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# EXPLICIT POLAR DECOMPOSITION OF COMPANION MATRICES*

P. VAN DEN DRIESSCHE† AND H. K. WIMMER‡

**Abstract.** An explicit formula for the polar decomposition of an $n \times n$ nonsingular companion matrix is derived. The proof involves the largest and smallest singular values of the companion matrix.

**Key words.** companion matrices, polar decomposition, singular values

**AMS(MOS) subject classification.** 15A23, 15A18

## 1. Introduction.

Let

$$f(z) = z^n - a_{n-1}z^{n-1} - \cdots - a_1 z - a_0, \quad a_0 \neq 0,$$

be a complex polynomial and

$$C = \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ a_0 & a_1 & \dots & a_{n-1} \end{bmatrix} \tag{1}$$

be an $n \times n$ nonsingular companion matrix associated with $f(z)$. Let $C = PU$ be the left polar decomposition of $C$ with positive-definite $P$ and unitary $U$. The singular values of $C$, i.e., the eigenvalues of $P$, are well known ([1], [5], [6]). They yield bounds for zeros and for products of zeros of $f(z)$ [6], and they are used for the computation of robustness measures in systems theory [5]. In view of the wide range of applications, both of the polar decomposition and of companion matrices, an explicit formula for $C = PU$ is useful. It is the purpose of this note to derive explicit expressions for the factors $P$ and $U$ in terms of the coefficients $a_\nu$ of $f(z)$. As companion matrices have been included in collections of test matrices (see e.g., Table I of [3]) our formula adds yet one more possibility for testing computational algorithms in numerical linear algebra. Our formula also shows that companion matrices belong to the class

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† Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia, Canada V8W 3P4 (pvdd@smart.math.uvic.ca). The work of this author was supported in part by Natural Sciences and Engineering Research Council of Canada grant A-8965 and by the University of Victoria Committee on Faculty Research and Travel.

‡ Mathematisches Institut, Universität Würzburg, D-97074 Würzburg, Germany (wimmer@mathematik.uni-wuerzburg.de).

of matrices for which the polar decomposition is finitely computable. Whether all complex square matrices have that property is an open problem, which has been studied in [2].

**2. Polar decomposition formula.** Our main result, Theorem 2.1, is the explicit formula for $P$ and $U$ in the left polar decomposition of a nonsingular companion matrix $C$ where the coefficients of the polynomial $f(z)$ form the last row.

THEOREM 2.1. *Let the companion matrix $C$ in* (1) *be partitioned as*

$$C = \begin{bmatrix} 0 & I_{n-1} \\ a_0 & d^* \end{bmatrix},$$

*with $a_0 \neq 0$ and $d^* = (a_1, \ldots, a_{n-1})$. Define*

$$w = \left[(|a_0|+1)^2 + |a_1|^2 + \cdots + |a_{n-1}|^2\right]^{\frac{1}{2}} = \left[(|a_0|+1)^2 + \|d\|^2\right]^{\frac{1}{2}} \tag{2}$$

*and*

$$v = \frac{a_0}{|a_0|w} \begin{bmatrix} -d \\ 1+|a_0| \end{bmatrix}. \tag{3}$$

*Then*

$$P = \frac{1}{w} \begin{bmatrix} wI_{n-1} - (w+|a_0|+1)^{-1}dd^* & d \\ d^* & w^2 - |a_0| - 1 \end{bmatrix} \tag{4}$$

*is positive definite and $P^2 = CC^*$. Assume $P = (p_1, \ldots, p_n)$ and set $U = (v, p_1, \ldots, p_{n-1})$. Then $U$ is unitary and $C = PU$ is the left polar decomposition of* (1).

To prove Theorem 2.1 we first consider the singular values of $C$, i.e. the nonnegative square roots of the eigenvalues of

$$CC^* = \begin{bmatrix} I_{n-1} & d \\ d^* & s \end{bmatrix}, \tag{5}$$

where

$$s = \sum_{i=0}^{n-1} |a_i|^2 = |a_0|^2 + \|d\|^2.$$

Set $a_n = 1$, and define

$$F(z) = z^2 - \left(\sum_{i=0}^{n} |a_i|^2\right) z + |a_0|^2. \tag{6}$$

The following result is known (see, e.g., [1, pp. 224–225], [5], [6]). To make our note self-contained we include a simple proof.

LEMMA 2.2. *Let* $0 < \sigma_1 \le \sigma_2 \le \cdots \le \sigma_n$ *be the singular values of* $C$. *Then* $\sigma_2 = \cdots = \sigma_{n-1} = 1$, *and* $\sigma_1^2, \sigma_n^2$ *are the zeros of* $F(z)$ *in* (6).

*Proof.* From (5) it follows that

$$
\begin{aligned}
(7)\quad \det(zI_n - CC^*) &= (z-s)\det[(z-1)I_{n-1}] - d^*\mathrm{adj}[(z-1)I_{n-1}]d \\
&= (z-1)^{n-2}\left[z^2 - (s+1)z + s - \|d\|^2\right] \\
&= (z-1)^{n-2}F(z).
\end{aligned}
$$

Thus $CC^*$ has 1 as eigenvalue of multiplicity at least $(n-2)$. Since the eigenvalues of the principal submatrix $I_{n-1}$ in (5) interlace those of $CC^*$, it follows that $\sigma_1^2 \le 1 \le \sigma_n^2$. □

Note that, as $F(z)$ in (6) is quadratic, the values of $\sigma_1^2, \sigma_n^2$ can be found explicitly in terms of $|a_0|^2$ and $\|d\|^2$, see [5, Th. 3.1]. Also $\sigma_1\sigma_n = |a_0|$ and $\sigma_1^2 + \sigma_n^2 = s+1$. These relations give $\sigma_1 + \sigma_n = w$, and $\|d\|^2 = s - \sigma_1^2\sigma_n^2 = -\left(\sigma_n^2 - 1\right)\left(\sigma_1^2 - 1\right)$. From (2) follows

$$
(8)\qquad \|d\|^2 = (w + |a_0| + 1)(w - |a_0| - 1).
$$

For the computation of the square root of $CC^*$ only a symmetric $2\times 2$ matrix has to be considered. The following can easily be verified.

LEMMA 2.3. *Let*

$$
H = \begin{bmatrix} 1 & \|d\| \\ \|d\| & |a_0|^2 + \|d\|^2 \end{bmatrix}.
$$

*Then,* $\det(zI - H) = F(z) = \left(z - \sigma_1^2\right)\left(z - \sigma_n^2\right)$, *and*

$$
H^{\frac{1}{2}} = w^{-1}\begin{bmatrix} 1 + |a_0| & \|d\| \\ \|d\| & w^2 - |a_0| - 1 \end{bmatrix}.
$$

*Proof of Theorem 2.1.* The case with $\sigma_1 = 1$ or $\sigma_n = 1$ is equivalent to $F(1) = 0$, or because of (7), equivalent to $d = 0$. In this case (5) implies

$$
P = (CC^*)^{\frac{1}{2}} = \mathrm{diag}(1, \ldots, 1, |a_0|).
$$

Furthermore $C = PU$ with $P$ as above and

$$
U = \begin{bmatrix} 0 & I_{n-1} \\ \frac{a_0}{|a_0|} & 0 \end{bmatrix},
$$

agreeing with (4) and (3).

In the case $\sigma_1 < 1 < \sigma_n$, that is $d \neq 0$, we define vectors

$$v_1 = \frac{1}{\|d\|}\begin{bmatrix} d \\ 0 \end{bmatrix}, \quad v_n = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then (5), and $CC^*v_1 = v_1 + \|d\|v_n$ and $CC^*v_n = \|d\|v_1 + sv_n$, imply

$$CC^*(v_1, v_n) = (v_1, v_n)\,H.$$

Now consider the eigenvalue 1 of $CC^*$ and let $y_2, \ldots, y_{n-1}$ be an orthonormal set of eigenvectors of $CC^*$ satisfying $CC^*y_i = y_i$, for $i = 2, \ldots, n-1$. Note that for each $y_i$ we have $y_i^* = (x_i^*, 0)$ and $d^*x_i = 0$. Then $V = (y_2, \ldots, y_{n-1}, v_1, v_n)$ is a unitary matrix, and

$$V^*CC^*V = \begin{bmatrix} I_{n-2} & 0 \\ 0 & H \end{bmatrix}.$$

Hence

$$P = (CC^*)^{\frac{1}{2}} = V\begin{bmatrix} I_{n-2} & 0 \\ 0 & H^{\frac{1}{2}} \end{bmatrix}V^*,$$

where $H^{\frac{1}{2}}$ is given in Lemma 2.3. Thus

$$P = I_n + (v_1, v_n)(H^{\frac{1}{2}} - I_2)\begin{bmatrix} v_1^* \\ v_n^* \end{bmatrix}.$$

From (8) it follows that

$$H^{\frac{1}{2}} - I_2 = w^{-1}\begin{bmatrix} -\|d\|^2(w + |a_0| + 1)^{-1} & \|d\| \\ \|d\| & w^2 - |a_0| - 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

On multiplication, the above expression for $P$ yields (4).

For a nonsingular companion matrix $C$ given by (1) it is well known that

$$C^{-1} = \begin{bmatrix} \frac{-d^*}{a_0} & \frac{1}{a_0} \\ I_{n-1} & 0 \end{bmatrix}$$

For the unitary factor of $C = PU$, we have $U = P(C^{-1})^*$. Hence

$$U = (p_1, \ldots, p_{n-1}, p_n)\begin{bmatrix} \frac{-d}{\bar{a}_0} & I_{n-1} \\ \frac{1}{\bar{a}_0} & 0 \end{bmatrix} = (v, p_1, \ldots, p_{n-1}),$$

with

$$v = \frac{1}{\bar{a}_0} P \begin{bmatrix} -d \\ 1 \end{bmatrix} = \frac{1}{\bar{a}_0 w} \begin{bmatrix} [-w + \|d\|^2 (w + |a_0| + 1)^{-1} + 1] d \\ -\|d\|^2 + w^2 - |a_0| - 1 \end{bmatrix}.$$

Using (8) yields (3) and completes the proof. □

It is well known (see, e.g., [4] or [7]) that for a given nonsingular matrix the unitary factors in the left and in the right polar decomposition are equal. Now define

$$\gamma = 1 - \frac{1}{w + |a_0| + 1}$$

and set

$$Q = \frac{1}{w} \begin{bmatrix} |a_0| + |a_0|^2 & \bar{a}_0 d^* \\ a_0 d & w I_{n-1} + \gamma d d^* \end{bmatrix}.$$

It is not difficult to verify that $Q$ is positive definite and $Q^2 = C^* C$. Hence if $C$ is nonsingular and $U$ is given as in Theorem 2.1, then $C = UQ$ is the right polar decomposition of (1).

Let $C_{lr}$, $C_{lc}$, $C_{fr}$, $C_{fc}$ be the companion matrices where the coefficients of the polynomial $f(z)$ form the last row, last column, first row, first column, respectively. So far in our note we have considered $C = C_{lr}$. Using the $n \times n$ permutation matrix (the reverse unit matrix) $K = (k_{ij})$ where $k_{i,n-i+1} = 1$, and 0 elsewhere, we note that

$$C_{lr} = C^T, \quad C_{fr} = KCK, \quad C_{fc} = KC^T K.$$

Hence the polar decompositions of the preceding three types of companion matrices are products that involve the matrices $U$, $K$, and $P$ or $Q$. For any real nonsingular $2 \times 2$ matrix the right polar decomposition in closed form is given in [8].

There is a relation between the singular values $\sigma_1$ and $\sigma_n$ of $C$ and the zeros $\lambda$ of the polynomial $f(z)$, namely $\sigma_1 \leq |\lambda| \leq \sigma_n$. Is it possible that the eigenvalues $e^{i\varphi_\nu}$, $\nu = 1, \ldots, n$, of the unitary factor $U$ also provide information on the geometry of the zeros of $f(z)$?

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