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THE q-NUMERICAL RANGE OF 3×3 TRIDIAGONAL MATRICES*

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Abstract. For $0 \leq q \leq 1$, we examine the q-numerical ranges of 3×3 tridiagonal matrices $A(b)$ that interpolate between the circular range $W_0(A(b))$ and the elliptical range $W_1(A(b))$ as q varies from 0 to 1. We show that for $q \leq (1-b)^2/(2(1+b^2))$, $W_q(A(b))$ is a circular disc centered at the origin with radius $(1+b^2)^{1/2}$, but $W_{4/5}(A(2))$ is not even an elliptical disc.

Key words. Tridiagonal matrix, Davis-Wielandt shell, q-numerical range.

AMS subject classifications. 15A60.

1. Introduction.

For a bounded linear operator T on a complex Hilbert space H, the *q-numerical range* $W_q(T)$ of T for $0 \leq q \leq 1$ is defined as

$$W_q(T) = \{\langle T\xi, \eta\rangle : \xi, \eta \in H, ||\xi|| = ||\eta|| = 1, \langle \xi, \eta \rangle = q\}.$$

In the paper [5], the authors of this paper give a bounded normal operator T on an infinite dimensional separable Hilbert space H defined by

$$T = (U + U^*)/2 + i\,\alpha\,(U - U^*)/(2i),$$

where U is a unitary operator on a Hilbert space H with $\sigma(U) = \{z \in \mathbf{C} : |z| = 1\}$, and $0 < \alpha < 1$, and show that

$$\text{closure}(W_q(T)) = \{x + iy : (x, y) \in \mathbf{R}^2, x^2 + \frac{y^2}{1 + \alpha^2 q^2 - q^2} \leq 1\}$$

is an elliptical disc which interpolates between the circular range $W_0(T)$ and the elliptical range $W(T) := W_1(T)$ as q varies from 0 to 1.

Various conditions for a bounded operator T are known which assure that the closure of the numerical range $W(T)$ is an elliptical disc (cf. [2],[3],[7]). It seems naturally to ask whether the conditions for elliptical range of $W(T)$ guarantee that $W_q(T)$ is also elliptical for $0 < q < 1$. If T is an $n \times n$ upper triangular nilpotent matrix

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associated with a tree graph, then the range $W_q(T)$ is circular for every $0 \le q \le 1$. Thus the question has a positive answer for such a special class of matrices. A special quadratic 3×3 matrix

$$A(\gamma, a + i\,b) = \begin{pmatrix} 0 & 1+\gamma & 0 \\ 1-\gamma & 0 & 0 \\ 0 & 0 & a+ib \end{pmatrix},$$

is another affirmative example (cf.[6]). It is shown [6] that if $\gamma > 0$, $a, b \in \mathbf{R}$, and $a^2 + (b/\gamma)^2 \le 1$ then

$$W_q(A(\gamma, a + i\,b)) = \{x + i\,y : \frac{x^2}{(1 + \gamma(1-q^2)^{1/2})^2} + \frac{y^2}{(\gamma + (1-q^2)^{1/2})^2} \le 1\}.$$

The main purpose of this note is to deal with the behavior of the q-numerical ranges of some 3×3 tridiagonal matrices A that interpolate between the circular range $W_0(A)$ and elliptical range $W_1(A)$ for $0 \le q \le 1$. We also give an example of a real 3×3 tridiagonal matrix which has a non-elliptical q-numerical range.

2. 3×3 **tridiagonal matrices.** The shapes of the classical numerical ranges of 3×3 matrices are tested and determined in [8],[10]. For tridiagonal matrices, it is proved in [1, Theorem 4] that if A is a nonnegative tridiagonal 3×3 matrix with 0 main diagonal:

$$A = \begin{pmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ 0 & a_{32} & 0 \end{pmatrix},$$

then the numerical range $W(A)$ is an elliptical disc centered at 0, the major axis on the real line. We show that the q-numerical range of this tridiagonal matrix is in general not an elliptical disc. Consider tridiagonal matrices of Toeplitz type

$$\begin{pmatrix} 0 & b & 0 \\ a & 0 & b \\ 0 & a & 0 \end{pmatrix},\ a \ne b.$$

We may assume that $a = 1$ and $b \ge 0,\ b \ne 1$

$$A = A(b) = \begin{pmatrix} 0 & b & 0 \\ 1 & 0 & b \\ 0 & 1 & 0 \end{pmatrix}. \tag{1}$$

First we compute the equation of the boundary of the *Davis-Wielandt shell* of A:

$$W(A, A^*A) = \{(\xi^* A\xi, \xi^* A^* A\xi) \in \mathbf{C} \times \mathbf{R} : \xi \in \mathbf{C}^3, \xi^*\xi = 1\}.$$

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We consider the form

$$F(t,x,y,z) = \det(tI_3 + x\Re(A) + y\Im(A) + zA^*A),$$

and find that

$$\begin{aligned}4F(1,x,y,z) &= 4 - 2x^2 - 4bx^2 - 2b^2x^2 - 2y^2 + 4by^2 - 2b^2y^2 + 8z \\ &+8b^2z - x^2z + 2b^2x^2z - b^4x^2z - y^2z + 2b^2y^2z - b^4y^2z \\ &+4z^2 + 8b^2z^2 + 4b^4z^2. \qquad (2)\end{aligned}$$

The surface $F(t,x,y,z) = 0$ in $\mathbf{CP}^3$ has an ordinary double singular point at

$$(t,x,y,z) = (1,0,0,-1/(1+b^2)).$$

Corresponding to this singular point, the boundary of the Davis-Wielandt shell $W(A, A^*A)$ has a flat portion on the horizontal plane

$$Z = 1 + b^2. \qquad (3)$$

The intersection of the shell $W(A, A^*A)$ and the horizontal plane (3) is the elliptical disc bounded by the ellipse

$$\begin{aligned}&1 - 4b^2 + 6b^4 - 4b^6 + b^8 - 4X^2 + 16bX^2 - 28b^2X^2 \\ &+32b^3X^2 - 28b^4X^2 + 16b^5X^2 - 4b^6X^2 - 4Y^2 - 16bY^2 \\ &-28b^2Y^2 - 32b^3Y^2 - 28b^4Y^2 - 16b^5Y^2 - 4b^6Y^2 = 0. \qquad (4)\end{aligned}$$

The Davis-Wielandt shell $W(A, A^*A)$ also provides information for $W_q(A)$. We define the height function

$$h(x+iy) = \max\{w \in \mathbf{R} : (x+iy, w) \in W(A, A^*A)\}. \qquad (5)$$

Then Tsing's circular union formula [11] is written as

$$W_q(A) = \{q\,z + \sqrt{1-q^2}w\,\Psi(z) : z \in W(A), w \in \mathbf{C}, |w| \le 1\}, \qquad (6)$$

where

$$\Psi(z) = \sqrt{h(z) - |z|^2}. \qquad (7)$$

The formula (6) leads to the convexity of the range $W_q(A)$.

Theorem 2.1 Let $A = A(b), b \ge 0$, be the matrix defined by (1).

(*i*) If $q \le (1-b)^2/(2(1+b^2))$ then $W_q(A)$ is the circular disc $x^2 + y^2 \le 1 + b^2$.

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(ii) If $(1-b)^2/(2(1+b^2)) < q < (1+b)^2/(2(1+b^2))$ then the boundary of $W_q(A)$ contains circular arcs on the circle $x^2+y^2 = 1+b^2$.

Proof. We prove (i) first. By (3), the function (7) restricted to the elliptical disc bounded by (4) becomes

$$\Psi(z) = \sqrt{1+b^2-|z|^2}.$$

Consider a point $z = x+iy \in W(A)$ for which

$$q\,z + r\,\sqrt{1-|q|^2}\,\Psi(z) \in \partial W_q(A)$$

for some $r \in \mathrm{C}, |r| = 1$. Moreover, if $\Psi(z)$ is continuously differentiable on a neighborhood of the point $x+iy$, then by [4, Theorem 2] the point $x+iy$ satisfies the equation

$$\Psi_x(x+i\,y)^2 + \Psi_y(x+i\,y)^2 = q^2/(1-q^2). \tag{8}$$

We compute that

$$\Psi_x(x+iy)^2 + \Psi_y(x+iy)^2 = (x^2+y^2)/(1+b^2-x^2-y^2). \tag{9}$$

From (8) and (9), we obtain

$$(x^2+y^2)/(1+b^2-x^2-y^2) = q^2/(1-q^2). \tag{10}$$

Substitute $R = x^2+y^2$ in (10). Then we have the relation

$$R = q^2 + b^2q^2.$$

Consider $X = 0$ in the ellipse (4), then the semi-minor of the ellipse is $(1-b)^2/(2(1+b^2)^{1/2})$. Suppose

$$R^{1/2} = (q^2+b^2q^2)^{1/2} \le (1-b)^2/(2(1+b^2)^{1/2}). \tag{11}$$

Then the circle $x^2+y^2 = q^2+b^2q^2$ is contains in the elliptical disc bounded by (4). The inequality (11) is rewritten as

$$q \le (1-b)^2/(2(1+b^2)).$$

Again, by [4, Theorem 2],

$$\{z \in W(A) : qz + r\sqrt{1-q^2}\Psi(z) \in \partial W_q(A(b)) \text{ for some } r \in \mathbf{C}, |r| = 1\}$$
$$\subset \{x+iy \in W(A)^\circ : (x^2+y^2)/(1+b^2-x^2-y^2) = q^2/(1-q^2)\},$$

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the boundary of $W_q(A)$ in (6) is expressed as

$$q\sqrt{1+b^2}qe^{i\theta}+\sqrt{1-q^2}e^{i\phi}\sqrt{1+b^2-(1+b^2)q^2},$$

which is a circle centered at the origin with radius $(1+b^2)^{1/2}$. This proves (i).

A similar argument of the proof of (i) is applicable to prove (ii). We consider $Y=0$ in the ellipse (4), then the semi-major of the ellipse is $(1+b)^2/(2\sqrt{1+b^2})$. Suppose

$$R^{1/2}=(q^2+b^2q^2)^{1/2}<(1+b)^2/(2\sqrt{1+b^2}),$$

i.e., $q<(1+b)^2/2(1+b^2)$. Then the intersection of the elliptical disc

$$\{x+iy:(x,y)\in\mathbf{R}^2,\frac{x^2}{(1+b)^4/(4(1+b^2))}+\frac{y^2}{(1-b)^4/(4(1+b^2))}\leq 1\}$$

and the set

$$\begin{aligned}&\{z\in W(A):qz+r\sqrt{1-q^2}\Psi(z)\in\partial W_q(A(b))\text{ for some }r\in\mathbf{C},|r|=1\}\\&\subset\{x+iy\in W(A)^\circ:(x^2+y^2)/(1+b^2-x^2-y^2)=q^2/(1-q^2)\},\end{aligned}$$

containing two arcs. Corresponding to these two arcs, the boundary of $W_q(A)$ in (6) contains arcs on the circle $x^2+y^2=1+b^2$. □

Although for $(1-b)^2/(2(1+b^2))<q<(1+b)^2/2(1+b^2)$, the boundary of $W_q(A)$ contains two arcs on the circle $x^2+y^2=1+b^2$, but $W_q(A)$ may not equal to the associated circular disc. Indeed, it may not even be an elliptical disc. We treat the case $b=2$ in the matrix (1). Then $(1-b)^2/(2(1+b^2))=1/10$ and $(1+b)^2/2(1+b^2)=9/10$. In the following, we show that $W_{4/5}(A(2))$ is not an elliptical disc. At first, we have the boundary equation of the Davis-Wielandt shell of $A(2)$.

Theorem 2.2 Let $A=A(2)$ be the matrix defined by (1) with $b=2$. Then every boundary point (X,Y,Z) of $W(A,A^*A)$ lies on the surface $G(X,Y,Z)=0$ of degree 10 or its multi-tangent $Z=5$ satisfying the inequality $20X^2+1620Y^2\leq 81$.

Proof. Let $G(X,Y,Z)=0$ be the dual surface of $F(t,x,y,z)=0$ (2). By [4], the boundary generating surface of the shell $W(A,A^*A)$ can be obtained by the following steps:

The dual surface $G(X,Y,Z)=0$ consists of the points (X,Y,Z) such that the plane $Xx+Yy+Zz+1=0$ is a tangent of the surface $F(1,x,y,z)=0$ at some

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non-singular point of this surface. Consider the polynomial

$$f(x, y : X, Y, Z) = Z^2\, F(1, x, y, -\frac{1}{Z} - \frac{Xx}{Z} - \frac{Yy}{Z}),$$

and eliminate the variables x and y from the equations

$$f(x, y : X, Y, Z) = 0, \quad \frac{\partial f}{\partial y}(x, y : X, Y, Z) = 0, \quad \frac{\partial f}{\partial x}(x, y : X, Y, Z) = 0.$$

Successive eliminations of x and y provide a performable method to this process. The polynomial $G(X, Y, Z)$ is obtained as a simple factor of the successive discriminants. $\square$

Partial terms of the polynomial $G(X, Y, Z)$ computed by the proof of Theorem 2.2 are given by

$$\begin{aligned}
G(X, Y, Z) = {} & (18432X^2 + 165888Y^2)Z^8 - (359424X^2 + 2571264Y^2)Z^7 \\
& +(-121856X^4 - 474624X^2Y^2 + 6925824Y^4 + 2750976X^2 \\
& +14805504Y^2)Z^6 + \text{ lower degree terms in } Z \\
& +(2000X^{10} + 490000X^8Y^2 + 40340000X^6Y^4 + 1142100000X^4Y^6 \\
& +2165130000X^2Y^8 + 1062882000Y^{10} - 1295975X^8 \\
& +256936100X^6Y^2 + 1698904150X^4Y^4 + 2621816100X^2Y^6 \\
& +1181144025Y^8 + 209935800X^6 + 528751800X^4Y^2 + 427696200X^2Y^4 \\
& +108880200Y^6 + 2624400X^4 + 5248800X^2Y^2 + 2624400Y^4).
\end{aligned}$$

The algebraic surface $G(X, Y, Z) = 0$ contains the three lines

$$\{(0, 0, Z) : Z \in \mathbf{C}\},\ \{(X, 0, (18 - X)/4) : X \in \mathbf{C}\},\ \{(X, 0, (18 + X)/4) : X \in \mathbf{C}\}$$

on the plane $Y = 0$. These lines do not lie on the boundary of the shell $W(A, A^*A)$. The equation $G(X, Y, Z) = 0$ gives the implicit expression of the hight function $Z = h(X + iY)$ (5). The Davis-Wielandt shell $W(A, A^*A)$ is symmetric with respect to the real axis $Y = 0$ and the imaginary axis $X = 0$:

$$G(X, -Y, Z) = G(X, Y, Z), \quad G(-X, Y, Z) = G(X, Y, Z).$$

By this property and the formula (6), the q-numerical range $W_q(A)$, $0 < q < 1$, is also symmetric with respect to the real and imaginary axes. For $\theta \in \mathbf{R}$, we define

$$M_\theta(q) = \max\{\Re(z \exp(-i\theta)) : z \in W_q(A)\}.$$

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By the symmetry property, we have that

$$M_{-\theta}(q) = M_\theta(q), \quad M_{\pi-\theta}(q) = M_\theta(q).$$

We denote $M_x = M_0$, $M_y = M_{\pi/2}$, $M_v = M_{\pi/4}$. By equation (6), we obtain

$$\begin{aligned} M_\theta(q) = \max\{&qx + \sqrt{1-q^2}\Phi_\theta(x) : \\ &\min_{z\in W(A)} \Re(z\exp(-i\theta)) \le x \le \max_{z\in W(A)} \Re(z\exp(-i\theta))\}, \end{aligned} \tag{12}$$

where

$$\Phi_\theta(x) = \max\{\sqrt{h(z)-|z|^2} : z \in W(A), \Re(z\exp(-i\theta)) = x\}.$$

If $z = X + iY$ lies on the elliptical disc

$$20X^2 + 1620Y^2 \le 81, \tag{13}$$

then the function $\Psi(z)$ is given by

$$\Psi(z) = \sqrt{5 - X^2 - Y^2}.$$

If $z \in W(A)$ does not belong to the disc (13), then $\Psi(z)$ satisfies

$$G(\Re(z), \Im(z), |z|^2 + \Psi(z)^2) = 0.$$

Suppose $\Im(z) = 0$. If $X = \Re(z) \in W(A)$ with $|X| > 9\sqrt{5}/10$ ($X \in W(A)$ implies $|X| \le 3\sqrt{2}/2$), then $W = \Phi_0(X)$ satisfies

$$\begin{aligned} &72W^8 + (288X^2 - 108)W^6 + (432X^4 - 791X^2 + 54)W^4 \\ &\quad +(288X^6 - 1258X^4 - 802X^2 - 9)W^2 \\ &\quad +(72X^8 - 575X^6 + 1144X^4 + 16X^2) = 0. \end{aligned}$$

Suppose $\Re(z) = 0$. If $Y = \Im(z) \in W(A)$ with $|Y| > \sqrt{5}/10$ ($iY \in W(A)$ implies $|Y| \le \sqrt{2}/2$), then $W = \Phi_{\pi/2}(Y)$ satisfies

$$\begin{aligned} &8W^8 + (32Y^2 - 108)W^6 + (48Y^4 - 71Y^2 + 486)W^4 + (32Y^6 + 182Y^4 \\ &-738Y^2 - 729)W^2 + (8Y^8 + 145Y^6 + 776Y^4 + 1296Y^2) = 0. \end{aligned} \tag{14}$$

If $|x| \le 9\sqrt{205}/410$, then the function $W = \Phi_{\pi/4}(x)$ is given by

$$\Phi_{\pi/4}(x) = \sqrt{5 - x^2}.$$

We have $\Phi_{\pi/4}(9\sqrt{205}/410) = \sqrt{4019/820}$. To express $W = \Phi_{\pi/4}(x)$ for $9\sqrt{205}/410 \le |x| \le \sqrt{10}/2$, we introduce

$$L(x, v, W) = G(\frac{x-v}{\sqrt{2}}, \frac{x+v}{\sqrt{2}}, W^2 + x^2 + y^2).$$

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The implicit expression of $W = \Phi_{\pi/4}(x)$ is obtained by the elimination of v from the equations

$$L(x, v, W) = 0, \quad \frac{\partial L}{\partial v}(x, v, W) = 0.$$

It is given by

$$\begin{aligned} T(x, W) = {} & 173946175488000000W^{28} + (2727476031651840000x^2 \\ & -3168477904896000000)W^{26} + (20313573530429030400x^4 \\ & -39881741313245184000x^2 + 23822488882380800000)W^{24} \\ & + \text{lower degree terms in } W \\ & +(2918332558536081408x^{28} + 30399297484750848000x^{26} \\ & -104417065668305747968x^{24} + 136940874704617472000x^{22} \\ & +532245897669408456704x^{20} - 927952166837110702080x^{18} \\ & -641779481622442147 8x^{16} + 2275950316355374284 80x^{14} \\ & -2031191223465228107776x^{12} + 3773060375254054993920x^{10} \\ & -1201269688073344516096x^{8} + 2485062193220818042880x^{6} \\ & +1261622913333890908 16x^{4} + 144996139555356672 0x^{2}). \end{aligned}$$

By using these equations, we prove the following theorem.

Theorem 2.3 Let $A(2)$ be the 3×3 tridiagonal matrix (1) with $b = 2$. Then

$$M_x = \max\{\Re(z) : z \in W_{4/5}(A)\} = \sqrt{5},$$

$$M_y = \max\{\Im(z) : z \in W_{4/5}(A)\} = \frac{27\sqrt{6}}{40},$$

and the quantity

$$M_v = \max\{(\Re(z) + \Im(z))/\sqrt{2} : z \in W_{4/5}(A)\},$$

is the greatest real root of the equation

$$R(v) = 33280000000v^6 - 257344640000v^4 + 547018156000v^2 - 194025305907 = 0, \tag{15}$$

and satisfying the inequality

$$M_v^2 > 3.91 > \frac{M_x^2 + M_y^2}{2} = \frac{6187}{1600}, \tag{16}$$

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and hence the boundary of the convex set $W_{4/5}(A)$ is not an ellipse.

Proof. Suppose that the inequality (16) is proved. If the boundary of $W_{4/5}(A)$ is an ellipse, by the symmetry of $W_{4/5}(A)$ with respect to the real and imaginary axes, the ellipse is given by

$$W_{4/5}(A) = \{x + iy : (x, y) \in \mathbf{R}^2, \frac{x^2}{M_x^2} + \frac{y^2}{M_y^2} \leq 1\},$$

and its support $ax + by + 1 = 0$ satisfies the equation

$$M_x^2 a^2 + M_y^2 b^2 = 1.$$

We may rewrite a support line

$$x \cos\theta + y \sin\theta - M_\theta = 0$$

as

$$-\frac{\cos\theta}{M_\theta}x - \frac{\sin\theta}{M_\theta}y + 1 = 0.$$

Then, we have the equation

$$M_x^2 \frac{\cos^2\theta}{M_\theta^2} + M_y^2 \frac{\sin^2\theta}{M_\theta^2} = 1,$$

and hence

$$M_\theta^2 = M_x^2 \cos^2\theta + M_y^2 \sin^2\theta. \tag{17}$$

As a special case $\theta = \pi/4$, we have

$$M_v^2 = \frac{M_x^2 + M_y^2}{2}.$$

Thus the inequality (16) implies that the boundary is not an ellipse.

Secondly, we determine the quantities M_x, M_y. By the equation (12), the quantities $M_x = M_0(4/5), M_y = M_{\pi/2}(4/5), M_v = M_{\pi/4}(4/5)$ are respectively the maximum of the function

$$qx + \sqrt{1-q^2}\Phi_\theta(x) = \frac{4}{5}x + \frac{3}{5}\Phi_\theta(x).$$

for $\theta = 0, \pi/2, \pi/4$. Each maximal point x_θ satisfies

$$\Phi_\theta'(x_\theta) = -\frac{q}{\sqrt{1-q^2}} = -\frac{4}{3}.$$

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for $\theta = 0, \pi/2, \pi/4$. For $\theta = 0$, x_θ is given by

$$x_\theta = x_0 = \frac{4\sqrt{5}}{5} < \frac{9\sqrt{5}}{10},$$

for which

$$M_x = \frac{4x_0}{5} + \frac{3}{5}\sqrt{5 - x_0^2} = \sqrt{5}.$$

For $\theta = \pi/2$, x_θ is given by

$$x_\theta = x_{\pi/2} = \frac{9\sqrt{6}}{40} \in [\frac{\sqrt{5}}{10}, \frac{\sqrt{2}}{2}]$$

at which the function

$$\frac{4}{5}x + \frac{3}{5}\Phi_{\pi/2}(x)$$

attains the maximum $M_y = 27\sqrt{6}/40$, where $\Phi_{\pi/2}(Y)$ is given by (14).

Thirdly, an implicit expression (15) of M_v can be obtained by the elimination of x from the equation

$$\tilde{T}(x, \tilde{W}) = \tilde{T}(x, \frac{4}{5}x + \frac{3}{5}W) = T(x, W) = 0, \frac{\partial}{\partial x}\tilde{T}(x, \tilde{W}) = 0,$$

and M_v is the greatest real root of the polynomial $R(v)$. Finding a numerical solution of the equation $R(v) = 0$, we obtain the inequality (16). □

The boundary points $x + iy$ of $W_{4/5}(A)$ are classified into the two classes. One class consists of points satisfying

$$x + iy = \frac{4}{5}(u + iv) + w\,\sqrt{5 - u^2 - v^2}$$

for some point $u+iv$ on the elliptical disc (13), and some $|w| = 1$. This class of points lies on the circle $x^2 + y^2 = 5$. Another class corresponds to points $u + iv$ satisfying $G(u, v, u^2 + v^2 + \Psi(u + iv)^2) = 0$, and lies on an algebraic curve $S(x, y) = 0$, where $S(x, y)$ is a polynomial in x and y of degree 16 which is decomposed as the product of a polynomial $S_1(x, y)$ of degree 6 and a polynomial $S_2(x, y)$ of degree 10. We perform the computation of $S(x, y)$. The curve $S_1(x, y) = 0$ is displayed in Figure 1. At the right-end corner of Figure 1, there is an intersection of two curves which is displayed in Figure 2. The convexity is connected by an arc of the circle $x^2 + y^2 = 5$ as shown in Figure 3. The curve $S_2(x, y) = 0$ is display in Figure 4, and the final boundary generating curve of $W_{4/5}(A)$ is displayed in Figure 5.

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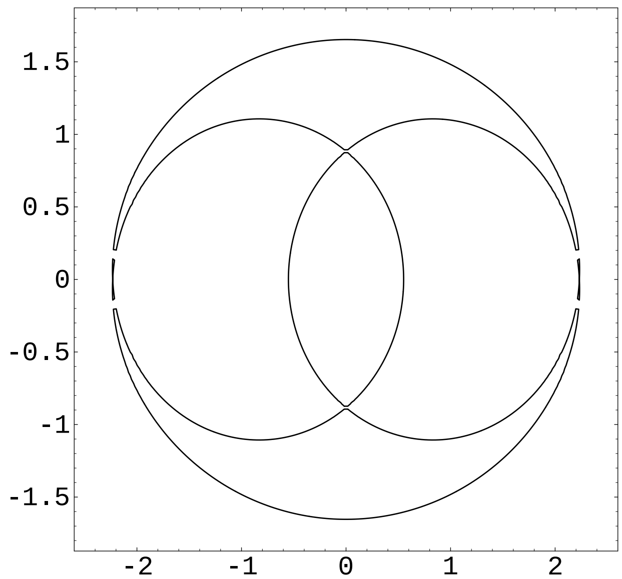

Fig. 1.

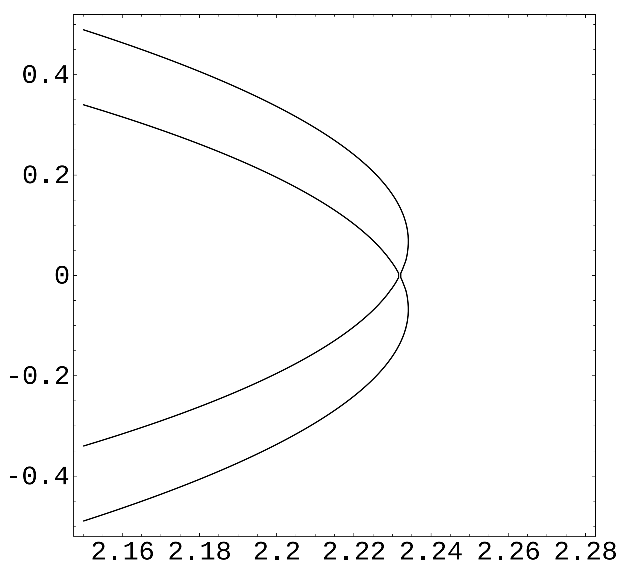

Fig. 2.

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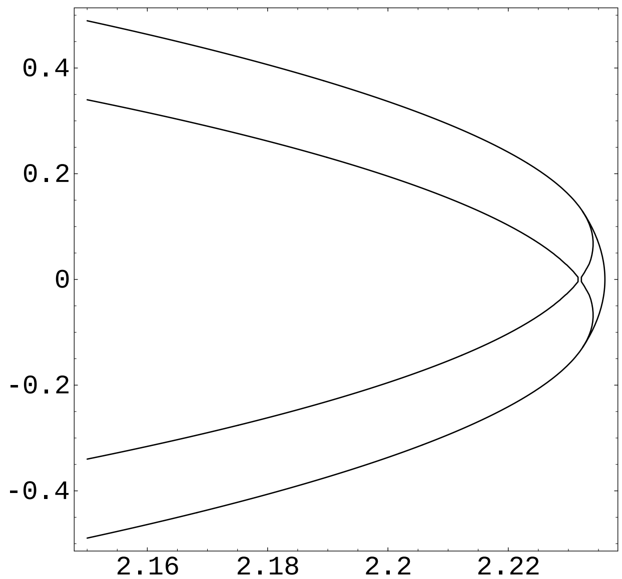

FIG. 3.

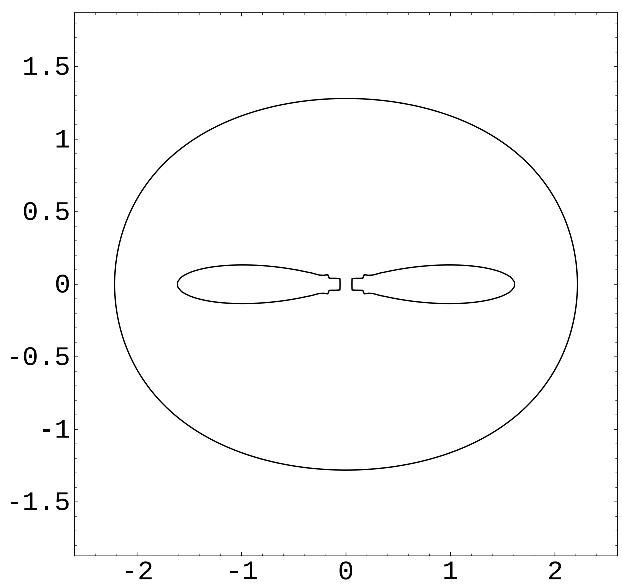

FIG. 4.

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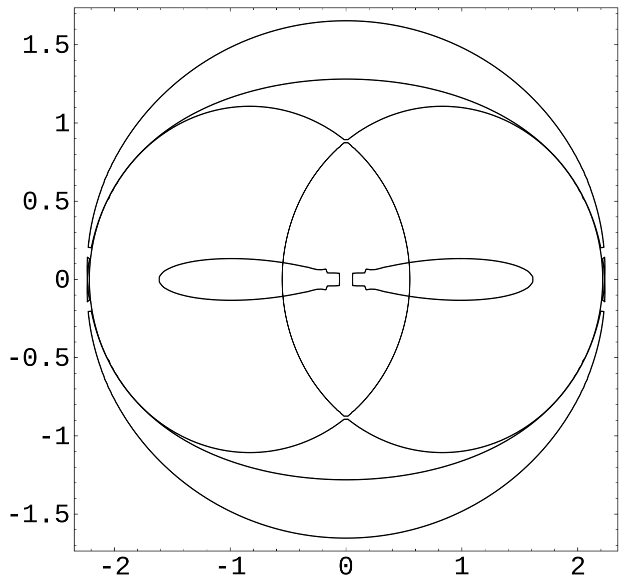

FIG. 5.

We have determined $W_q(A(b))$ in Theorem 2.1 for $q < (1+b)^2/2(1+b^2)$, and demonstrated in Theorem 2.3 that $W_q(A(2))$ is not an elliptical disc if $q = 4/5$. A similar method can be applied to show that for $q = 12/13 > (1+b)^2/2(1+b^2)|_{b=2}$, $W_{12/13}(A(2))$ is also not an elliptical disc. Indeed, the boundary of $W_{12/13}(A(2))$ lies on a polynomial curve of degree 16.

3. The functions Φ_θ. By a duality theorem in [9], the function Ψ on $W(A)$ is reflected to the elliptical range property of the family of $\{W_q(A) : 0 \le q \le 1\}$. Can we find another criterion for the elliptical range of $\{W_q(A) : 0 \le q \le 1\}$? The following example suggests that the functions Φ_θ can not play the role.

We consider an example:

$$B = \begin{pmatrix} 0 & 8/5 \\ 2/5 & 0 \end{pmatrix}.$$

The set $W_q(B)$ is an elliptical disc for $0 \le q \le 1$. The function $\Phi_0(x)$ on $[-1,1]$ is given by

$$\Phi_0(x) = \frac{3 + 5\sqrt{1-x^2}}{5},$$

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and the function $\Phi_{\pi/2}(x)$ on $[-3/5, 3/5]$ is given by

$$\Phi_{\pi/2}(x) = \frac{5 + \sqrt{9 - 25x^2}}{5}.$$

Moreover, the function $W = \Phi_{\pi/4}(x)$ on $[-\sqrt{17}/5, \sqrt{17}/5]$ has an implicit expression

$$\begin{aligned} T(x, W) = {} & 112890625W^8 + (325781250x^2 - 341062500)W^6 \\ & +(412890625x^4 - 374000000x^2 + 141280000)W^4 \\ & +(300000000x^6 - 340000000x^4 + 51200000x^2 - 20889600)W^2 \\ & +(100000000x^8 - 20480000x^4 + 1048576) = 0. \end{aligned} \tag{18}$$

For every $0 < q < 1$, the respective maxima $M_0(q)$, $M_{\pi/2}(q)$, $M_{\pi/4}(q)$ of the function

$$q\,x + \sqrt{1 - q^2}\Phi_\theta(x)$$

for $\theta = 0, \pi/2, \pi/4$ satisfy

$$M_{\pi/4}(q)^2 = \frac{M_0(q)^2 + M_{\pi/2}(q)^2}{2}$$

for all q. For instance, the respective maxima

$$M_0(\frac{12}{13}) = \frac{16}{13}, M_{\pi/2}(\frac{12}{13}) = \frac{64}{65}$$

are attained at $x = 12/13 = 1 \times 12/13$ and $x = 36/65 = 3/5 \times 12/13$. By (17), we have

$$M_{\pi/4}(\frac{12}{13}) = ((M_0(\frac{12}{13}) + M_{\pi/2}(\frac{12}{13}))/2)^{1/2} = \frac{8\sqrt{82}}{65}.$$

Since $\Phi_{\pi/4}(x)$ satisfies (18)

$$T(x, \Phi_{\pi/4}(x)) = 0,$$

The maximal point x_0 of the function $\Phi_{\pi/4}(x)$ is located at $\Phi'_{\pi/4}(x_0) = 0$. We compute

$$\Phi'_{\pi/4}(x) = -\frac{\partial_x T(x, w)}{\partial_w (x, w)}$$

for $w = \Phi_{\pi/4}(x)$, and obtain that $x_0 = 222\sqrt{82}/2665$ which is different from $\sqrt{17}/5 \times 12/13$. The maximum M_θ of the functions Φ_θ satisfies property (17):

$$M_\theta^2 = M_x^2 \cos^2\theta + M_y^2 \sin^2\theta,$$

but this property is not reflected to a simple relation among the functions $\Phi_0, \Phi_{\pi/2}$ and Φ_θ for $0 < \theta < \pi/2$.

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