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RANGES OF SYLVESTER MAPS AND A MINIMAL RANK PROBLEM*

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Abstract. It is proved that the range of a Sylvester map defined by two matrices of sizes $p \times p$ and $q \times q$, respectively, plus matrices whose ranks are bounded above, cover all $p \times q$ matrices. The best possible upper bound on the ranks is found in many cases. An application is made to a minimal rank problem that is motivated by the theory of minimal factorizations of rational matrix functions.

Key words. Sylvester maps, Invariant subspaces, Rank.

AMS subject classifications. 15A06, 15A99.

1. Introduction.

Let $\mathbb{F}$ be a (commutative) field. We let $\mathbb{F}^{p\times q}$ stand for the set of $p \times q$ matrices with entries in $\mathbb{F}$; $\mathbb{F}^{p\times 1}$ will be abbreviated to $\mathbb{F}^p$. The following minimal rank problem was stated in [8, Section 6] for the case when $\mathbb{F}$ is the complex field $\mathbb{C}$:

PROBLEM 1.1. *Given $A \in F^{n\times n}$, and given an A-invariant subspace $\mathcal{M} \subseteq \mathbb{F}^n$, find the smallest possible rank, call it $\mu(A, \mathcal{M})$, for the difference $A - Z$, where Z runs over the set of all $n \times n$ matrices with entries in $\mathbb{F}$ for which there is a Z-invariant subspace $\mathcal{N} \subseteq \mathbb{F}^n$ complementary to $\mathcal{M}$. Also, find structural properties, or description, of such matrices Z.*

The problem (for $\mathbb{F} = \mathbb{C}$) is intimately connected with minimal factorizations of rational matrix functions, in particular, if certain additional symmetry properties of A, $\mathcal{M}$, and Z are assumed; see [8] for more details. Pairs of matrices (A, Z) that have a pair of complementary subspaces $\mathcal{M}$, $\mathcal{N}$, of which the first is A-invariant and the second is Z-invariant, but without explicit rank conditions on $A - Z$, are studied in [1, 2], for example, in connection with complete minimal factorization of rational matrix functions.

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In the general formulation, Problem 1.1 appears to be difficult, even intractable, especially the part concerning properties or description of all matrices Z. To illustrate, assume $\mathbb{F}$ is algebraically closed, and let

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \mathcal{M} = \text{Span} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then $\mu(A, \mathcal{M}) = 1$, and a matrix $Z = \begin{pmatrix} x & w \\ y & z \end{pmatrix}$, $x, y, z, w \in \mathbb{F}$, has the properties that rank $(A - Z) = 1$ and Z has an invariant subspace complemented to $\mathcal{M}$ if and only if Z is not a nonzero scalar multiple of A and the equality $xz + y(1 - w) = 0$ holds.

If A and $\mathcal{M}$ are as in Problem 1.1, by applying a similarity transformation we can assume without loss of generality that $\mathcal{M}$ is spanned by first p unit coordinate vectors in $\mathbb{F}^n$; thus A has the block form

$$A = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix},$$

where $A_1 \in \mathbb{F}^{p\times p}$, $A_2 \in \mathbb{F}^{(n-p)\times(n-p)}$. If the minimal polynomials of A_1 and A_2 are coprime, then it is easy to see that $\mu(A, \mathcal{M}) = 0$, i.e., A has an invariant subspace $\mathcal{N}$ complementary to $\mathcal{M}$. Indeed, such $\mathcal{N}$ is spanned by the columns of $\begin{pmatrix} Q \\ I \end{pmatrix}$, where the matrix Q satisfies the equation

$$\begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} Q \\ I \end{pmatrix} = \begin{pmatrix} Q \\ I \end{pmatrix} A_2,$$

or

$$QA_2 - A_1Q = A_{12}. \tag{1.1}$$

It is well known that the *Sylvester map* $Q \mapsto QA_2 - A_1Q$ is invertible if and only if the minimal polynomials of A_1 and A_2 are coprime. See, e.g., [5, 7] for this fact; although this was established in [5, 7] only for the complex field, the extension to any algebraically closed field is immediate, and to prove this fact for the general field $\mathbb{F}$ one considers the algebraic closure of $\mathbb{F}$. Hence (1.1) can be solved for Q for any given A_{12}, and $\mu(A, \mathcal{M}) = 0$ is established. This example shows close connections of Problem 1.1 with properties of Sylvester maps. There is a large literature (in mathematical and engineering journals) on numerical analysis involving Sylvester maps; see, e.g., [3, 4] and the references cited there.

Connected to the Sylvester map the following problem appears in the theory of control for coordination (see [6]). Given is a linear system $\dot{x}(t) = Ax(t)$, where the

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matrix A has the form

$$A = \begin{pmatrix} A_{11} & 0 & A_{1c} \\ 0 & A_{22} & A_{2c} \\ 0 & 0 & A_{cc} \end{pmatrix}$$

with respect to a fixed decomposition $X = X_1 \dot{+} X_2 \dot{+} X_c$ of the state space. One allows transformations $A \mapsto S^{-1}AS$, where S is of the form

$$S = \begin{pmatrix} I & 0 & S_1 \\ 0 & I & S_2 \\ 0 & 0 & I \end{pmatrix}.$$

Then

$$S^{-1}AS = \begin{pmatrix} A_{11} & 0 & A_{11}S_1 - S_1A_{cc} + A_{1c} \\ 0 & A_{22} & A_{22}S_2 - S_2A_{cc} + A_{2c} \\ 0 & 0 & A_{cc} \end{pmatrix}.$$

¿From the point of view of communicating as little as possible between the coordinator acting in X_c and the subsystems acting in X_1 and X_2, it is of interest to study when the ranks of $A_{ii}S_i - S_iA_{cc} + A_{ic}$ are as small as possible for $i = 1, 2$. It is precisely this problem we shall discuss in the next section.

2. Ranges of Sylvester maps. We recall the definition of invariant polynomials. For $A \in \mathbb{F}^{p\times p}$, we let

$$\lambda I - A = E(\lambda)\text{diag}\,(\phi_{A,1}(\lambda), \ldots, \phi_{A,p}(\lambda))F(\lambda),$$

where $E(\lambda)$, $F(\lambda)$ are everywhere invertible matrix polynomials, and $\phi_{A,j}$ are scalar monic polynomials such that $\phi_{A,j}$ is divisible by $\phi_{A,j+1}$, for $j = 1, \ldots, p-1$. The polynomials $\phi_{A,j}$ are called the *invariant polynomials* of A; $\phi_{A,1}$ is in fact the minimal polynomial of A.

For two matrices A_1 and A_2 over the field $\mathbb{F}$ of sizes $p \times p$ and $q \times q$, respectively, define the nonnegative integer $s(A_1, A_2)$ as

$$\max\{j \mid 1 \le j \le \min\{p, q\},\ \phi_{A_1,j}(\lambda) \text{ and } \phi_{A_2,j}(\lambda) \text{ are not coprime}\}.$$

The maximum of the empty set in this formula is assumed to be zero. Clearly, $s(A_1, A_2) = 0$ if and only if the minimal polynomials of A_1 and A_2 are coprime.

If all eigenvalues of A_1 and A_2 are in $\mathbb{F}$ (in particular if $\mathbb{F}$ is algebraically closed), then

$$s(A_1, A_2) = \max_{\lambda\in\mathbb{F}} \min\{\dim \text{Ker}\,(A_1 - \lambda I), \dim \text{Ker}\,(A_2 - \lambda I)\}.$$

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Consider the linear Sylvester map (for fixed A_1 and A_2) $T : \mathbb{F}^{p\times q} \longrightarrow \mathbb{F}^{p\times q}$ defined by

$$T(S) = SA_2 - A_1S, \qquad S \in \mathbb{F}^{p\times q},$$

THEOREM 2.1.

(a) *Every matrix* $X \in \mathbb{F}^{p\times q}$ *can be written in the form*

$$X = T(S) + Y,$$

for some $S \in \mathbb{F}^{p\times q}$ *and some* $Y \in \mathbb{F}^{p\times q}$ *with* $\operatorname{rank} Y \le s(A_1, A_2)$.

(b) *Assume that* $s(A_1, A_2) \ne 0$, *and that the greatest common divisor of the minimal polynomials of* A_1 *and* A_2 *have all their roots in* $\mathbb{F}$ *(in particular, this condition is always satisfied if* $\mathbb{F}$ *is algebraically closed). Then for fixed* A_1 *and* A_2, *there is a Zariski open nonempty set* Ω *of* $\mathbb{F}^{p\times q}$ *such that for every* $X \in \Omega$, *there is no representation of* X *in the form*

$$X = T(S) + Y,$$

where $S, Y \in \mathbb{F}^{p\times q}$ *are such that* $\operatorname{rank} Y < s(A_1, A_2)$.

Theorem 2.1 can be thought of as a generalization of the well known fact that the Sylvester map is a bijection if and only if the minimal polynomials of A_1 and A_2 are coprime.

Consider the following example to illustrate Theorem 2.1. Let

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The invariant polynomials are

$$\phi_{A_1,1}(\lambda) = \phi_{A_2,1}(\lambda) = \lambda(\lambda - 1),$$

$$\phi_{A_1,2}(\lambda) = \lambda, \quad \phi_{A_1,2}(\lambda) = \lambda - 1, \quad \phi_{A_1,3}(\lambda) = \phi_{A_2,3}(\lambda) = 1.$$

We have $s(A_1, A_2) = 1$. The range of the Sylvester map is easy to find:

$$\operatorname{Range} T = \left\{ \begin{bmatrix} 0 & * & * \\ 0 & * & * \\ * & 0 & 0 \end{bmatrix} \right\},$$

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where by * we denote arbitrary entries which are independent free variables. For every $X = [x_{i,j}] \in \mathbb{F}^{3\times 3}$, we have

$$X = \begin{bmatrix} x_{1,1} \\ x_{2,1} \\ 1 \end{bmatrix} \begin{bmatrix} 1 & x_{3,2} & x_{3,3} \end{bmatrix} + \begin{bmatrix} 0 & * & * \\ 0 & * & * \\ * & 0 & 0 \end{bmatrix}.$$

3. Proof of Theorem 2.1. Part (a). We use the rational canonical forms for A_1 and A_2 (see, e.g., [5]), together with the invariance of the statement of the theorem and of its conclusions under similarity transformations

$$A_1 \longrightarrow G_1^{-1}A_1G_1, \quad A_2 \longrightarrow G_2^{-1}A_2G_2, \quad G_1 \text{ and } G_2 \text{ invertible.}$$

We may assume therefore without loss of generality that

$$A_1 = \operatorname{diag}(A_1^{(1)}, \ldots, A_1^{(u)}) \quad \text{and} \quad A_2 = \operatorname{diag}(A_2^{(1)}, \ldots, A_2^{(v)}), \tag{3.1}$$

where

$$A_1^{(j)} = \operatorname{diag}(A_1^{(j,1)}, \ldots, A_1^{(j,\gamma_{1,j})}), \quad j = 1, \ldots, u,$$

and where the characteristic polynomials of $A_1^{(j,k)}$, $k = 1, \ldots, \gamma_{1,j}$, are all powers of the same monic irreducible polynomial $f_{1,j}$. The matrices $A_1^{(j,k)}$ are nonderogatory, i.e., the minimal and the characteristic polynomials of $A_1^{(j,k)}$ coincide. In addition, we require that the irreducible polynomials $f_{1,1}, \ldots, f_{1,u}$ are all distinct.

Similarly,

$$A_2^{(j)} = \operatorname{diag}(A_2^{(j,1)}, \ldots, A_2^{(j,\gamma_{2,j})}), \quad j = 1, \ldots, v,$$

where the characteristic polynomials of $A_2^{(j,k)}$, $k = 1, \ldots, \gamma_{2,j}$, are all powers of the same monic irreducible polynomial $f_{2,j}$, and the polynomials $f_{2,1}, \ldots, f_{2,v}$ are all distinct. Again, the matrices $A_2^{(j,k)}$ are nonderogatory.

Moreover, we arrange the blocks $A_1^{(1)}, \ldots, A_1^{(u)}$ and $A_2^{(1)}, \ldots, A_2^{(v)}$ so that

$$f_{1,1} = f_{2,1}, \ldots, f_{1,\ell} = f_{2,\ell},$$

but the irreducible polynomials

$$f_{1,1}, \ldots, f_{1,u}, f_{2,\ell+1}, \ldots, f_{2,v}$$

are all distinct. (The case when $\ell = 0$, i.e., the polynomials $f_{1,j}$ and $f_{2,j}$ are all distinct, is not excluded; in this case the subsequent arguments should be modified in

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obvious ways.) Note that since powers of distinct irreducible polynomials are coprime, it follows that the characteristic polynomials of the matrices

$$A_1^{(1)}, \dots, A_1^{(u)}, A_2^{(\ell+1)}, \dots, A_2^{(v)} \tag{3.2}$$

are pairwise coprime.

We assume in addition that $A_1^{(j,k)}$ are companion matrices. To set up notation, we let e_j be a row with 1 in the jth position and zeros in all other positions (the number of components in e_j will be evident from context), and analogously let e_j^T (the transpose of e_j) be the column with 1 in the jth position and zeros in all other positions. Let $\xi_{1,j,k}$ (resp., $\xi_{2,j,k}$) be the size of the matrix $A_1^{(j,k)}$ (resp., $A_2^{(j,k)}$). Thus, we let

$$A_1^{(j,k)} = \begin{bmatrix} e_2 \\ e_3 \\ \vdots \\ e_{\xi_{1,j,k}} \\ \alpha_{1,j,k} \end{bmatrix} \tag{3.3}$$

or

$$A_1^{(j,k)} = \begin{bmatrix} e_2^T & e_3^T & \dots & e_{\xi_{1,j,k}}^T & \alpha_{1,j,k}^T \end{bmatrix} \tag{3.4}$$

for some row $\alpha_{1,j,k}$ (with entries in $\mathbb{F}$), and analogously,

$$A_2^{(j,k)} = \begin{bmatrix} e_2 \\ e_3 \\ \vdots \\ e_{\xi_{2,j,k}} \\ \alpha_{2,j,k} \end{bmatrix} \tag{3.5}$$

or

$$A_2^{(j,k)} = \begin{bmatrix} e_2^T & e_3^T & \dots & e_{\xi_{2,j,k}}^T & \alpha_{2,j,k}^T \end{bmatrix} \tag{3.6}$$

for some row $\alpha_{2,j,k}$.

The forms (3.3) and (3.5) will be used if $\gamma_{1,j} \le \gamma_{2,j}$, and the forms (3.4) and (3.6) will be used if $\gamma_{1,j} > \gamma_{2,j}$.

We return to the Sylvester map T. Conformably with (3.1), we partition

$$S = [S_{j_1,j_2}]_{j_1=1,\dots,u; j_2=1,\dots,v}.$$

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Thus,

$$T(S) = SA_2 - A_1 S = [S_{j_1,j_2} A_2^{(j_2)} - A_1^{(j_1)} S_{j_1,j_2}]_{j_1=1,\ldots,u; j_2=1,\ldots,v}.$$

Also, if $X \in \mathbb{F}_{p\times q}$ is an arbitrary matrix, then we partition again conformably with (3.1):

$$X = [X_{j_1,j_2}]_{j_1=1,\ldots,u; j_2=1,\ldots,v}.$$

We will show that for any given $X \in \mathbb{F}^{p\times q}$, there exist

$$S_{j,j} \in \mathbb{F}^{(\xi_{1,j,1}+\cdots+\xi_{1,j,\gamma_{1,j}})\times(\xi_{2,j,1}+\cdots+\xi_{2,j,\gamma_{2,j}})}, \quad j = 1,\ldots,\ell,$$

with the property that

$$X_{j,j} = Y_{j,j} + \left(S_{j,j} A_2^{(j)} - A_1^{(j)} S_{j,j}\right), \tag{3.7}$$

for some matrix $Y_{j,j}$ such that

$$\operatorname{rank} Y_{j,j} \le \min\{\gamma_{1,j}, \gamma_{2,j}\}, \qquad j = 1,\ldots,\ell. \tag{3.8}$$

Assuming that we have already shown the existence of $S_{j,j}$ satisfying (3.8) and (3.7), we can easily complete the proof of Part (a).

Indeed, let

$$\mu = \max_{j=1,\ldots,\ell} (\min\{\gamma_{1,j}, \gamma_{2,j}\}),$$

and notice that $\mu = s(A_1, A_2)$. Now let

$$Y_{j,j} = W_{j,j} Z_{j,j}, \quad j = 1,\ldots,\ell$$

be a rank decomposition, where the matrix $W_{j,j}$ (resp., $Z_{j,j}$) has μ columns (resp., μ rows). We also put formally $W_{j,j} = 0$ for $j = \ell+1,\ldots,u$, and and $Z_{j,j} = 0$ for $j = \ell+1,\ldots,v$. Using the property that the characteristic polynomials of matrices (3.2) are pairwise coprime, and that the Sylvester map $S \mapsto SB_2 - B_1 S$ is onto if the characteristic polynomials of B_1 and B_2 are coprime, we find

$$S_{j_1,j_2} \in \mathbb{F}^{(\xi_{1,j_1,1}+\cdots+\xi_{1,j_1,\gamma_{1,j_1}})\times(\xi_{2,j_2,1}+\cdots+\xi_{2,j_2,\gamma_{2,j_2}})}$$

for

$$j_1 = 1,\ldots,u, \quad j_2 = 1,\ldots,v, \quad (j_1,j_2) \notin \{(1,1)\ldots,(\ell,\ell)\},$$

such that

$$X_{j_1,j_2} = W_{j_1,j_1} Z_{j_2,j_2} + \left(S_{j_1,j_2} A_2^{(j_2)} - A_1^{(j_1)} S_{j_1,j_2}\right).$$

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Now letting

$$Y = \begin{bmatrix} W_{1,1} \\ W_{2,2} \\ \vdots \\ W_{u,u} \end{bmatrix} \begin{bmatrix} Z_{1,1} & Z_{2,2} & \cdots & Z_{v,v} \end{bmatrix},$$

we have $X = T(S) + Y$, and obviously, $\operatorname{rank} Y \leq \mu$.

Thus, it remains to show the existence of $S_{j,j}$ satisfying (3.7) and (3.8). We fix j, $j = 1, \ldots, \ell$. We assume that $\gamma_{1,j} \leq \gamma_{2,j}$, thus (3.3) and (3.5) will be used; if $\gamma_{1,j} > \gamma_{2,j}$ the proof is completely analogous using (3.4) and (3.6). Choose rows $\alpha'_{1,j,k}$ $(k = 1, \ldots, \gamma_{1,j})$ so that the characteristic polynomials of the matrices

$$B_1^{(j,k)} := \begin{bmatrix} e_2 \\ e_3 \\ \vdots \\ e_{\xi_{1,j,k}} \\ \alpha'_{1,j,k} \end{bmatrix}, \quad k = 1, \ldots, \gamma_{1,j},$$

are coprime to $f_{2,j} = f_{1,j}$, and let

$$B_1^{(j)} = \operatorname{diag}\left(B_1^{(j,1)}, \ldots, B_1^{(j,\gamma_{1,j})}\right).$$

Therefore, we can find $S_{j,j}$ so that

$$X_{j,j} = S_{j,j} A_2^{(j)} - B_1^{(j)} S_{j,j}.$$

Now (3.7) holds with

$$Y_{j,j} = \left(B_1^{(j)} - A_1^{(j)}\right) S_{j,j},$$

and since the matrix $Y_{j,j}$ has at most $\gamma_{1,j}$ nonzero rows, we have $\operatorname{rank} Y_{j,j} \leq \gamma_{1,j}$, as required.

Part (b). We assume that A_1 and A_2 have the form (3.1), and use the notation introduced in the proof of Part (a). We have $\ell \geq 1$. Let j_0 be such that $\mu = \min\{\gamma_{1,j_0}, \gamma_{2,j_0}\}$. Without loss of generality we may assume $j_0 = 1$. Let $p_1 \times p_1$ and $q_1 \times q_1$, be the size of

$$A_1^{(1)} = \operatorname{diag}(A_1^{(1,1)}, \ldots, A_1^{(1,\gamma_{1,1})}) \quad \text{and} \quad A_2^{(1)} = \operatorname{diag}(A_2^{(1,1)}, \ldots, A_2^{(1,\gamma_{2,1})}),$$

respectively. It is easy to see that it suffices to find a Zariski open nonempty set Ω_1 of $\mathbb{F}^{p_1 \times q_1}$ such that for every $X_1 \in \Omega_1$, there is no representation of X_1 in the form $X_1 =$

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$(SA_2^{(1)} - A_1^{(1)}S) + Y$, where $S, Y \in F^{p_1 \times q_1}$ and $\operatorname{rank} Y < \min\{\gamma_{1,1}, \gamma_{2,1}\}$. Because of the hypotheses of Part (b), we may assume that every matrix $A_k^{1,w}$, $w = 1, 2, \ldots, \gamma_{k,1}$, $k = 1, 2$, is an (upper triangular) Jordan block with the same eigenvalue λ; let the size of this block be $p_{k,w} \times p_{k,w}$. Consider a matrix of the form $SA_2^{(1)} - A_1^{(1)}S$, $S \in \mathbb{F}^{p_1 \times q_1}$, which is partitioned

$$SA_2^{(1)} - A_1^{(1)}S = [Q_{\alpha,\beta}]_{\alpha=1,\ \beta=1}^{\gamma_{1,1},\ \gamma_{2,1}}, \tag{3.9}$$

where the block $Q_{\alpha,\beta}$ has the size

$$\left(\text{size of } A_1^{1,\alpha}\right) \times \left(\text{size of } A_2^{1,\beta}\right).$$

Since the $A_k^{1,w}$'s are Jordan blocks, the bottom left corners of the blocks $Q_{\alpha,\beta}$ are all zeros. Now partition

$$X_1 = [X_{\alpha,\beta}]_{\alpha=1,\ \beta=1}^{\gamma_{1,1},\ \gamma_{2,1}} \in \mathbb{F}^{p_1 \times q_1}$$

comformably with the right hand side of (3.9). The Zariski open set Ω_1 consists of exactly those matriceds X_1 for which the $\gamma_{1,1} \times \gamma_{2,1}$ matrix formed by the bottom left corners of the $X_{\alpha,\beta}$'s has the full rank, equal to $\min\{\gamma_{1,1}, \gamma_{2,1}\}$. □

4. A special case of the minimal rank problem.

Given a subspace $\mathcal{M} \subseteq \mathbb{F}^n$ and a matrix $Z \in \mathbb{F}^{n \times n}$, we say that $\mathcal{M}$ is a *complementary Z-invariant subspace* if $\mathcal{M}$ is Z-invariant and some direct complement to $\mathcal{M}$ in $\mathbb{F}^n$ is also Z-invariant. Denote by $\mathcal{CI}(\mathcal{M})$ the set of all matrices Z for which $\mathcal{M}$ is a complementary invariant subspace.

The following problem is closely related to Problem 1.1.

Problem 4.1. *Given a matrix $A \in \mathbb{F}^{n \times n}$ and its invariant subspace $\mathcal{M} \subseteq \mathbb{F}^n$, find the smallest possible rank of the differences $A - Y$, where Y is an arbitrary matrix in $\mathcal{CI}(\mathcal{M})$, and find a matrix $Z \in \mathcal{CI}(\mathcal{M})$ such that the difference $A - Z$ has this smallest possible rank.*

In fact, Problem 4.1 requires an extra condition for Z in comparison with Problem 1.1, namely, that $\mathcal{M}$ is an invariant subspace for Z.

Using similarity, we assume without loss of generality that

$$\mathcal{M} = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} : x \in F^p \right\}, \qquad A = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix}, \qquad A_1 \in \mathbb{F}^{p \times p}, \quad A_2 \in \mathbb{F}^{q \times q}.$$

Theorem 2.1 sheds some light on Problem 4.1, as follows.

Theorem 4.2. *Let $\kappa := s(A_1, A_2)$. There exists a matrix $Z \in \mathcal{CI}(\mathcal{M})$ such that*

$$\operatorname{rank}(A - Z) \le \kappa. \tag{4.1}$$

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Moreover, Z can be taken in the form

$$Z = \begin{pmatrix} A_1 & W \\ 0 & A_2 \end{pmatrix} \tag{4.2}$$

for some W.

Proof. We use Theorem 2.1. Indeed, if Z is in the form (4.2), then $Z \in \mathcal{CI}(\mathcal{M})$ if and only if for some matrix $Q \in \mathbb{F}^{p\times q}$ the subspace Span $\begin{pmatrix} Q \\ I \end{pmatrix}$ is Z-invariant, i.e., the equation

$$\begin{pmatrix} A_1 & W \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} Q \\ I \end{pmatrix} = \begin{pmatrix} Q \\ I \end{pmatrix} A_2$$

holds, or equivalently,

$$QA_2 - A_1Q - (W - A_{12}) = A_{12}.$$

By Theorem 2.1, such Q exists for some W with the property that rank $(A_{12}-W) \le \kappa$. Since obviously

$$\operatorname{rank}(A - Z) = \operatorname{rank}(A_{12} - W),$$

the result follows. □

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