1 + \mu_{n_i})n_2}}{2}$$

and

$$\overline{\lambda}_i = \frac{r_2 + \mu_{n_i} - \sqrt{(r_2 - \mu_{n_i})^2 + 4(r_1 + \mu_{n_-})n_2}}{2}.$$

Note that $\frac{r_2+\mu_{n_i} \pm \sqrt{(r_2-\mu_{n_i})^2+4(r_1+\mu_{n_i})n_2}}{2} = r_2$ if and only if $\mu_i = -r_1$. So λ_i or $\overline{\lambda}_i$ is r_2 if and only if G_1 is bipartite (note that at most one of λ_i is r_2). If G_1 is bipartite and the bipartition of its vertex set is $V_1 \cup V_2$, then by Lemma 2.2 and some computations, we obtain that $(\mathbf{j}, -\mathbf{j}, 0, ..., 0)^T$ (1 on V_1, -1 on V_2, and 0 on all copies of G_2) is an eigenvector of G corresponding the eigenvalue $-r_1$.

Observe that if λ_i and $\overline{\lambda}_i$ are not equal to r_2 then λ_i and $\bar{\lambda}_i$ are eigenvalues of G corresponding to the eigenvectors $F_i = (X_i, ..., \frac{X_i(s)+X_i(t)}{\lambda_i - r_2}, ...)^T$ and $\overline{F}_i = (X_i, ..., \frac{X_i(s)+X_i(t)}{\overline{\lambda}_i - r_2}, ...)^T$, respectively (see Figure 2, picture in the middle). In fact, it needs only to be checked that characteristic equations $\sum_{v \sim u} F_i(v) = \lambda_i F_i(u)$ (resp. $\sum_{v \sim u} \bar{F}_i(v) = \overline{\lambda}_i \bar{F}_i(u)$) hold for every vertex u in G.

For any vertex u in k-copy of G_2, let edge $e_k = st$, then $F_i(u) = \frac{X_i(s)+X_i(t)}{\lambda_i - r_2}$. Furthermore,

$$\sum_{v \sim u} F_i(v) = r_2 F_i(u) + X_i(s) + X_i(t) = \lambda_i F_i(u).$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 20, pp. 586-594, September 2010

ELA
http://math.technion.ac.il/iic/ela

For any vertex u in G_1,

$$\begin{aligned}\sum_{v\sim u} F_i(v) &= \sum_{v\sim u, v\in V(G_1)} F_i(v) + \sum_{v\sim u, v\notin V(G_1)} F_i(v)\\ &= \mu_i X_i(u) + \frac{r_1 n_2 X_i(u)}{\lambda_i - r_2} + \frac{n_2}{\lambda_i - r_2}\sum_{v\sim u, v\in V(G_1)} F_i(v)\\ &= \lambda_i X_i(u) = \lambda_i F_i(u).\end{aligned}$$

Therefore we obtain $2n_1$ eigenvalues and corresponding eigenvectors of G if G_1 is not bipartite and $2n_1 - 1$ eigenvalues and corresponding eigenvectors of G if G_1 is bipartite.

Let $Y_1, Y_2, ..., Y_b$ be a maximal set of independent solution vectors of linear system $R(G_1)Y = 0$. Then $b = m_1 - n_1$ if G_1 is not bipartite and $b = m_1 - n_1 + 1$ if G_1 is bipartite. For $i = 1, 2, ..., b$, let $H_i = (0, Y_i(e_1)\mathbf{j}, ..., Y_i(e_m)\mathbf{j})^T$ (see Figure 2, picture on the right). We can obtain that H_i is an eigenvector of G corresponding to eigenvalues $r_2 = \eta_{n_2}$. Thus these $Y_i' s$ provide b eigenvalues and corresponding eigenvectors of G.

Therefore we obtain $n_1 + m_1 n_2$ eigenvalues and corresponding eigenvectors of G and it is easy to see that these eigenvectors of G are linearly independent. Hence the proof is completed. □

Next we consider the Laplacian spectrum of $G_1 \diamond G_2$.

Let $L(G_1)$ and $L(G_2)$ be the Laplacian matrices of the graphs G_1 and G_2, respectively, and $R(G_1)$ be the vertex-edge incidence matrix of G_1. Then the Laplacian matrix of $G = G_1 \diamond G_2$ is

$$L(G) = \left(\begin{array}{c|c} L(G_1) + r_1 n_2 I_{n_1} & -R(G_1)\otimes \mathbf{j}_{n_2} \\ \hline -(R(G_1)\otimes \mathbf{j}_{n_2})^T & \mathbf{I}_{m_1}\otimes(2\mathbf{I}_{n_2} + L(G_2)) \end{array}\right).$$

In the following, we give a complete characterization of the Laplacian eigenvalues and eigenvectors of $G_1 \diamond G_2$.

Let G_1 be an r_1-regular graph and G_2 be any graph and

$$S(G_1) = (\theta_1, \theta_2, ..., \theta_{n_1}),\quad S(G_2) = (\tau_1, \tau_2, ..., \tau_{n_2}) \tag{2.2}$$

be their Laplacian spectra, respectively. If G_1 is 1-regular then $G_1 = K_2$ as G_1 is connected. In this case, $G_1 \diamond G_2$ is the complete product of K_2 and G_2 (by [5]), or by some direct computations, we can obtain that the Laplacian spectrum of $G = G_1 \diamond G_2$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 20, pp. 586-594, September 2010

ELA
http://math.technion.ac.il/iic/ela

is

$$S(G) = (0, \tau_2 + 2, ..., \tau_{n_2} + 2, n_2 + 2, n_2 + 2\ \ ).$$

THEOREM 2.4. *Let G_1 be an r_1-regular ($r_1 \geq 2$) graph and G_2 be any graph and their Laplacian spectra are written as in* (2.2). *Let*

$$\beta_i, \bar{\beta}_i = \frac{r_1 n_2 + \theta_i + 2 \pm \sqrt{(r_1 n_2 + \theta_i + 2)^2 - 4(n_2 + 2)\theta_i}}{2}$$

for every θ_i. Then the Laplacian spectrum $S(G)$ of G is

$$\begin{pmatrix} \tau_1 + 2, & \tau_2 + 2, & \cdots, & \tau_{n_2} + 2, & \beta_1, & \bar{\beta}_1, & \cdots, & \beta_{n_1}, & \bar{\beta}_{n_1} \\ m_1 - n_1 & m_1 & \cdots & m_1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}$$

where entries in the first row are the eigenvalues with the number of repetitions written below, respectively.

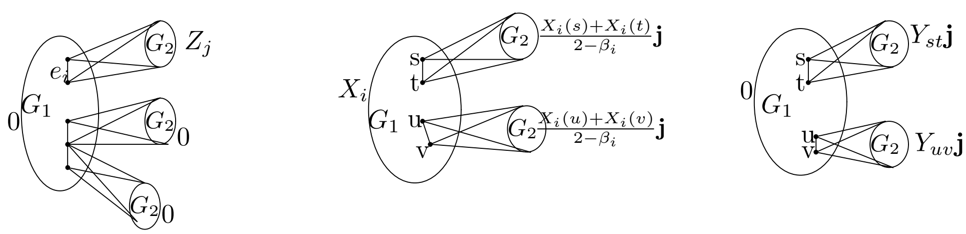

Figure 3: Description of Laplacian eigenvectors

Proof. Let $Z_1, Z_2, ..., Z_{n_2}$ be the eigenvectors of $L(G_2)$ corresponding to the eigenvalues $0 = \tau_1, \tau_2, ..., \tau_{n_2}$. Note that $Z_j \perp \mathbf{j}$ for $j = 2, ..., n_2$. Then for $i = 1, 2, ..., m_1$ and for $j = 2, 3, ..., n_2$, we have that $(n_1 + m_1 n_2)$-dimension vectors $(0, 0, ..., 0, Z_j, 0, ..., 0)^T$, where $(i+1)$-th block is Z_j are eigenvectors of $L(G)$ corresponding to eigenvalue $\tau_j + 2$ (see Figure 3, picture on the left). Thus we obtain $m_1(n_2 - 1)$ eigenvalues and corresponding eigenvectors of $L(G)$.

Let $X_1, X_2, ..., X_{n_1}$ be the orthogonal eigenvectors of $L(G_1)$ corresponding to the eigenvalues $\theta_1, \theta_2, ..., \theta_{n_1}$, respectively. For $i = 1, 2, ..., n_1$, note that:

$$\beta_i, \bar{\beta}_i = \frac{r_1 n_2 + \theta_i + 2 \pm \sqrt{(r_1 n_2 + \theta_i + 2)^2 - 4(n_2 + 2)\theta_i}}{2} = \frac{r_1 n_2 + \theta_i + 2 \pm \sqrt{(r_1 n_2 + \theta_i - 2)^2 + 4n_2(2r_1 - \theta_i)}}{2}$$

since $r_1 \geq 2, n_2 \geq 1$, $\beta_i \neq 2$. Note that $\theta_i \leq 2r_1$ and the equality holds if and only if G_1 is bipartite. Note that $\bar{\beta}_i = 2$ implies that $\theta_i = 2r_1$. That is, $\bar{\beta}_i = 2$ appears only if G_1 is bipartite and $i = n_1$. Moreover, if G_1 is bipartite and the bipartition of its vertex set is $V_1 \cup V_2$, then it is easy to check that $(\mathbf{j}, -\mathbf{j}, 0, ..., 0)^T$ (1 on V_1,

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 20, pp. 586-594, September 2010

-1 on V_2, and 0 on all copies of G_2) is an eigenvector corresponding the eigenvalue $(n_1+2)r_1 = \beta_{n_1}$ of $L(G)$.

Observe that if β_i and $\bar{\beta}_i$ are not equal to 2, then β_i and $\bar{\beta}_i$ are eigenvalues of $L(G)$ and $F_i = (X_i, ..., \frac{X_i(s)+X_i(t)}{2-\beta_i}, ...)^T$ and $\bar{F}_i = (X_i, ..., \frac{X_i(s)+X_i(t)}{2-\bar{\beta}_i}, ...)^T$ are eigenvectors of β_i and $\bar{\beta}_i$ respectively (see Figure 3, picture in the middle). In fact, it needs only to be checked that characteristic equations $d_G(u)F_i(u) - \sum_{v\sim u} F_i(v) = \beta_i F_i(u)$ (resp. $d_G(u)\bar{F}_i(u) - \sum_{v\sim u} \bar{F}_i(v) = \bar{\beta}_i \bar{F}_i(u)$) hold for every vertex u in G, where $d_G(u)$ is the degree of the vertex u in G.

For every vertex u in k-copy of G_2, let the edge $e_k = st$, then $d_G(u) = d_{G_2}(u) + 2$ and $F_i(u) = \frac{X_i(s)+X_i(t)}{2-\beta_i}$. Further,

$$\begin{aligned} d_G(u)F_i(u) - \sum_{v\sim u} F_i(v) &= d_{G_2}(u) + 2)F_i(u) - d_{G_2}(u)\frac{X_i(s)+X_i(t)}{2-\beta_i} - (X_i(s)+X_i(t)) \\ &= \beta_i F_i(u). \end{aligned}$$

For every vertex u in G_1, note that

$$r_1 X_i(u) - \sum_{\substack{v\sim u \\ v\in V(G_1)}} X_i(v) = \theta_i X_i(u).$$

We have

$$\begin{aligned} &d_G(u)F_i(u) - \sum_{v\sim u} F_i(v) = (r_1 + r_1 n_2)F_i(u) - \sum_{v\sim u, v\in V(G_1)} F_i(v) + \sum_{v\sim u, v\notin V(G_1)} F_i(v) \\ &= (r_1 + r_1 n_2)X_i(u) - \sum_{v\sim u, v\in V(G_1)} X_i(v) - \sum_{v\sim u, v\in V(G_1)} \frac{n_2}{2-\beta_i}(X_i(u)+X_i(v)) \\ &= \frac{(r_1+r_1n_2)(2-\beta_i) - 2n_2r_1 + n_2\theta_i}{2-\beta_i} X_i(u) + (\theta_i - r_1)X_i(u) \\ &= \beta_i X_i(u) = \beta_i F_i(u). \end{aligned}$$

Therefore we obtain $2n_1$ eigenvalues and corresponding eigenvectors of $L(G)$ if G_1 is not bipartite, and $2n_1 - 1$ eigenvalues and corresponding eigenvectors of $L(G)$ if G_1 is bipartite.

Let $Y_1, Y_2, ..., Y_b$ be a maximal set of independent solution vectors of the linear system $R(G_1)Y = 0$. Then $b = m_1 - n_1$ if G_1 is not bipartite, and $b = m_1 - n_1 + 1$ if G_1 is bipartite. For $i = 1, 2, ..., b$, let $H_i = (0, Y_i(e_1)\mathbf{j}, ..., Y_i(e_m)\mathbf{j})^T$ (see Figure 3, picture on the right). We can obtain that H_i is an eigenvector corresponding the eigenvalue 2 $(= \tau_1 + 2)$ of $L(G)$. Thus these $Y_i's$ provide b eigenvalues and corresponding eigenvectors of $L(G)$.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 20, pp. 586-594, September 2010

ELA
http://math.technion.ac.il/iic/ela

Therefore we obtain $n_1 + m_1n_2$ eigenvalues and corresponding eigenvectors of $L(G)$ and it is easy to see that these eigenvectors of $L(G)$ are linearly independent. Hence the proof is completed. □

As an application of the above results, we give the number of spanning trees of the edge corona of two graphs.

Let G be a connected graph with n vertices and Laplacian eigenvalues $0 = \theta_1 < \theta_2 \leq \cdots \leq \theta_n$. Then the number of spanning trees of G is

$$t(G) = \frac{\theta_2\theta_3 \cdots \theta_n}{n}.$$

By Theorem 2.4 we have

Proposition 2.5. *For a connected r_1-regular graph G_1 and arbitrary graph G_2, let the number of spanning trees of G_1 be $t(G_1)$ and the Laplacian spectra of G_2 be $0 = \tau_1 \leq \tau_2 \leq \cdots \leq \tau_{n_2}$. Then the number of spanning trees of $G_1 \diamond G_2$ is*

$$t(G_1 \diamond G_2) = 2^{m_1-n_1+1}(n_2+2)^{n_1-1}t(G_1)(\tau_2+2)^{m_1}\cdots(\tau_{n_2}+2)^{m_1}.$$

Proof. Following the notions in Theorem 2.4, note that $\beta_i\bar{\beta}_i = (n_2 + 2)\theta_i$ for $i = 1, 2, ..., n_1$ and $\beta_1 = r_1n_2 + 2, \bar{\beta}_1 = 0$. Thus

$$\begin{aligned} t(G_1 \diamond G_2) &= \frac{2^{m_1-n_1}(r_1n_2+2)(n_2+2)^{n_1-1}\prod_{i=2}^{n_2}(\tau_i+2)^{m_1}\prod_{j=2}^{n_1}\theta_j}{n_1+m_1n_2} \\ &= \frac{n_1 2^{m_1-n_1}(r_1n_2+2)(n_2+2)^{n_1-1}t(G_1)\prod_{i=2}^{n_2}(\tau_i+2)^{m_1}}{n_1+m_1n_2} \\ &= 2^{m_1-n_1+1}(n_2+2)^{n_1-1}t(G_1)(\tau_2+2)^{m_1}\cdots(\tau_{n_2}+2)^{m_1}. \end{aligned}$$

The last equality follows from $n_1 + m_1n_2 = \frac{n_1(2+r_1n_2)}{2}$. □

By Proposition 2.5, we have $t(G \diamond K_1) = 2^{m-n+1}3^{n-1}t(G)$ for a regular graph G. In fact $t(G \diamond K_1) = 2^{m-n+1}3^{n-1}t(G)$ holds for arbitrary graph G by the following proposition.

Proposition 2.6. *Let G be a connected graph with n vertices and m edges. Then the number of spanning trees of $G \diamond K_1$ is $2^{m-n+1}3^{n-1}t(G)$, where $t(G)$ is the number of spanning trees of G.*

Proof. Note that the Laplacian matrix of $G \diamond K_1$ is

$$L(G \diamond K_1) = \begin{pmatrix} L(G) + D(G) & -R \\ -R^T & 2I_m \end{pmatrix}.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 20, pp. 586-594, September 2010

Let $(L(G))_{11}$ be the reduced Laplacian matrix of G obtained by removing the first row and first column of $L(G)$ and R_1 be the matrix obtained by removing the first row of the vertex-edge incidence matrix R. By the Matrix-Tree theorem [2], we have

$$\begin{aligned} t(G \diamond K_1) = \det(L(G \diamond K_1))_{11} &= \det \begin{pmatrix} (L(G)+D(G))_{11} & -R_1 \\ -R_1^T & 2I_m \end{pmatrix} \\ &= 2^m \det[(L(G)+D(G))_{11} - \frac{1}{2} R_1 R_1^T], \end{aligned}$$

since $RR^T = D(G) + A(G)$, $R_1 R_1^T = (D(G) + A(G))_{11}$. Thus

$$t(G \diamond K_1) = 2^m \det(\frac{3}{2}(D(G) - A(G))_{11} = 2^{m-n+1} 3^{n-1} t(G). \ \square$$

Acknowledgments. The authors would like to express their sincere gratitude to the referee for a very careful reading of the paper and for all his or her insightful comments and valuable suggestions, which made a number of improvements on this paper

REFERENCES

[1] S. Barik, S. Pati, and B. K. Sarma. The spectrum of the corona of two graphs. *SIAM. J. Discrete Math.,* 24:47–56, 2007.

[2] D. Cvetković, M. Doob, and H. Sachs. *Spectra of Graphs: Theory and Application,* Johann Ambrosius Barth Verlag, Heidelberg-Leipzig, 1995.

[3] R. Frucht and F. Harary. On the corona two graphs. *Aequationes Math.,* 4:322–325, 1970.

[4] F. Harary. *Graph Theory.* Addition-Wesley Publishing Co., Reading, MA/Menlo Park, CA/London, 1969.

[5] R. Merris. Laplacian matrices of graphs: a survey. *Linear Algebra Appl.,* 197/198:143–176, 1994.

[6] B. Mohar. The Laplacian spectrum of graphs, *Graph Theory, Combinatorics and Applications.* John Wiley, New York, pp. 871–898, 1991.